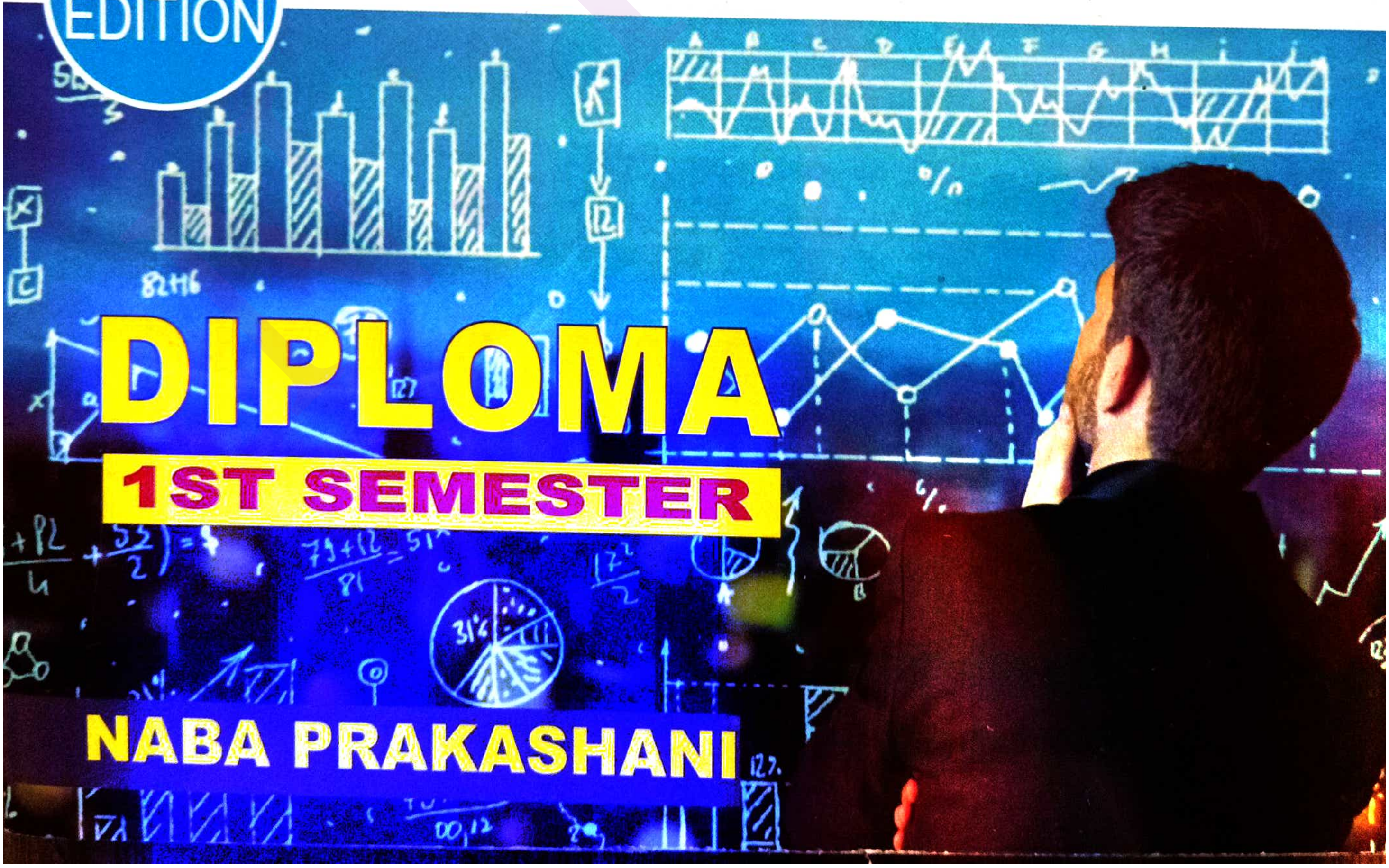
NEW SYLLABUS

1

# MATHEMATICS

A. SARKAR

NEW EDITION



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# MATHEMATICS - 1

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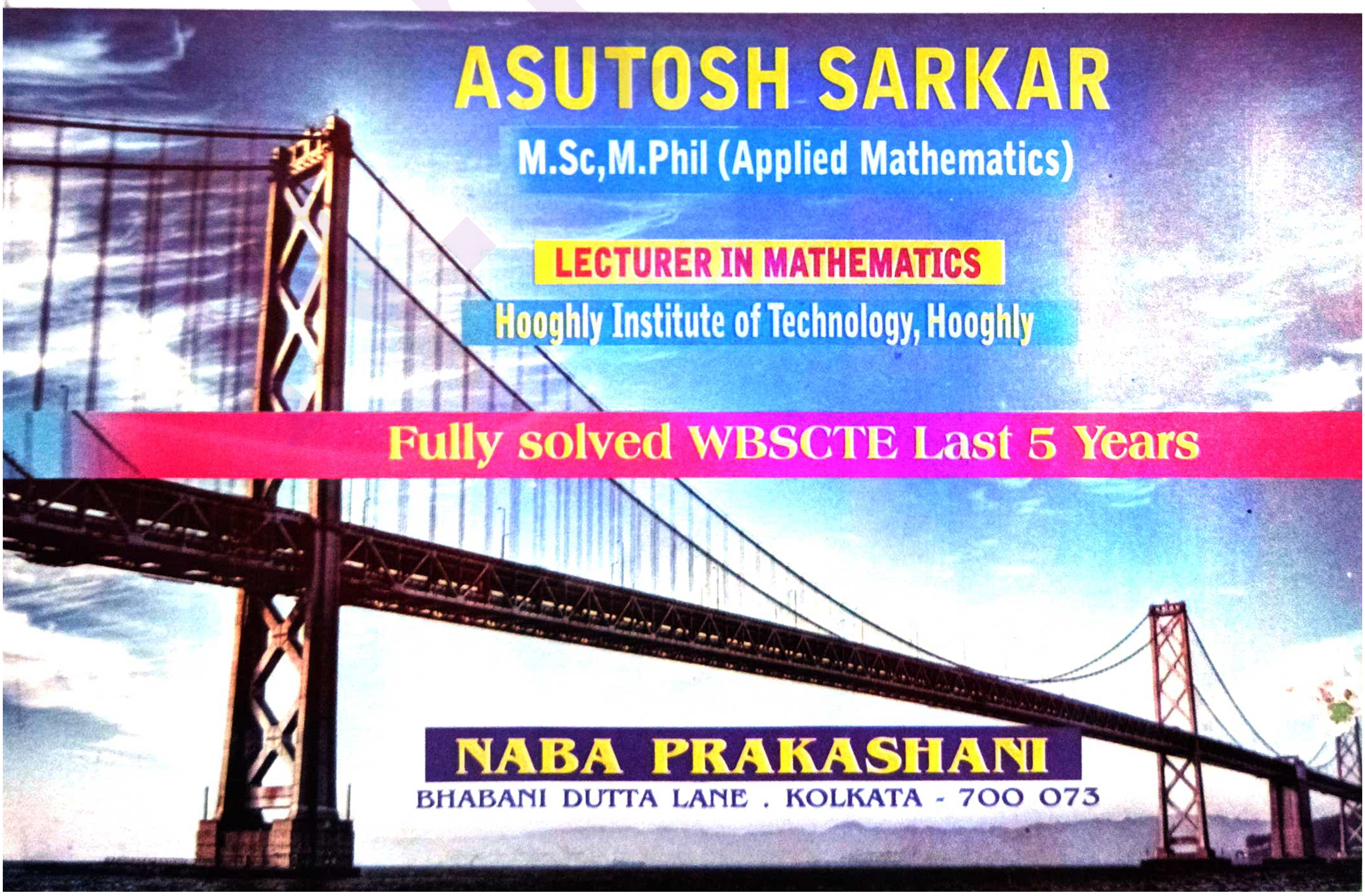


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Fully solved WBSCTE Last 5 Years


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# PREFACE TO THE NEW REVISED EDITION

This book presents the whole of the subject matter covered by the Revised Syllabus of the Diploma Courses in Engineering and Technology from the Academic Session **2020 – 2021** of **MATHEMATICS – I** of West Bengal State Council of Technical & Vocational Education and Skill Development.

The contents of the text are divided into Four Units as follows:

**Unit 1:: ALGEBRA**

**Unit 2:: VECTOR ALGEBRA**

**Unit 3:: TRIGONOMETRY**

**Unit 4:: FUNCTION, LIMIT AND CONTINUITY, DERIVATIVE**

The distinguishable features of this book are:

1. All necessary discussions, theorems, properties and formulae have been given in the beginning of respective chapters followed by PROBLEM SET, ANSWERS of all the sums and SOLUTION OF THE PROBLEMS MARKED BY '*' at the end of respective PROBLEM SET.
2. All the sums of the previous Examination Papers upto 2018 of West Bengal State Council of Technical Education are included.
3. Number of sums have been collected from Higher Secondary, Joint Entrance Examination papers.
4. Moreover attempt has been made for **TYPICAL REARRANGEMENT** of the sums and at least one sum from each type from all the chapters are worked out.

The method of presentation of sums is solely based on my teaching experience. Hope it will encourage the students to solve the sums on their own by going through the **SELF LEARNING** process.

Any suggestion for further improvement of this book is welcome.

I thank NABA PRAKASHANI for their sincere co-operation and publishing this book in a short span of time.

Kolkata, November, 2020

*A. Sarkar*

# ENGINEERING MATHEMATICS - 1

## SYLLABUS

### Unit 1 :: ALGEBRA

**1.1 Logarithm**

1.1.1 Definition of natural and common Logarithm.

1.1.2 General properties of Logarithm and Simple problems.

**1.2 Complex Numbers**

1.2.1 Definition of Complex numbers, Real and Imaginary parts of a complex number. Equality of two complex numbers. Conjugate of a complex number.

1.2.2 Modulus and Argument of a complex number and simple problems.

1.2.3 Polar and Cartesian forms of a complex number and their relation.

1.2.4 Algebraic operations (Addition, Subtraction, Multiplication, Division) of complex numbers.

1.2.5 De Moivre's Theorem (without proof) and simple problems

1.2.6 Cube roots of unity and their properties with problems.

**1.3 Quadratic Equations.**

1.3.1 Definition of Quadratic Equations.

1.3.2 Finding the roots of a quadratic equation, conjugate roots and simple problems.

1.3.3 Nature of the roots using discriminant and problems.

1.3.4 Relation between roots & coefficients and problems.

1.3.5 Formation of a quadratic equations if roots are given.

**1.4 Binomial Theorem**

1.4.1 Definition of factorial of a number, permutation($^np_r$) and combination($^nc_r$) with formula only.

1.4.2 Binomial theorem(without proof) for any index, simple problems on positive index only.

1.4.3 General Term and Middle Term and problems.

1.4.4 Expansion of $(1 + x)^{-1}$, $(1 - x)^{-1}$ where $|x| < 1$, exponential and logrithimic series only(no problem).

# SYLLABUS

## Unit 2 :: VECTOR ALGEBRA

2.1 Definition of vector and types of a vectors.

2.2 Concept of a Position vector and Ratio formula and simple problems.

2.3 Rectangular resolution of a vector.

2.4 Equality, addition, subtraction of vectors and multiplication of a vector by a scalar.

2.5 Scalar(dot) and Vector(cross) product of two vectors with properties and simple problems.

2.6 Application of dot product -- work done by a force, projection of a vector upon another.

2.7 Application of cross product -- finding area of a triangle and parallelogram, moment of a force.

## Unit 3 :: TRIGONOMETRY

3.1 Concept of Trigonometrical Angles, measurement of angles in degree, radian and grade and their relations only.

3.2 Trigonometrical Ratios of angles, associated angles, Trigonometrical Ratios of some standard angles, problems.

3.3 Compound angles formula(without proof), multiple and submultiple angles and simple problems.

3.4 Solutions of Trigonometrical Equations, simple problems (angle lies between 0 and 2p)

3.5 Inverse Circular Function and simple problems.

3.6 Properties of Triangle, basic formulae only.

# SYLLABUS

## Unit 4 :: Function, Limit and Continuity, Derivative.

### 4.1 Function

- 4.1.1. Definitions of variables and Constants.
- **4.1.2** Definition of function with examples, domain and range of a function.
- **4.1.3** Types of functions(even-odd, increasing-decreasing, inverse, periodic) with simple examples.
- **4.1.4** Graph of trigonometric functions sinx, cosx, tanx only.

### 4.2 Limit and Continuity

- **4.2.1** Definition of Limit(with left hand limit and right hand limit). Fundamental theorem on limit (only statement), standard limits and simple problems.
- **4.2.2** Continuity of functions, elementary test for continuity of functions(finite limit)

### 4.3 Derivative

- **4.3.1** Definition of derivative.
- **4.3.2** Derivative of standard functions.
- **4.3.3** Rules for differentiation of sum, difference, product and quotient of functions.
- **4.3.4** Derivative of composite functions (Chain rule).
- **4.3.5** Derivative of inverse circular functions, implicit functions, Logarithmic differentiation.
- **4.3.6** Derivative of parametric functions, derivative of a function with respect to another function.
- **4.3.7** Second order derivatives with simple problems.
- **4.3.8** Application of derivatives —— Physical and Geometrical interpretation of derivative, checking increasing-decreasing functions, finding velocity and acceleration, Maxima-Minima of function of single variable with simple problems..

# CONTENTS





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## UNIT – 4 :: FUNCTION, LIMIT & CONTINUITY, DERIVATIVE



## ABBREVIATIONS USED IN THE BOOK

1. **WBSC** — West Bengal State Council Examination
2. **HS** — Higher Secondary Examination
3. **JEE** — Joint Entrance Examination

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# FEW IMPORTANT FORMULA AND RESULTS

## ALGEBRA

### 1. INDICES :

(i) $a^m.a^n = a^{m+n}$ (ii) $\frac{a^m}{a^n} = a^{m-n}, m > n$ (iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^m = a^m.b^m$ (v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ (vi) $a^0 = 1$

(vii) $a^n = \frac{1}{a^{-n}}$ (viii) $a^{-\frac{p}{q}} = \frac{1}{\sqrt[q]{a^p}}$ (ix) $a^m = a^n \Rightarrow m = n$

(x) $a^m = b^m \Rightarrow a = b$

### 2. PROGRESSION :

(i) nth term of an A. P. $t_n = a + (n - 1)d$ (ii) Sum of first n terms $= \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$

(iii) $1 + 2 + 3 + ........................ + n = \frac{n(n+1)}{2}$ (iv) $1^2 + 2^2 + 3^2 + ........................ + n^2 = \frac{n(n+1)(2n+1)}{6}$

(v) $1^3 + 2^3 + 3^3 + ........................ + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

(vi) nth term of a G. P. a, ar, $ar^2$, $ar^3$, ........= $a\,r^{n-1}$

(vii) $a + ar + ar^2 + ar^3 + ..................... + ar^{n-1} = a \cdot \frac{r^n - 1}{r - 1}$ when $r > 1$ and $= a \cdot \frac{1 - r^n}{1 - r}$ when $r < 1$

(viii) $a + ar + ar^2 + ar^3 + .......... + ar^{n-1} + ..........$ to infinity. $= \frac{a}{1 - r}, -1 < r < 1.$

(ix) $a + ar + ar^2 + ar^3 + .......... + ar^{n-1} + ..........$ to infinity. $= \frac{a}{1 - r}, -1 < r < 1.$

### 3. LOGARITHM :

(i) $\log 1 = 0$ (ii) $\log_a a = 1$ (iii) $e^{m \log x} = x^m$

(iv) $\log_a b \times \log_b a = 1$ (v) $\log_a(mn) = \log_a m + \log_a n$ (vi) $\log_a \quad = \log_a m - \log_a n$

(vii) $\log_a m^n = n.\log_a m.$ (viii) $\log_a m = \log_b m \times \log_a b$

N.Tute

## 4. PERMUTATION and COMBINATION :

**(A)** (i) $n! = 1.2.3......(n-2).(n-1).n$; (ii) $n! = n.(n-1)!$; $0! = 1$

**(B)** (i) ${}^nP_r = \frac{n!}{(n-r)!}$ (ii) ${}^nP_n = n!$ (iii) ${}^nC_r = \frac{n!}{r!\cdot(n-r)!}$

(iv) ${}^nC_r = {}^nC_{n-r}$ (v) ${}^nC_p = {}^nC_q \Rightarrow p = q$ or $p + q = n$.

(vi) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (vii) ${}^nC_0 = {}^nC_n = 1$ (viii) ${}^nC_1 = {}^nC_{n-1} = n$

(ix) ${}^nP_r = r!.{}^nC_r$ (x) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

## 5. BINOMIAL THEOREM :

**(A) Expansion of $(a + x)^n$ and $(1 + x)^n$ :**

$$(a+x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + {}^nC_3a^{n-3}x^3 + ..................$$

$$........... + {}^nC_ra^{n-r}x^r + ............. + x^n$$

$$= a^n + n.a^{n-1}x + \frac{n(n-1)}{2!}.a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}.a^{n-3}x^3 + ........$$

$$+ \frac{n(n-1)(n-2)........................(n-r+1)}{r!}a^{n-r}x^r + ...... + x^n,$$ for all values of a and x.

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + .......+ {}^nC_rx^r + ........ + x^n$$

$$= 1 + n.x + \frac{n(n-1)}{2!}.x^2 + \frac{n(n-1)(n-2)}{3!}.x^3 + ..................$$

$$+ \frac{n(n-1)(n-2)........................(n-r+1)}{r!}.x^r + ............. + x^n,$$ for all values of x.

**(B) Expansion of $(a - x)^n$ and $(1 - x)^n$ :**

$(a-x)^n = a^n - {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 - {}^nC_3a^{n-3}x^3 + ............ + (-1)^r.{}^nC_ra^{n-r}x^r + ........ + (-1)^n.x^n$, for all values of a and x.

$$(1-x)^n = 1 - {}^nC_1.x + {}^nC_2.x^2 - {}^nC_3.x^3 + .....+ (-1)^r.{}^nC_r.x^r + ......+ (-1)^n.x^n$$

**(C)(i)** If n is negative or fraction and $-1 < x < 1$ then,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}.x^2 ................. + \frac{n(n-1)(n-2)......(n-r+1)}{r!}.x^r + .................$$

(ii) Expansion of $(a + x)^n$, $a > 0$ when n is a negative integer or fraction:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + .......... + \frac{n(n-1)(n-2)....(n-r+1)}{r!}a^{n-r}x^r + ........$$

**(D) Few important Expansion :**

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots \infty$

(ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 - \ldots\ldots\ldots\ldots \infty$

(iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots \infty$

(iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots \infty$

(v) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + \ldots\ldots\ldots \infty$

(vi) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + \ldots\ldots\ldots\ldots \infty$

(vii) $(1-x)^{-4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + \ldots\ldots\ldots\ldots \infty$

(viii) $(1+x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + \ldots\ldots\ldots\ldots \infty$

**(E) Exponential Series :**

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^r}{r!} + \ldots\ldots \text{to } \infty$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots + (-1)^r . \frac{x^r}{r!} + \ldots\ldots \text{to } \infty$$

$$e^{x.\log a} = 1 + \frac{x.\log_e a}{1!} + \frac{x^2.(\log_e a)^2}{2!} - \frac{x^3.(\log_e a)^3}{3!} + \ldots\ldots \text{to } \infty$$

$$a^x = 1 + \frac{x}{1!}(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 - \frac{x^3}{3!}(\log_e a)^3 + \ldots\ldots \text{to } \infty$$

**(F) LOGARITHMIC SERIES :**

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{to } \infty, \quad -1 < x \le 1$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots \text{to } \infty, \quad -1 \le x < 1$$

$$\frac{1}{2}.\log\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots \text{to } \infty, \quad -1 < x < 1.$$

## TRIGONOMETRY

**1.** LIMITS TO THE VALUES OF TRIGONOMETRICAL RATIOS:

$-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1$; $\sec\theta \ge 1$, $\sec\theta \le -1$; $\operatorname{cosec}\theta \ge 1$, $\operatorname{cosec}\theta \le -1$

$\tan\theta$ and $\cot\theta$ can have any real values.

**2.** Trigonometrical ratios of $-\theta$

$\sin(-\theta) = -\sin\theta$, $\quad \cos(-\theta) = \cos\theta \quad \tan(-\theta) = -\tan\theta$,

$\cot(-\theta) = -\cot\theta \quad \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta \quad \sec(-\theta) = \sec\theta$

**3.** $\sin\theta \operatorname{cosec}\theta = 1 \quad \cos\theta \sec\theta = 1 \quad \tan\theta \cot\theta = 1$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} \quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

**4.** (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$ (ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$ (iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

**5.** (i) $\sin(A + B).\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A + B).\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

**6.** (i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A.\tan B}$ (ii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A.\tan B}$

(iii) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$ (iv) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

**7.** (i) $\sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

$= \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C)$

(ii) $\cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C$

$= \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(iv) $\cot(A+B+C) = \dfrac{\cot A.\cot B.\cot C - \cot A - \cot B - \cot C}{\cot A \cot B + \cot B \cot C + \cot C \cot A - 1}$

**8.** (i) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (ii) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$, $C > D$

(iii) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (iv) $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$, $C < D$

**9.** (i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ (ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ (iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

**10.** (i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ (iv) $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$ (v) $\sin 3A = 3\sin A - 4\sin^3 A$

(vi) $\cos 3A = 4\cos^3 A - 3\cos A$ (vii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$ (viii) $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

**11.** (i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(iii) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$ (iv) $\cot\theta = \dfrac{\cot^2\frac{\theta}{2} - 1}{2\cot\frac{\theta}{2}}$ (v) $\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$

(vi) $\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$ (vii) $\tan\theta = \dfrac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1 - 3\tan^2\frac{\theta}{3}}$ (viii) $\cot\theta = \dfrac{\cot^3\frac{\theta}{3} - 3\cot\frac{\theta}{3}}{3\cot^2\frac{\theta}{3} - 1}$

**12.** (i) $\sin\frac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$ (ii) $\cos\frac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$ (iii) $\tan\frac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

(iv) $\tan\frac{\theta}{2} = \dfrac{-1 \pm\sqrt{1+\tan^2\theta}}{\tan\theta}$ (v) $2\sin\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}$

(vi) $2\cos\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \mp \sqrt{1-\sin\theta}$

**13.** $\sin 18^0 = \dfrac{\sqrt{5}-1}{4}$; $\cos 18^0 = \frac{1}{4}\sqrt{10+2\sqrt{5}}$ $\cos 36^0 = \dfrac{\sqrt{5}+1}{4}$;

$\sin 36^0 = \frac{1}{4}\sqrt{10-2\sqrt{5}}$ $\sin 54^0 = \dfrac{\sqrt{5}+1}{4}$; $\cos 54^0 = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

$\sin 72^0 = \cos 18^0 = \frac{1}{4}\sqrt{10+2\sqrt{5}}$; $\cos 72^0 = \sin 18^0 = \dfrac{\sqrt{5}-1}{4}$

**14.** $\sin 3^0 = \frac{1}{16}\left[(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \sqrt{10+2\sqrt{5}}\,(\sqrt{6}-\sqrt{2})\right]$

$\cos 3^0 = \frac{1}{16}\left[\sqrt{10+2\sqrt{5}}\,(\sqrt{6}+\sqrt{2}) + (\sqrt{5}-1)(\sqrt{6}-\sqrt{2})\right]$

**15.** $\cos 15^0 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ $\sin 15^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

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**16.** $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$, $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $\tan^{-1}x = \cot^{-1}\frac{1}{x}$

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

**17.** (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ (iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

**18.** (i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

(iii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left\{xy + \sqrt{(1-x^2)(1-y^2)}\right\}$

(iv) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left\{xy - \sqrt{(1-x^2)(1-y^2)}\right\}$

(v) $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \cos^{-1}\left\{xyz - z\sqrt{(1-x^2)(1-y^2)} - y\sqrt{(1-z^2)(1-x^2)} - x\sqrt{(1-y^2)(1-z^2)}\right\}$

(vi) $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \sin^{-1}\left\{x\sqrt{(1-y^2)(1-z^2)} + y\sqrt{(1-z^2)(1-x^2)} + z\sqrt{(1-x^2)(1-y^2)} - xyz\right\}$

(vii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$ (viii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

(ix) $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\frac{xy-1}{y+x}$ (x) $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\frac{xy+1}{y-x}$

(xi) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(xii) $\cot^{-1}x + \cot^{-1}y + \cot^{-1}z = \cot^{-1}\left(\frac{xyz-x-y-z}{xy+yz+zx-1}\right)$

(xiii) $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \cos^{-1}\left(1-2x^2\right)$

(xiv) $2\cos^{-1}x = \cos^{-1}\left(2x^2-1\right)$

(xv) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (xvi) $2\cot^{-1}x = \cot^{-1}\left(\frac{x^2-1}{2x}\right)$

(xvii) $3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right)$ (xviii) $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right)$

(xix) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ (xx) $3\cot^{-1}x = \cot^{-1}\left(\frac{x^3-3x}{3x^2-1}\right)$

(xxi) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

**19.** (i) In any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

(ii) $a = b\cos C + c\cos B$ (iii) $b = c\cos A + a\cos C$

(iv) $c = a\cos B + b\cos A$

(v) $a^2 = b^2 + c^2 - 2bc\cos A$ (vi) $b^2 = c^2 + a^2 - 2ca\cos B$

(vii) $c^2 = a^2 + b^2 - 2ab\cos C$

(viii) $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

(ix) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(x) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

## DIFFERENTIAL CALCULUS

**1. Algebra of Limits :**

(i) $\underset{x\to a}{\text{Lt}}\{c\cdot f(x)\} = c\cdot\underset{x\to a}{\text{Lt}} f(x)$, c is a constant.

(ii) $\underset{x\to a}{\text{Lt}}\{f(x)\pm\phi(x)\} = \underset{x\to a}{\text{Lt}} f(x) \pm \underset{x\to a}{\text{Lt}}\phi(x)$

(iii) $\underset{x\to a}{\text{Lt}}\{f(x)\times\phi(x)\} = \underset{x\to a}{\text{Lt}} f(x) \times \underset{x\to a}{\text{Lt}}\phi(x)$

(iv) $\underset{x\to a}{\text{Lt}}\left[\frac{f(x)}{\phi(x)}\right] = \frac{\underset{x\to a}{\text{Lt}} f(x)}{\underset{x\to a}{\text{Lt}}\phi(x)}$ provided, $\underset{x\to a}{\text{Lt}}\phi(x)\neq 0$

**2.** (i) $\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x} = 1$ (ii) $\underset{x\to 0}{\text{Lt}}\frac{e^x-1}{x} = 1$

(iii) $\underset{x\to 0}{\text{Lt}}\frac{1}{x}\log_e(1+x) = 1$ (iv) $\underset{x\to 0}{\text{Lt}}(1+x)^{\frac{1}{x}} = e$

(v) $\underset{x\to 0}{\text{Lt}}\frac{a^x-1}{x} = \log_e a$, $(a > 0)$ (vi) $\underset{x\to\infty}{\text{Lt}}\left(1+\frac{1}{x}\right)^x = e$

**3. Continuity**

For a given function $y = f(x)$, if $\underset{x\to a}{\text{Lt}} f(x) = f(a)$

i.e., $\underset{x\to a-}{\text{Lt}} f(x) = \underset{x\to a+}{\text{Lt}} f(x) = f(a)$ then $y = f(x)$ will be continuous at $x = a$.

**4. Differentiation :**

$$\frac{dy}{dx} = f'(x) = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}, \text{ provided this limit exists.}$$

(i) The derivative of the function $y = f(x)$ at $x = a$ is denoted by $f'(a)$

or, $\left(\frac{dy}{dx}\right)_{x=a}$ and, $f'(a) = \left(\frac{dy}{dx}\right)_{x=a} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}$

**5. Rules of differentiation :**

Rule 1. $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)] = c \cdot f'(x)$, where c is a constant.

Rule 2. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)] = f'(x) \pm g'(x)$.

Rule 3. $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$; $\frac{d}{dx}(u \cdot v \cdot w) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

Rule 4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$, where $v \neq 0$.

6. (i) $\frac{d}{dx}(c) = 0$, c = constant

(ii) $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, n is a rational number.

(iii) $\frac{d}{dx}(x) = 1$, $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(iv) $\frac{d}{dx}\left(e^x\right) = e^x$

(v) $\frac{d}{dx}\left(a^x\right) = a^x \log_e a$, $(a > 0)$

(vi) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$, $(x > 0)$

(vii) $\frac{d}{dx}(\log_a x) = \frac{1}{x} \cdot \log_a e$, $(x > 0)$

(viii) $\frac{d}{dx}(\sin x) = \cos x$

(ix) $\frac{d}{dx}(\cos x) = -\sin x$

(x) $\frac{d}{dx}(\tan x) = \sec^2 x$

(xi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

(xiii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

(xiv) $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$, $(|x| < 1)$

(xv) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$, $(|x| < 1)$

(xvi) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$, $(-\infty < x < \infty)$

(xvii) $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$, $(-\infty < x < \infty)$

(xviii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$, $(|x| > 1)$

(xix) $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$, $(|x| > 1)$

N. Iute

# UNIT – 1

## ALGEBRA

PAGE NO



N. Tute

# LOGARITHM

## 1.1 Definition of Logarithm :

If x, a, m ($m > 0, a > 0, a \neq 1$) be three such numbers that $a^x = m$, then x is called the logarithm of the number m with respect to the base a and is written as $x = \log_a m$. For example, we know

$2^6 = 64$, therefore, from definition $6 = \log_2 64$ $\quad$ $4^3 = 64$, therefore, from definition $3 = \log_4 64$

$8^2 = 64$, therefore, from definition $2 = \log_8 64$ $\quad$ $3^{-5} = \frac{1}{243}$ therefore, from definition $-5 = \log_3\left(\frac{1}{243}\right)$

$10^{-3} = \frac{1}{1000} = 0.001$ therefore, from definition $-3 = \log_{10}(0.001)$ etc.

Conversely, if $x = \log_a m$ then $a^x = m$.

### Natural logarithm :

The natural logarithm is the logarithm to the base e, where e is an irrational and transcendental constant approximately equal to 2.718281828.

The natural logarithm is generally written as ln x, $\log_e x$ or sometimes, if the base of e is implicit, as simply log x.

The natural logarithm of a number x is the power to which e would have to be raised to equal x.

For example, ln(7.389...) is 2, because $e^2$=7.389....

The natural log of e itself ln(e) is 1 because $e^1 = e$, while the natural logarithm of 1 i. e., ln(1) is 0, since $e^0 = 1$.

### Common Logarithm:

The common logarithm is the logarithm with base 10. It is indicated by $\log_{10}(x)$.

**NOTE :**

(i) From above examples, it is clear that, logarithmic values of a given number (64) are different (6, 3, 2) for different bases (2, 4, 8) and hence it is clear that, without base logarithm of a number has no meaning.

(ii) For real a and m if $a^x = -m$ ($a > 0, m > 0$) then the value of x will be imaginary, i.e., logarithmic value of a negative number is imaginary.

(iii) $\because a^0 = 1, (a \neq 0)$ and if $a > 0, a \neq 1$ then from definition we get, $\log_a 1 = 0$. Therefore, logarithm of 1 to any finite non-zero base is zero.

(iv) $\because a^1 = a$, then from definition we get, $\log_a a = 1, a \neq 1, a > 0$. Therefore logarithm of a positive number to the same base is always 1.

## 1.2 Laws of logarithm :

(i) $\log_a(mn) = \log_a m + \log_a n$ (ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$ (iii) $\log_a m^n = n.\log_a m.$

(iv) $\log_a m = \log_b m \times \log_a b$ (v) $a^{\log_a m} = m$

(vi) $\log_a 1 = 0$, a be any non-zero finite real quantity.

(vii) $\log_a a = 1$, a is any positive number.

## PROBLEM SET

[ ..... Problems with '*' marks are solved at the end of the problem set ..... ]

**1.** Find the value of :

*(i) $2.\log_{\sqrt{2}} 2$ **[WBSC – 01]** *(ii) $\log_{3\sqrt{2}} 324$ (iii) $2.\log_{\sqrt{7}} 343$ **[WBSC – 00]**

*(iv) $\log_3 5 \times \log_{25} 27$ *(v) $\log_{\sqrt{7}} 343$ **[WBSC – 07]** (vi) $\log_{2\sqrt{3}} 1728$ **[WBSC – 03]**

*(vii) $\frac{1}{6}\left[\frac{3.\log 1728}{\frac{1}{2}.\log 36 + \frac{1}{3}.\log 8}\right]^{\frac{1}{2}}$ (viii) $\log_3 \log_2 \log_{\sqrt{3}} 81$ *(ix) $\log_2 \log_2 \log_3 \log_3 27^3$

*(x) $\frac{\log\sqrt{27} + \log 8 - \log\sqrt{1000}}{\log 1.2}$ (xi) $\log_5 5 \log_4 9 \log_3 2$

**2.** *(i) Prove that $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$

(ii) Prove that $\log_b a \times \log_c b \times \log_a c = 1$. Hence show that, $\log_{\sqrt{a}} b \times \log_{\sqrt{b}} c \times \log_{\sqrt{c}} a = 8$

**3.** Prove that :

(i) $7\log_{10}\left(\frac{10}{9}\right) + 3\log_{10}\left(\frac{81}{80}\right) = 2\log_{10}\left(\frac{25}{24}\right) + \log_{10} 2$

(ii) $7\log_{10}\left(\frac{15}{16}\right) + 6\log_{10}\left(\frac{8}{3}\right) + 5\log_{10}\left(\frac{2}{5}\right) + \log_{10}\left(\frac{32}{25}\right) = \log_{10} 3$

*(iii) $\log_{10} 2 + 16\log_{10}\left(\frac{16}{15}\right) + 12\log_{10}\left(\frac{25}{24}\right) + 7\log_{10}\left(\frac{81}{80}\right) = 1$

(iv) $\log_{10}\left(\frac{384}{5}\right) + \log_{10}\left(\frac{81}{32}\right) + 3\log_{10}\left(\frac{5}{3}\right) + \log_{10}\left(\frac{1}{9}\right) = 2$

**4.** Find the value of

(i) $\log_8 27$ if $\log_2 3 = a$ *(ii) $a^{\frac{2}{r}}$ if $\log_{10} a = r$

*(iii) $\log_a x = 0.3$, $\log_a 3 = 0.4$ find the value of $\log_3 x$ **[WBSC – 07]**

**5.** Find the value of

(i) $\log\left\{(16)^{\frac{1}{5}} \times 5^2 \div (108)^3\right\}$ if $\log 2 = 0.3010$ and $\log 3 = 0.4771$

*(ii) $\log\left\{(2.7)^2 \times (0.81)^{\frac{4}{5}} \times (90)^{\frac{5}{4}}\right\}$ given, $\log 3 = 0.4771213$

6. *(i) If $a^{2-x} b^{5x} = a^{x+3} b^{3x}$ show that, $x.\log\left(\frac{b}{a}\right) = \frac{1}{2}.\log a$

(ii) If $a^{3-x} b^{5x} = a^{5+x} b^{3x}$ show that, $x.\log\left(\frac{b}{a}\right) = \log a$

7. *(i) If $a^2 + b^2 = 7ab$, show that, $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}.(\log a + \log b)$

(ii) If $a^2 + b^2 = 14ab$, show that, $\log\left(\frac{a+b}{4}\right) = \frac{1}{2}.(\log a + \log b)$

8. (i) If $\log\left(\frac{x+y}{5}\right) = \frac{1}{2}.(\log x + \log y)$ show that, $\frac{x}{y} + \frac{y}{x} = 23$

*(ii) If $\log\left(\frac{x+y}{7}\right) = \frac{1}{2}.(\log x + \log y)$ show that, $\frac{x}{y} + \frac{y}{x} = 47$

*9. If $\log_a k$, $\log_b k$, $\log_c k$ be three consecutive terms of an A.P. then prove that, $c^2 = (ac)^{\log_a b}$. [WBSC – 92]

*10. If $\log_p x = a$ and $\log_q x = b$, prove that, $\log_{\frac{p}{q}} x = \frac{ab}{a-b}$

11. If $l = x y^{p-1}$, $m = x y^{q-1}$, $n = x y^{r-1}$, prove that $(q-r).\log l + (r-p).\log m + (p-q).\log n = 0$.

*12. If $x = \log_a (bc)$, $y = \log_b (ca)$, $z = \log_c (ab)$ then show that

(i) $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$ **[WBSC – 07, 09, 12, 14, 17]** (ii) $x + y + z + 2 = xyz$

13. If $\log_4 10 = x$, $\log_2 20 = y$ and $\log_5 8 = z$, show that, $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$

*14. If $a^x = bc$, $b^y = ca$ and $c^z = ab$, show that, $\frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} = 2$ **[WBSC – 10]**

*15. If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, show that, $xyz + 1 = 2yz$. **[WBSC – 96**

*16. If $x = \log_c b + \log_b c$, $y = \log_a c + \log_c a$ and $z = \log_a b + \log_b a$, show that $x^2 + y^2 + z^2 - xyz = 4$

17. If $x = \log_{15} 2$, $y = \log_{10} 3$, $z = \log_6 5$, prove that $xy + yz + zx + 2xyz = 1$

18. *(i) Show that, $\frac{1}{\log_a (bc) + 1} + \frac{1}{\log_b (ca) + 1} + \frac{1}{\log_c (ab) + 1} = 1$ **[WBSC – 04]**

**(ii)** Show that, $\frac{1}{\log_{xy} (xyz)} + \frac{1}{\log_{yz} (xyz)} + \frac{1}{\log_{zx} (xyz)} = 2$

19. *(i) If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, show that $x^x y^y z^z = 1$ **[WBSC – 08]**

*(ii) If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, show that $xyz = 1$. **[WBSC – 05, 07]**

*(iii) If $\frac{\log x}{ry-qz} = \frac{\log y}{pz-rx} = \frac{\log z}{qx-py}$, show that $x^p y^q z^r = 1$

(iv) If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$, show that

(a) $x^a y^b z^c = 1$ (b) $x^{b+c}.\ y^{c+a}.\ z^{a+b} = 1$

(c) $x^{b+c}.y^{c+a}.\ z^{a+b} = x^a y^b z^c$ (d) $x^{b^2+bc+c^2} \times y^{c^2+ca+a^2} \times z^{a^2+ab+b^2} = 1$

*(v) If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, show that $x^{y+z} y^{z+x} z^{x+y} = 1$. **[WBSC – 06]**

20. *(i) If $\frac{pq.\log(pq)}{p+q} = \frac{qr.\log(qr)}{q+r} = \frac{rp.\log(rp)}{r+p}$, show that, $p^p = q^q = r^r$

*(ii) If $\frac{a(b+c-a)}{\log a} = \frac{b(c+a-b)}{\log b} = \frac{c(a+b-c)}{\log c}$, prove that, $b^a a^b = c^a a^c = b^c c^b$ **[WBSC – 91]**

*21. If $\frac{x+y}{\log z} = \frac{y+z}{\log x} = \frac{z+x}{\log y}$ prove that, $\left(\frac{z}{y}\right)^x \times \left(\frac{x}{z}\right)^y \times \left(\frac{y}{x}\right)^z = 1$

22. Show that the value of

*(i) $\log_{10}3$ lies between $\frac{1}{2}$ and $\frac{2}{5}$ (ii) $\log_{10}2$ lies between $\frac{1}{4}$ and $\frac{1}{3}$

23. Find the number of digits in

*(i) $(875)^{15}$ **[WBSC – 84]** (ii) $3^{43}$ (iii) $(108)^{10}$

24. Solve :

*(i) $\log_x \log_2 \log_3 81 = 1$ **[WBSC – 84]** (ii) $\log_2 \log_2 \log_2 x = 1$

*(iii) $\frac{1}{\log_3 x} = \frac{1}{9}$ **[WBSC – 06]** (iv) $\log_x(8x-3) - \log_x 4 = 2$

(v) $x^{\log_{10} x} = 100x$ *(vi) $\frac{1}{\log_x 10} + 2 = \frac{2}{\log_5 10}$ **[WBSC – 94]**

*(vii) $\log_x 2 \times \log_{\frac{x}{16}} 2 = \log_{\frac{x}{64}} 2$ *(viii) $\log_5\left(5^{\frac{1}{x}} + 125\right) = \log_5 6 + 1 + \frac{1}{2x}$

(ix) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$ (x) $\log_x 3 \times \log_{\frac{x}{81}} 3 = \log_{\frac{x}{729}} 3$

25. *(i) Determine b satisfying, $\log_e 2 . \log_b 625 = \log_{10} 16 . \log_e 10$ **[WBSC – 89, 04]**

*(ii) Determine b in terms of a if, $\log_{10} a . \log_{10} b = \log_{10}(a+b)$ **[WBSC – 06]**

## ANSWERS

**1.** (i) 4 (ii) 4 (iii) 12 (iv) $\frac{3}{2}$ (v) 6 (vi) 6 (vii) $\frac{1}{2}$ (viii) 1 (ix) 0 (x) $\frac{3}{2}$ (xi) 1

**4.** (i) a (ii) 100 (iii) 0.75 **5.** (i) – 4.4611 (ii) 3.23232 **23.** (i) 45 (ii) 21 (iii) 21

**24.** (i) 2 (ii) 16 (iii) $3^9$ (iv) $\frac{3}{2}, \frac{1}{2}$ (v) 100, $\frac{1}{10}$ (vi) $\frac{1}{4}$ (vii) 4, 8 (viii) $\frac{1}{2}, \frac{1}{4}$ (ix) 8 (x) 9, 27 **25.** (i) 5 (ii) $\frac{a}{a-1}$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1.(i) Solution:** $2.\log_{\sqrt{2}} 2 = 2.\log_{\sqrt{2}}\left(\sqrt{2}\right)^2 = 4$ **(Ans)**

**1.(ii) Solution:** $\log_{3\sqrt{2}} 324 = \log_{3\sqrt{2}}\left(3\sqrt{2}\right)^4 = 4.\log_{3\sqrt{2}}\left(3\sqrt{2}\right) = 4$ **(Ans)**

**1.(iv) Solution:** $\log_3 5 \times \log_{25} 27 = \frac{\log 5}{\log 3} \times \frac{\log 27}{\log 25} = \frac{\log 5}{\log 3} \times \frac{\log 3^3}{\log 5^2} = \frac{\log 5}{\log 3} \times \frac{3.\log 3}{2.\log 5} = \frac{3}{2}$ **(Ans)**

**1.(v) Solution:** $\log_{\sqrt{7}} 343 = \log_{\sqrt{7}}\left(\sqrt{7}\right)^6 = 6\log_{\sqrt{7}}\left(\sqrt{7}\right) = 6 \times 1 = 6$ **(Ans)**

**1.(vii) Solution:** Given, $\frac{1}{6}\left[\frac{3.\log 1728}{\frac{1}{2}.\log 36 + \frac{1}{3}.\log 8}\right]^{\frac{1}{2}} = \frac{1}{6}\left[\frac{3.\log(12)^3}{\frac{1}{2}.\log 6^2 + \frac{1}{3}.\log 2^3}\right]^{\frac{1}{2}} = \frac{1}{6}\left[\frac{9.\log 12}{\log 6 + \log 2}\right]^{\frac{1}{2}} = \frac{1}{6}\left[\frac{9.\log 12}{\log 12}\right]^{\frac{1}{2}}$

$$= \frac{1}{6} \times 9^{\frac{1}{2}} = \frac{1}{6} \times 3 = \frac{1}{2} \text{ (Ans)}$$

**1.(ix) Solution:** Given, $\log_2\log_2\log_3\ \log_3 27^3 = \log_2\log_2\log_3\log_3 3^9$

$$= \log_2\log_2\ \log_3 3^2 = \log_2\ \log_2 2 = \log\ _2(1) = 0 \ \textbf{(Ans)}$$

**1.(x) Solution:** Given, $\dfrac{\log\sqrt{27}+\log 8-\log\sqrt{1000}}{\log 1.2} = \dfrac{\frac{1}{2}\log 3^3+\log 2^3-\frac{1}{2}\log 10^3}{\log 1.2}$

$$= \frac{\frac{3}{2}(\log 3+\log 4-\log 10)}{\log 1.2} = \frac{\frac{3}{2}(\log 12-\log 10)}{\log 1.2} = \frac{3}{2}\cdot\frac{\log\left(\frac{12}{10}\right)}{\log 1.2} = \frac{3}{2}\cdot\frac{\log 1.2}{\log 1.2} = \frac{3}{2} \ \textbf{(Ans)}$$

**2. (i) Solution:** L.H.S = log (1 + 2 + 3) = log6 = log (1 × 2 × 3) = log1 + log2 + log3 = R.H.S. **(Proved)**

**3.(iii) Solution:** Given, $\log_{10} 2+16\log_{10}\left(\frac{16}{15}\right)+12\log_{10}\left(\frac{25}{24}\right)+7\log_{10}\left(\frac{81}{80}\right)$

$$= \log_{10} 2+\log_{10}\left(\frac{16}{15}\right)^{16}+\log_{10}\left(\frac{25}{24}\right)^{12}+\log_{10}\left(\frac{81}{80}\right)^{7} \quad [\because\ a.\log x = \log x^a]$$

$$= \log_{10}\left[2\times\left(\frac{16}{15}\right)^{16}\times\left(\frac{25}{24}\right)^{12}\times\left(\frac{81}{80}\right)^{7}\right] = \log_{10}\left[\frac{2\times(2^4)^{16}\times(5^2)^{12}\times(3^4)^7}{(3\times5)^{16}\times(3\times2^3)^{12}\times(5\times2^4)^7}\right] = \log_{10}\left[\frac{2\times2^{64}\times5^{24}\times3^{28}}{3^{16}\times5^{16}\times3^{12}\times2^{36}\times5^7\times2^{28}}\right]$$

$$= \log_{10}\left[\frac{2^{1+64-36-28}\times5^{24-16-7}}{3^{16+12-28}}\right] = \log_{10}\left[\frac{2^1\times5^1}{3^0}\right] = \log_{10}10 = 1 \ \textbf{(Proved)}$$

**4.(ii) Solution:** Given, $\log_{10}a = r \quad \therefore\ a = 10^r$

$$\therefore\ a^{\frac{2}{r}} = \left(10^r\right)^{\frac{2}{r}} = 10^2 = 100 \ \textbf{(Ans)}$$

**4.(iii) Solution:** Given, $\log_a x = 0.3$ and $\log_a 3 = 0.4$

Therefore, $\log_3 x = \dfrac{\log_a x}{\log_a 3} = \dfrac{0.3}{0.4} = \dfrac{3}{4} = 0.75$ **(Ans)**

**5.(ii) Solution:** $\log\left\{(2.7)^2\times(0.81)^{\frac{4}{5}}\times(90)^{\frac{5}{4}}\right\} = \log\left\{\left(\frac{27}{10}\right)^2\times\left(\frac{81}{100}\right)^{\frac{4}{5}}\times9^{\frac{5}{4}}\times(10)^{\frac{5}{4}}\right\}$

$$= \log\left[\frac{3^6\times3^{\frac{16}{5}}\times3^{\frac{5}{2}}\times10^{\frac{5}{4}}}{10^2\times10^{\frac{8}{5}}}\right] = \log\left[\frac{3^{6+\frac{16}{5}+\frac{5}{2}}}{10^{2+\frac{8}{5}-\frac{5}{4}}}\right] = \log\left[\frac{3^{\frac{117}{10}}}{10^{\frac{47}{20}}}\right] = \frac{117}{10}\log 3-\frac{47}{20}\log 10$$

$= 11.7\times(0.4771213) - 2.35\times1 = 3.23232$ **(appx). (Ans)**

**6.(i) Solution:** Given, $a^{2-x}.b^{5x} = a^{x+3}\ b^{3x}$ or, $\dfrac{b^{5x}}{b^{3x}} = \dfrac{a^{x+3}}{a^{2-x}}$

or, $b^{5x-3x} = a^{x+3-2+x}$ or, $b^{2x} = a^{2x}.a$ or, $\dfrac{b^{2x}}{a^{2x}} = a$ or, $\left(\dfrac{b}{a}\right)^{2x} = a$

or, $\log\left(\frac{b}{a}\right)^{2x} = \log a$ [taking log of both sides] or, $2x.\log\left(\frac{b}{a}\right) = \log a$ or, $x.\log\left(\frac{b}{a}\right) = \frac{1}{2}\log a$ **(Proved)**

**7.(i) Solution :** Given, $a^2+b^2 = 7ab$, or, $(a+b)^2 - 2ab = 7ab$ or, $(a+b)^2 = 9ab$, or, $\left(\frac{a+b}{3}\right)^2 = ab$

or, $\log\left(\frac{a+b}{3}\right)^2 = \log(ab)$ [taking log of both sides]

or, $2.\log\left(\frac{a+b}{3}\right) = \log a+\log b$

or, $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a+\log b)$ **(Proved)**

**8.(ii) Solution:** Given, $\log\left(\frac{x+y}{7}\right)=\frac{1}{2}(\log x+\log y)$ or, $2.\log\left(\frac{x+y}{7}\right)=\log(xy)$ or, $\log\left(\frac{x+y}{7}\right)^2=\log(xy)$ or, $\left(\frac{x+y}{7}\right)^2=xy$

or, $(x+y)^2=49xy$ or, $x^2+y^2+2xy=49xy$ or, $x^2+y^2=47xy$

or, $\frac{x^2}{xy}+\frac{y^2}{xy}=47$ or, $\frac{x}{y}+\frac{y}{x}=47$ **(Proved)**

**9. Solution:** Given, $\log_a k$, $\log_b k$, $\log_c k$, are in A. P.

$\therefore$ $2\log_b k=\log_a k+\log_c k$ or, $\frac{2.\log k}{\log b}=\frac{\log k}{\log a}+\frac{\log k}{\log c}$ or, $\frac{2}{\log b}=\frac{\log a+\log c}{\log a.\log c}$ or, 2 loga.logc = logb.log(ac)

or, $2.\log c=\frac{\log b}{\log a}\times\log(ac)=\log_a b\times\log(ac)$

or, $\log c^2=\log(ac)^{\log_a b}$ or, $c^2=(ac)^{\log_a b}$ **(Proved)**

**10. Solution:** $\because$ $\log_p x=a$ or, $\log_x p=\frac{1}{a}$ and $\log_q x=b$ or, $\log_x q=\frac{1}{b}$

$\therefore$ $\log_{\frac{p}{q}} x=\frac{1}{\log_x\left(\frac{p}{q}\right)}=\frac{1}{\log_x p-\log_x q}=\frac{1}{\frac{1}{a}-\frac{1}{b}}=\frac{ab}{a-b}$ **(Proved)**

**12. Solution:** Given, $x=\log_a(bc)$, or, $x+1=\log_a(bc)+\log_a a=\log_a(abc)$

$\therefore$ $\frac{1}{x+1}=\log_{abc}a$. Similarly, $\frac{1}{y+1}=\log_{abc}b$, $\frac{1}{z+1}=\log_{abc}c$

$\therefore$ $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=\log_{abc}a+\log_{abc}b+\log_{abc}c=\log_{abc}abc=1$ $\therefore$ $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$ **[Proved (i)]**

or, $\frac{1}{x+1}+\frac{1}{y+1}=1-\frac{1}{z+1}$ or, $\frac{x+1+y+1}{(x+1)(y+1)}=\frac{z+1-1}{z+1}$ or, $\frac{x+y+2}{xy+x+y+1}=\frac{z}{z+1}$

or, $(x+y+2)(z+1)=z(xy+x+y+1)$ or, $xz+yz+2z+x+y+2=xyz+zx+zy+z$

or, $x+y+z+2=xyz$ **[Proved (ii)]**

**14. Hints :** $a^x=bc$ or, $x\log a=\log(bc)$ [taking log of both sides]

or, $x=\frac{\log(bc)}{\log a}=\log_a(bc)$ $\therefore$ $x+1=\log_a(bc)+\log_a a=\log_a(bac)$

$\therefore$ $\frac{x}{x+1}=\frac{\log_a(bc)}{\log_a(abc)}=\log_{abc}(bc)$

Similarly, $\frac{y}{y+1}=\log_{abc}(ca)$, $\frac{z}{z+1}=\log_{abc}(ab)$ etc.

**15. Solution:** Given, $x=\log_{2a}a$, $y=\log_{3a}2a$, $z=\log_{4a}3a$

$\therefore$ $xyz=\log_{2a}a\times\log_{3a}2a\times\log_{4a}3a=\frac{\log a}{\log 2a}\times\frac{\log 2a}{\log 3a}\times\frac{\log 3a}{\log 4a}=\frac{\log a}{\log 4a}=\log_{4a}a$

or, $xyz+1=\log_{4a}a+\log_{4a}4a=\log_{4a}(4a^2)=\log_{4a}(2a)^2=2\log_{4a}2a=2\log_{3a}2a\times\log_{4a}3a=2yz$

$\therefore$ $xyz+1=2yz$ **(Proved)**

**16. Solution:** Let $p=\log_c b$, $q=\log_a c$, $r=\log_b a$ $\therefore$ $pqr=1$

$\therefore$ $x=p+\frac{1}{p}$, $y=q+\frac{1}{q}$, $z=r+\frac{1}{r}$

$\therefore$ $xyz=\left(p+\frac{1}{p}\right)\left(q+\frac{1}{q}\right)\left(r+\frac{1}{r}\right)=\frac{(1+p^2)(1+q^2)(1+r^2)}{pqr}$

$$=1+(p^2+q^2+r^2)+p^2q^2+q^2r^2+r^2p^2+(pqr)^2=1+p^2+q^2+r^2+\frac{1}{r^2}+\frac{1}{p^2}+\frac{1}{q^2}+1\left[\begin{array}{l}\because pqr=1\\ \therefore pq=\frac{1}{r}\text{ etc}\end{array}\right]$$

or, $xyz = \left(p + \frac{1}{p}\right)^2 - 2 + \left(q + \frac{1}{q}\right)^2 - 2 + \left(r + \frac{1}{r}\right)^2 - 2 + 2$

or, $xyz = x^2 + y^2 + z^2 - 6 + 2$ or, $x^2 + y^2 + z^2 - xyz = 4$ **(Proved)**

**18.(i) Solution:** L. H. S $= \frac{1}{\log_a(bc)+1} + \frac{1}{\log_b(ca)+1} + \frac{1}{\log_c(ab)+1}$

$= \frac{1}{\log_a(bc)+\log_a a} + \frac{1}{\log_b(ca)+\log_b b} + \frac{1}{\log_c(ab)+\log_c c}$ $[\log_a a = 1$ etc.$]$

$= \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)} = \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc}(abc) = 1 =$ R. H. S. **(Proved)**

**19.(i) Solution:** Given, $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$ (say)

$\therefore \log x = k(y - z), \log y = k(z - x), \log z = k(x - y)$

$\therefore x \log x + y\log y + z\log z = xk(y - z) + yk(z - x) + zk(x - y)$

or, $\log x^x + \log y^y + \log z^z = k(xy - zx + yz - xy + zx - yz)$

or, $\log(x^x y^y z^z) = k \times 0 = 0$ or, $x^x y^y z^z = e^0 = 1$ **(Proved)**

**19.(ii) Solution:** Given, $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$ (say)

$\therefore \log x = k(y - z), \log y = k(z - x), \log z = k(x - y)$

$\therefore \log x + \log y + \log z = k(y - z) + k(z - x) + k(x - y)$

or, $\log(xyz) = k(y - z + z - x + x - y)$ or, $\log(x\,y\,z) = k \times 0 = 0$ or, $xyz = e^0 = 1$ **(Proved)**

**19.(iii) Solution:** Given, $\frac{\log x}{ry-qz} = \frac{\log y}{pz-rx} = \frac{\log z}{qx-py} = k$ (say)

$\therefore \log x = k(ry - qz), \log y = k(pz - rx), \log z = k(qx - py)$

$\therefore p \log x + q \log y + r \log z = k\{p(ry - qz) + q(pz - rx) + r(qx - py)\}$

or, $\log x^p + \log y^q + \log z^r = k\{pry - pqz + pqz - qrx + qrx - pry\}$

or, $\log(x^p.y^q.z^r) = k \times 0 = 0 \therefore x^p y^q z^r = e^0 = 1$ **(Proved)**

**19.(v) Solution:** Given, $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$ (say)

$\therefore \log x = k(y - z), \log y = k(z - x), \log z = k(x - y)$

$\therefore (y + z) \log x + (z + x) \log y + (x + y) \log z = k(y + z)(y - z) + k(z + x)(z - x) + k(x + y)(x - y)$

or, $\log x^{y+z} + \log y^{z+x} + \log z^{x+y} = k(y^2 - z^2 + z^2 - x^2 + x^2 - y^2)$

or, $\log(x^{y+z} z^{x+y} y^{z+x}) = k \times 0$ or, $x^{y+z} z^{x+y} y^{z+x} = e^0 = 1$ **(Proved)**

**20.(i) Solution:** Given, $\frac{pq.\log(pq)}{p+q} = \frac{qr.\log(qr)}{q+r} = \frac{rp.\log(rp)}{r+p} = k$ (say)

$\therefore \log(pq) = k.\frac{p+q}{pq}$ or, $\log p + \log q = k.\left(\frac{1}{p} + \frac{1}{q}\right)$ ----------------------------------(1)

Similarly, $\log q + \log r = k.\left(\frac{1}{q} + \frac{1}{r}\right)$ -----------(2) and $\log r + \log p = k.\left(\frac{1}{p} + \frac{1}{r}\right)$ -----------------------(3)

Adding (1), (2) and (3) we get,

$2(\log p+\log q+\log r)=2k.\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$ or, $\log p+\log q+\log r=k.\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$ ---------------------- (4)

$\therefore$ from (1) and (4) we get, $\log r=k.\frac{1}{r}$ [subtracting (1) from (4)]

or, $r\log r=k$ or, $\log r^r=k$

Similarly, from (2) and (4) we get, $\log p^p=k$

and from (3) and (4) we get, $\log q^q=k$

$\therefore \log p^p=\log q^q=\log r^r$

$\therefore p^p=q^q=r^r$ **(Proved)**

**20.(ii) Solution:** Given, $\frac{a(b+c-a)}{\log a}=\frac{b(c+a-b)}{\log b}=\frac{c(a+b-c)}{\log c}=\frac{1}{k}, k\neq 0$ (say)

$\therefore \log a=ka(b+c-a)$ ----------(1)

$\log b=kb(c+a-b)$ ----------(2)

$\log c=kc(a+b-c)$ ------------(3)

From (1) and (2) we get $[(1)\times b+(2)\times a]$

$b\log a+a\log b=kab(b+c-a+c+a-b)=2kabc$

or, $\log a^b+\log b^a=2kbac$ or, $\log(a^b.b^a)=2kabc$ ----(4)

Similarly from (2) and (3) we get $[(2)\times c+(3)\times b]$ $\log(b^c.c^b)=2kabc$ -----------(5)

and from (1) and (3) we get $[(1)\times c+(3)\times a]$ $\log(a^c.c^a)=2kabc$ -----------(6)

From (4), (5) and (6) we get,

$\log(a^b.b^a)=\log(b^c.c^b)=\log(c^a.a^c)$ or, $a^b.b^a=b^c.c^b=c^a.a^c$ **(Proved)**

**21. Solution:** Given, $\frac{x+y}{\log z}=\frac{y+z}{\log x}=\frac{z+x}{\log y}=\frac{1}{k}, k\neq 0$ (say)

$\therefore \log x=k(y+z)$ ------------(1) $\log y=k(z+x)$ ------------(2) $\log z=k(x+y)$ ------------(3)

$\therefore (y-z).\log x=k(y^2-z^2)$, $(z-x)\log y=k(z^2-x^2)$ and $(x-y)\log z=k(x^2-y^2)$

$\therefore (y-z)\log x+(z-x)\log y+(x-y)\log z=k(y^2-z^2+z^2-x^2+x^2-y^2)=0$

or, $\log x^{y-z}+\log y^{z-x}+\log z^{x-y}=0$ or, $\log(x^{y-z}.y^{z-x}.z^{x-y})=0$

or, $x^{y-z}.y^{z-x}.z^{x-y}=e^0=1$

or, $\frac{x^y}{x^z}.\frac{y^z}{y^x}.\frac{z^x}{z^y}=1$

or, $\left(\frac{z}{y}\right)^x\times\left(\frac{x}{z}\right)^y\times\left(\frac{y}{x}\right)^z=1$ **(Proved)**

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**22.**(i) **Solution:** We know, $3^5 > 10^2$ or, $\log_{10}3^5 > \log_{10}10^2$ or, $5\log_{10}3 > 2$ or, $\log_{10}3 > \frac{2}{5}$ ---------- (1)

Again $3^2 < 10$ or, $\log_{10}3^2 < \log_{10}10$ or, $2\log_{10}3 < 1$ or, $\log_{10}3 < \frac{1}{2}$ ---------- (2)

Combining (1) and (2) we get, $\frac{2}{5} < \log_{10}3 < \frac{1}{2}$

Hence, $\log_{10}3$ lies between $\frac{1}{2}$ and $\frac{2}{5}$ **(Proved)**

**23.**(i) **Solution:** Let $x = (875)^{15}$ or, $\log x = \log(875)^{15} = 15\log(875)$

$= 15\log(5^3 \times 7) = 15(3\log 5 + \log 7)$

$= 15\{3 \times 0.69897 + 0.8455098\} = 44.1301208$

Since the characteristic of logx is 44, the required number of digits in $(875)^{15}$ is $(44 + 1) = 45$. **(Ans)**

**24.**(i) **Solution:** Given, $\log_x \log_2 \log_3 81 = 1$ or, $\log_x \log_2 \log_3 3^4 = 1$ or, $\log_x \log_2 4 = 1$ [$\because \log_3 3 = 1$]

or, $\log_x \log_2 2^2 = 1$ or, $\log_x 2 = 1$ [$\because \log_2 2 = 1$] or, $2 = x^1$ or, $x = 2$ **(Ans)**

**24.**(iii) **Solution:** Given, $\frac{1}{\log_3 x} = \frac{1}{9}$ or, $\log_3 x = 9$ or, $x = 3^9$ **(Ans)**

**24.**(vi) **Solution:** Given, $\frac{1}{\log_x 10} + 2 = \frac{2}{\log_5 10}$ or, $\log_{10}x + 2 = 2\log_{10}5$

or, $\log_{10}x - \log_{10}5^2 = -2$ or, $\log_{10}\frac{x}{25} = -2$

or, $\frac{x}{25} = 10^{-2}$ or, $x = \frac{25}{100} = \frac{1}{4}$ **(Ans)**

**24.**(vii) **Solution:** Given, $\log_x 2 \times \log_{\frac{x}{16}} 2 = \log_{\frac{x}{64}} 2$

or, $\frac{1}{\log_2 x} \times \frac{1}{\log_2\left(\frac{x}{16}\right)} = \frac{1}{\log_2\left(\frac{x}{64}\right)}$

or, $\log_2 x \times \log_2\left(\frac{x}{16}\right) = \log_2\left(\frac{x}{64}\right)$

or, $\log_2 x.(\log_2 x - \log_2 16) = \log_2 x - \log_2 64$

or, $a(a - \log_2 2^4) = a - \log_2 2^6$ [let $a = \log_2 x$]

or, $a(a - 4) = a - 6$ [ $\log_2 2 = 1$]

or, $a^2 - 4a - a + 6 = 0$ or, $a^2 - 5a + 6 = 0$

or, $(a - 3)(a - 2) = 0$.

$\therefore$ either $a = 3$ or, $a = 2$ or, $\log_2 x = 3$ or, $x = 2^3 = 8$

or, $\log_2 x = 2$ or, $x = 2^2 = 4$

$\therefore x = 8, x = 4$ are the required solutions. **(Ans)**

**24.(viii) Solution:** Given, $\log_5\left(5^{\frac{1}{x}}+125\right)=\log_5 6+1+\frac{1}{2x}$

or, $\log_5\left(5^{\frac{1}{x}}+125\right)-\log_5 6=1+\frac{1}{2x}$ or, $\log_5\left(\frac{5^{\frac{1}{x}}+125}{6}\right)=1+\frac{1}{2x}$ or, $\frac{5^{\frac{1}{x}}+125}{6}=5^{1+\frac{1}{2x}}$ or, $5^{\frac{1}{x}}+125=6.5.5^{\frac{1}{2x}}$

or, $a^2+125=30a$ $\left[\text{let, } a=5^{\frac{1}{2x}}\right]$ or, $a^2-30a+125=0$ or, $(a-5)(a-25)=0$

$\therefore$ either, $a=5$ or, $5^{\frac{1}{2x}}=5$ or, $\frac{1}{2x}=1$ or, $x=\frac{1}{2}$

or, $a=25$ or, $5^{\frac{1}{2x}}=25=5^2$ or, $\frac{1}{2x}=2$ or, $x=\frac{1}{4}$

$\therefore$ $x=\frac{1}{2},\frac{1}{4}$ are the required solutions. **(Ans)**

**25. (i) Solution:** Given, $\log_e 2.\ \log_b 625=\log_{10}16.\log_e 10$

or, $\log_e 2\times\frac{\log_e 5^4}{\log_e b}=\frac{\log_e 2^4}{\log_e 10}\times\log_e 10$ or, $\frac{\log_e 2\times 4.\log_e 5}{\log_e b}=4.\log_e 2$ or, $\log_e b=\log_e 5$ $\therefore$ $b=5$ **(Ans)**

**25. (ii) Solution:** Given, $\log_{10}a+\log_{10}b=\log_{10}(a+b)$ or, $\log_{10}(ab)=\log_{10}(a+b)$

or, $ab=a+b$ or, $b(a-1)=a$ or, $b=\frac{a}{a-1}$ **(Ans)**

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# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. The logarithm of 1728 to the base $2\sqrt{3}$ is: (a) 3, (b) 6, (c) 9, (d) None of these. [WBSC – 03]
2. The value of log of 324 to the base $3\sqrt{2}$ is – (a) 8, (b) 4, (c) 1, (d) 2. [WBSC – 10]
3. The value of $\log_{\sqrt{7}} 343$ is – (a) 3, (b) 6, (c) 4, (d) none of these. [WBSC – 07, 12]
4. If $2\log_{\sqrt{2}} 2 = a$, then a is – (a) 2 (b) 3 (c) 4 (d) none of these. [WBSC – 11]
5. If $\log_e 2 \log_x 625 = \log_{10} 16 \log_2 10$, then the values of x is (a) 4, (b) 5, (c) $\frac{1}{5}$, (d) None. [WBSC – 04, 11]
6. $\log_5 5 \log_4 9 \log_3 2$ simplifies to – (a) 2, (b) 1, (c) 5, (d) none of these. [WBSC – 05]
7. The value of $\log_3 \log_2 \log_{\sqrt{3}} 81$ is – (a) 3, (b) 1, (c) $\frac{1}{3}$, (d) none of these. [WBSC – 08, 10]
8. If $\log_5 5 \log_2 x \log_{10} 2 = 2$ then x is – (a) 10, (b) 100, (c) 1000, (d) none of these. [WBSC – 09]
9. If $\log_2 \log_x 81 = 1$, find x. [WBSC – 12]
10. If $\log_{16} x + \log_4 x + \log_2 x = 14$, then x is – (a) 16 (b) 256 (c) 32 (d) 64 [WBSC – 09]
11. The value of x for $\frac{1}{\log_3 x} = \frac{1}{9}$ is – (a) 3, (b) $\frac{1}{3}$, (c) $\sqrt{3}$, (d) none of these. [WBSC – 06, 07]
12. If $\log_a x = 0.3$, $\log_a 3 = 0.4$ then $\log_3 x$ is – (a) 0.12 (b) 0.7 (c) 0.75 (d) $\frac{4}{3}$ [WBSC – 07, 08]
13. If $\log_{10} a + \log_{10} b = \log_{10}(a + b)$, then b can be expressed as (a) $\frac{a}{a-1}$, (b) $\frac{a}{a+1}$, (c) $\frac{a-1}{a}$, (d) none of these. [WBSC – 06]
14. If $\frac{1}{2}(\log a + \log b) = \log \frac{a+b}{2}$ then the relation is – (a) a = b (b) $a = \frac{b}{2}$ (c) a = 2b (d) $a = \frac{b}{3}$ [WBSC – 08]
15. The equation $\log_e x + \log_e(1 + x) = 0$ may be written as –

    (a) $x^2 + x - 1$, (b) $x^2 + x + 1$, (c) $x^2 + x - e = 0$, (d) none is true. [WBSC – 08]
16. The value of $5^{\log_5 3}$ is – (a) 5 (b) $5^3$ (c) 3 (d) $3^5$ [WBSC – 07]

## SUBJECTIVE TYPE

1. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$ show that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$ [WBSC – 07, 09, 12, 14, 17]
2. Show that, $\frac{1}{\log_a(bc)+1} + \frac{1}{\log_b(ca)+1} + \frac{1}{\log_c(ab)+1} = 1$ [WBSC – 04]
3. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$ prove that xyz = 1 [WBSC – 05, 07]
4. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, prove that abc = 1 [WBSC – 09]

5. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$ prove that $x^x y^y z^z = 1$ [WBSC – 08]

6. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$ show that $x^{y+z} z^{x+y} y^{z+x} = 1$ [WBSC – 06, 18, 19]

7. If $\frac{\log x}{b+c} = \frac{\log y}{c+a} = \frac{\log z}{a+b}$, prove that $x^{b-c} y^{c-a} z^{a-b} = 1$ [WBSC – 08, 11]

8. If $a^x = bc$, $b^y = ca$, $c^z = ab$ show that $\frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} = 2$ [WBSC – 10]

9. If $\frac{\log x}{y+z} = \frac{\log y}{z+x} = \frac{\log z}{x+y}$ prove that, $\left(\frac{x}{y}\right)^z \times \left(\frac{y}{z}\right)^x \times \left(\frac{z}{x}\right)^y = 1$ [WBSC – 16]

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Chapter 2

# COMPLEX NUMBERS

## 2.1 Introduction :

It follows from the properties of real numbers that the square of a real number is never negative.

For example: The squares of the real numbers $2, -\frac{3}{2}, -\frac{\sqrt{7}}{3}, \sqrt{3}$ are respectively $4, \frac{9}{4}, \frac{7}{9}$ and 3 which are not negative.

But the squares of the numbers $\sqrt{-1}, \sqrt{-7}$ etc. are respectively $-1, -7$ which are negative and hence, the numbers $\sqrt{-1}, \sqrt{-7}$ etc. are not real.

Consequently, there are no real numbers which satisfy the equations of the forms $x^2 + 1 = 0$, $x^2 + 6 = 0$ etc. i.e., $x^2 = -1$ or, $x^2 = -6$ etc. So the system of real numbers is not sufficient to solve all the algebraic equation.

In order to solve this type of equations, i.e., to find the square root of negative quantities we introduce a new class of numbers known as **imaginary** or **complex numbers**.

## 2.2 Definitions :

- Let us imagine that there exist a number i which is equal to $\sqrt{-1}$. Therefore, we have $i = \sqrt{-1}$ or, $i^2 = -1$. This i is not a real number and is called the **fundamental imaginary unit**.
- If an orderd pair (a, b) is represented by the symbol $a + i\,b$, where a and b are real numbers and $i = \sqrt{-1}$, then the orderd pair (a, b) is called a **complex number**.
- Let, $z = (a, b) = a + i\,b$. When $b = 0$, then $z = a$, which is purely real number. When $a = 0$, then $z = i\,b$, which is purely imaginary number.

  Again if $a = 0$ and $b = 1$ then $z = (0, 1) = 0 + i.1 = i$, i.e., i represents the unit of a complex quantity.
- If $z = (a, b) = a + i\,b$, then a is called the **real part** and b is called the **imaginary part** of the complex number $a + i\,b$ and are respectively denoted by $Re(z)$ and $Im(z)$ i. e., $Re(z) = a$ and $Im(z) = b$.

**Hence,** the real numbers are particular cases of complex numbers. In other words, complex number is the generalized form of number system.

### • Conjugate complex quantities :

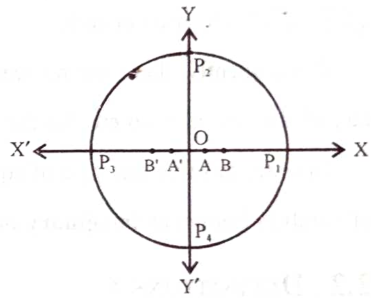
Two complex quantities, which differ only in the sign before the imaginary part, are called **conjugate complex** quantities. Thus, $a + ib$ and $a - ib$ are two conjugate complex quantities, each being conjugate to the other.

- **Negative of a complex number:**

If $z = a + ib$ where a and b are real, be a given complex number then the complex number $(-a) + i(-b)$ is called the **negative** of z and is denoted by $(-z)$.

## 2.3 Geometrical representation of Complex Numbers :

Let $\overleftrightarrow{XX'}$ be any straight line in the plane of the paper and O be any point on it. The point O represents the number 0 (zero). Now take a point A on $\overrightarrow{OX}$ such that OA = 1 unit. Then the point A represents the number 1, i.e., $A \equiv 1$. Take another point B on $\overrightarrow{OX}$ such that $OB = 2 \, . \, OA = 2.1 = 2$; then the point B represents the number 2 i.e., $B \equiv 2$. In general, take a point $P_1$ on $\overrightarrow{OX}$ such that $OP_1 = a.OA = a.1 = a$; then the point $P_1$ represents the real positive number a i.e., $P_1 \equiv a$. In this way each **positive** real number can be represented by a unique point on $\overrightarrow{OX}$, Conversely, a point on $\overrightarrow{OX}$ always represents a positive real number.




Again, take a point A′ on $\overrightarrow{OX'}$ such that OA = OA′. Then the point A′ represents the number −1 i.e., $A' \equiv -1$. Take another point B′ on $\overrightarrow{OX'}$ such that OB = OB′; then the point B′ represents the number −2 i.e., $B' \equiv -2$.

In general, take a point $P_3$ on $\overrightarrow{OX'}$ such that $OP_1 = OP_3$; then the point $P_3$ represents the negative real number $-a$ $(a > 0)$ i.e., $P_3 \equiv -a$. In this way each **negative** real number can be represented by a unique point on $\overrightarrow{OX'}$. Conversely, a point on $\overrightarrow{OX'}$ always represents a negative real number.

Therefore, a point on $\overleftrightarrow{XX'}$ always represents one and only one real number. Conversely, each real number can be represented by one and only one point on $\overleftrightarrow{XX'}$. For this reason $\overleftrightarrow{XX'}$ is called the **real axis.**

Now, the point $P_3 \equiv -a = (-1).a$ on $\overrightarrow{OX'}$ is obtained from the point $P_1 \equiv a$ on $\overrightarrow{OX}$ by rotating $OP_1$ through $180^{O}$ about O in the anticlock wise direction. So multiplication by (−1) may be looked upon as an operation which rotates a point $180^{O}$ in the anti-clock wise direction.

Again, $-1 = i^2 = i \cdot i \Rightarrow -a = (-1) \cdot a = i^2 \cdot a = i \cdot (i \cdot a)$

As multiplication by $-1 = i^2 = i \cdot i$ rotates the number through $180^{O}$, multiplication by i may be taken as the rotation of the point through $90^{O}$ in anticlock wise direction.

Let us consider now, $\overleftrightarrow{YY'}$ be perpendicular to $\overleftrightarrow{XX'}$ at O. If we rotate the point $P_1$ on $\overrightarrow{OX}$ through $90^{O}$ then it will reach to the point $P_2$ on $\overrightarrow{OY}$ such that $OP_1 = OP_2$. As $P_1$ represents the number a, $P_2$ represents the number i.a i.e., $P_2 \equiv$ i.a.

As $P_2$ represents the number i.a, $P_3$ represents the number i.(i.a) [since, $P_3$ is obtained by rotating $P_2$ through $90^{O}$ in anti-clock wise direction such that $OP_2 = OP_3$] i.e., $P_3 \equiv i^2 \cdot a \Rightarrow P_3 = -a$ [since, $i^2 = -1$]

Similarly, if the point $P_3$ on $\overrightarrow{OX'}$ rotates through $90^{\circ}$ in the anti-clock wise direction it will reach to the point $P_4$ such that $OP_3 = OP_4$ and then $P_4$ represents the number $i \cdot (-a) = -i \cdot a$ i.e. $P_4 \equiv -i \cdot a$ and so on.

**Geometrically, the symbol i is equivalent to the rotation of a point through $90^{\circ}$ in anticlock wise direction.**

So, for a real positive number a one and only one point on $\overrightarrow{OY}$ represents i.a and one and only one point on $\overrightarrow{OY'}$ represents – i.a. Therefore, a point on $\overleftrightarrow{YY'}$ will always represent one and only one purely imaginary number. For this reason $\overleftrightarrow{YY'}$ is called the **imaginary axis.**

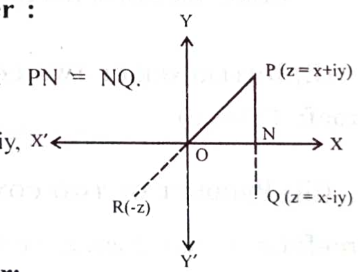
The plane of the two axes is called the **Argand plane.** The diagram showing points which represents complex numbers is called the **Argand diagram.**

Let $P(x, y)$ be any point in the complex plane with respect to the real axis $\overleftrightarrow{XOX'}$ and imaginary axis $\overleftrightarrow{YOY'}$ (as shown in the figure).

Then the point P geometrically represents the complex number $z = x + iy$. For a given complex number we shall get unique position of P representing the complex number and conversely, each position of P in the complex plane represents a definite complex number. If $y = 0$, then $z = x$, is real and the point lies on the real axis $\overleftrightarrow{XOX'}$. If $x = 0$, then $z = iy$, is purely imaginary and the point lies on the imaginary axis YOY′.

The point (0, 0) represents the number 0 (zero).

- **Geometrical representation of Conjugate Complex Number :**




Draw PN perpendicular to OX and produce it backwards to Q such that PN = NQ.

Then the point Q in the Argand plane represents the complex number $\bar{z} = x - iy$, the conjugate of z.

- **Geometrical Representation of negative of a Complex Number:**

Again, join OP and produce it to R so that OP = OR. Then the point R in the Argand plane represents the complex number (–z).

## 2.4 Properties of Complex Numbers :

**(i)** If a, b are real and $a + ib = 0$ then $a = 0$ and $b = 0$.

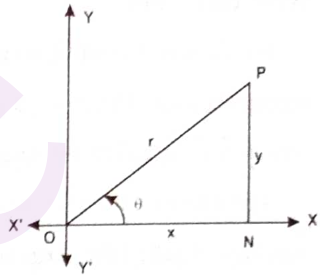
**(ii)** If a, b, c, d are real and $a + ib = c + id$, then $a = c$ and $b = d$.

**(iii)** The algebric sum, difference, product and ratio of two complex numbers is a complex number.

**(iv)** The sum and product of two conjugate complex numbers are both real.

## 2.5 Polar and Exponential form of complex number :

Taking the origin as the pole and the positive direction of real axis $\left(\overrightarrow{OX}\right)$ as the initial line, let $(r, \theta)$ be the polar coordinates of P whose cartesian co-ordinates are $(x, y)$ referred to two perpendicular lines as axes, the first co-ordinate axis $\left(\overleftrightarrow{XOX'}\right)$ being called the real axis and the second $\left(\overleftrightarrow{YOY'}\right)$ is the imaginary axis.




Then $x = r\cos\theta,\ y = r\sin\theta$ ....... (1)

Therefore, $z = x + iy = r(\cos\theta + i\sin\theta)$.

This is called the **polar form** or **modulus-amplitude** form of the complex number.

Again we know $e^{i\theta} = \cos\theta + i\sin\theta$ [known as Euler's formula]

Therefore, $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is called the **Exponential form** of complex number.

## 2.6 Algebra of Complex numbers:

### (i) Sum of two complex numbers is a complex number.

**Proof:** Let, $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers where a, b, c and d are real.

Then their sum $= z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$, which is of the form $A + iB$ where $A = a + c$ and $B = b + d$ are both real.

Hence, the sum of two complex numbers is a complex number.

### (ii) Difference of two complex numbers is a complex number.

**Proof:** Follow (i).

### (iii) Product of two complex numbers is a complex number.

**Proof:** Let, $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers where a, b, c and d are real.

Then their product $= z_1.z_2 = (a + ib).(c + id) = (ac - bd) + i.(ad + bc)$, [since $i^2 = -1$] $= A + iB$, where A and B are both real.

Hence, the product of two complex numbers is a complex number.

### (iv) Quotient of two complex numbers is a complex number.

**Proof:** Let, $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers where a, b, c and d are real.

Now, $\dfrac{z_1}{z_2} = \dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{(c+id)(c-id)} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$= \dfrac{ac+bd}{c^2+d^2} + i\dfrac{bc-ad}{c^2+d^2} = A + iB$ where $A = \dfrac{ac+bd}{c^2+d^2}$, $B = \dfrac{bc-ad}{c^2+d^2}$ are both real.

Hence, the quotient of two complex numbers is a complex number.

**(v) Any integral power of a complex number is a complex number.**

**Proof :** Let $z = a + ib$ be a complex number where a and b are real.

**Case I: n is a positive integer.**

Since n is a positive integer, therefore

$z^n = (a + ib)^n = (a + ib).(a + ib).(a + ib)$............................ to n factors $= A + iB$, where A and B are real.

(since the product of two or more complex numbers can be expressed in the form of $A + iB$; A and B are real.)

**Case II: n is a negative integer.**

Since n is a negative integer, let $n = -m$, m is a positive integer.

Then, $z^n = (a + ib)^n = (a + ib)^{-m} = \frac{1}{(a+ib)^m} = \frac{1}{C+iD}$ [following **Case I**]

$$= \frac{C-iD}{(C+iD)(C-iD)} = \frac{C-iD}{C^2+D^2} = \frac{C}{C^2+D^2} + i\frac{-D}{C^2+D^2} = A + iB;\ A = \frac{C}{C^2+D^2} \text{ and } B = \frac{-D}{C^2+D^2} \text{ are both real.}$$

Hence, any integral power of a complex number is a complex number.

**(vii) Any root of a complex number is a complex number.**

**Proof :** Let $z = a + ib$ be a complex number where a and b are real.

Again let the nth root of $a + ib$ be x. $\therefore\ (a+ib)^{\frac{1}{n}} = x \Rightarrow a + ib = x^n$ ............ (1)

If x is real then, $x^n$ also real. Equating real and imaginary parts of both sides we get, $a = x^n$ and $b = 0$. For $b = 0$, $a + ib$ would become a real quantity, which is not the case here. Hence x is a complex quantity.

Hence, any root of a complex number is a complex number.

## 2.7 Modulus and Argument of Complex Numbers :

Let $z = a + ib$. Then the positive square root of $(a^2 + b^2)$ is called the modulus of z and is denoted by $|z|$ or, mod(z). Thus if $z = a + ib$, then $|z| = \sqrt{a^2+b^2}$.

* Let $z = a + ib$. Then the unique value of $\theta$ satisfying $x = |z| \cos\theta$, $y = |z|. \sin\theta$ and $-\pi \le \theta \le \pi$ is called the **principal value** of argument (or, amplitude) of z is denoted by **arg(z)** or, **amp(z).**

## 2.8 Square root of a complex quantity :

Let $a + ib$ be a complex quantity.

Suppose $\sqrt{a+ib} = x + iy$, x, y are real. or, $a + ib = x^2 - y^2 + 2ixy$ (squaring).

Equating real and imaginary parts of the two sides we have, $x^2 - y^2 = a$ -------------(i) and $2xy = b$ --------(ii)

Solving (i) and (ii) we get, $x = \pm\frac{1}{\sqrt{2}}\left[\sqrt{a^2+b^2}+a\right]^{\frac{1}{2}}$, $y = \pm\frac{1}{\sqrt{2}}\left[\sqrt{a^2+b^2}-a\right]^{\frac{1}{2}}$

From (ii) we see that xy have the same sign as b.

So, (a) if b is positive, both x and y will have the same sign; either both positive or both negative.

And (b) if b is negative, x and y will have opposite signs; one positive and other negative.

# PROBLEM SET – I

[Problems with '*' marks are solved at the end of the problem set]

Simplify or Express in the form A + *i*B (A, B are real).

**1.** (a) (i) $i^{457}$ *(ii) $(-i)^{4n+3}$ (n, a positive integer)

*(iii) If $\left(\frac{1+i}{1-i}\right)^n = 1$ find the smallest integral value of n.

(b) (i) $\frac{1}{4-\sqrt{-3}}$ *(ii) $\frac{2\sqrt{-5}+3\sqrt{-3}}{2\sqrt{-5}-3\sqrt{-3}}$ (iii) $\frac{2\sqrt{2}-\sqrt{-3}}{\sqrt{2}-i\sqrt{3}}$

(c) (i) $(1-i)^3$ (ii) $(1+i)^{-1}$ *(iii) $\left(\frac{1+i}{1-i}\right)^3$

*(iv) $(1-i)\left(1-\frac{1}{i}\right)$ (v) $\frac{1+i}{1-i}$ (vi) $\frac{1-i}{1+i}$

(vii) $\left(\frac{1+i}{1-i}\right)^2+\left(\frac{1-i}{1+i}\right)^2$

(d) (i) $\frac{i}{1+i}+\frac{1+i}{i}$ *(ii) $\frac{i}{2+i}+\frac{3}{1+4i}$ (iii) $\frac{3}{1+i}-\frac{2}{2-i}+\frac{2}{1-i}$

(e) (i) $\frac{1+i}{1-i}-\frac{1-i}{1+i}$ (ii) $\frac{3-i}{2+i}+\frac{3+i}{2-i}$

(iii) $\frac{a+ib}{a-ib}-\frac{a-ib}{a+ib}$ *(iv) $\frac{2a+ib}{a+2ib}-\frac{2a-ib}{a-2ib}$

**2.** *(i) If $\frac{\sqrt{3}+i\sqrt{2}}{2\sqrt{3}+i\sqrt{2}} = x+iy$ where x, y are real and $i=\sqrt{-1}$, find x and y.

(ii) If $\frac{2+i}{2-3i} = A+iB$ (A, B are real, $i=\sqrt{-1}$, find the value of $A^2+B^2$.

(iii) x, y are real and $x+iy = \frac{5}{-3+4i}$ find x and y.

*(iv) If x, y are real and x + 3i and –2 + iy are conjugate to each other then find x and y. **[WBSC – 03]**

**3.** If x = 2 + 3i and y = 2 – 3i find the value of :

(i) $\frac{x^3-y^3}{x^3+y^3}$ (ii) $\frac{x^2+xy+y^2}{x^2-xy+y^2}$

**4.** (i) If $x = 2+i\sqrt{5}$ and xy = 9, find the value of $\frac{x^2-xy+y^2}{x^2+xy+y^2}$

(ii) If $x = -1+i\sqrt{2}$, find the value of $x^4+4x^3+6x^2+4x+9$.

*(iii) If $x = -2+\sqrt{-3}$, find the value of $x^4+8x^3+24x^2+32x+16$.

*(iv) If $x = 2-i\sqrt{3}$, find the value of k if $2x^4-5x^3-3x^2+41x+k = 0$.

(v) If $a = \frac{1+i}{\sqrt{2}}$ show that, $a^6+a^4+a^2+1 = 0$

*(vi) $x\sqrt{2} = 1+\sqrt{-1}$ find the value of $x^6+x^4+x^2+2$.

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5. Find modulus of the following :

*(i) $\frac{3+4i}{4-3i}$ **[WBSC – 07]** (ii) $\frac{1+2i}{2-i}$ **[WBSC – 97]** *(iii) $\frac{8+i}{1+8i}$ **[WBSC – 00]**

*(iv) $\frac{1}{1+i}$ **[WBSC – 95]** *(v) $\sqrt{12}+6\left(\frac{1-i}{1+i}\right)$ (vi) $\frac{1}{1-i}$

*(vii) $(a-ib)^2$ (viii) $\frac{20+15i}{12+5i}$ (ix) $\frac{x-iy}{-a+ib}$

(x) $\frac{2}{4+3i}+\frac{1}{3-4i}$

6. Find Amplitude of the following :

*(i) $\sqrt{3}-i$ *(ii) $\sqrt{12}+6\left(\frac{1-i}{1+i}\right)$ **[WBSC – 04]** *(iii) $\frac{i}{1-i}$

*(iv) $\frac{1}{1+i}$ **[WBSC – 95]** (v) $(a-ib)^2$ (vi) $\frac{20+15i}{12+5i}$

7. Find the square root of the following :

*(i) $\frac{7-24i}{3+4i}$ **[WBSC – 04]** (ii) $16-30i$ (iii) $-8-6i$

*(iv) $i$ **[WBSC – 99]** (v) $\frac{1+i}{1-i}$ (vi) $-i$

(vii) $24-10i$ (viii) $-1+\sqrt{-3}$ (ix) $1+2\sqrt{-6}$

(x) $4+6\sqrt{-5}$ (xi) $-11-60\sqrt{-1}$ *(xii) $-7-24i$

*(xiii) $3+4i$ **[WBSC – 05]**

8. Show that

*(i) $\sqrt{i}+\sqrt{-i}$ is a real number. **[WBSC – 06]**

(ii) $\sqrt{i}-\sqrt{-i}=\pm\sqrt{-2}$ (iii) $(1+i)^{\frac{1}{2}}-(1-i)^{\frac{1}{2}}=i\sqrt{2\sqrt{2}-2}$

9. *(i) If $|z_1|=|z_2|$ and $\operatorname{amp}z_1-\operatorname{amp}z_2=\pi$, then show that, $z_1+z_2=0$ **[WBSC – 94]**

**(ii)** If $z_1=2+4i$ and $z_2=3-i$, prove that $|2z_1-z_2|^2=2|z_1|^2+|z_2|^2+32$

10. (i) Express the following as the sum of two squares:

(a) $(a^2+b^2)(c^2+d^2)$ *(b) $(x^2+a^2)(y^2+b^2)(z^2+c^2)$

**(ii)** Resolve into factors :

*(a) $x^2+xy+y^2$ (b) $x^3+y^3$ *(c) $a^3+b^3+c^3-3abc$

## 6.7 THE CUBE ROOTS OF UNITY:

Let us assume that the cube root of 1 is x i.e., $\sqrt[3]{1} = x$. Then $x^3 = 1$ or, $x^3 - 1 = 0$ or, $(x-1)(x^2+x+1) = 0$

$\therefore$ either $x - 1 = 0 \Rightarrow x = 1$. or, $x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

Therefore, there are three cube roots of unity viz. 1, $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$, $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Of the three cube roots of unity, one is real and other two are conjugate complex numbers.

Here, $-\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ are called imaginary cube roots of unity.

### PROPERTIES OF CUBE ROOTS OF UNITY:

**(i)** One imaginary cube root of unity is the square of the other.

Two complex cube roots of unity are $-\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. Let $\alpha = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $\beta = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ .................... (1)

Now $\alpha^2 = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2 = \frac{1}{4}\left[1 - 2.i\sqrt{3} - 3\right] = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \beta$ [from (1)]

and $\beta^2 = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2 = \frac{1}{4}\left[1 + 2.i\sqrt{3} - 3\right] = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = \alpha$ [from (1)]

$\therefore \alpha^2 = \beta, \beta^2 = \alpha$, which shows that, one imaginary cube root of unity is the square of the other.

**NOTE :**

**(i) (a)** If one imaginary cube root of unity be $\omega$ then the other would be $\omega^2$.

**(b)** The three cube roots of unity are generally represented by 1, $\omega$, $\omega^2$, where $\omega$ is an imaginary cube root of unity.

**(ii)** The sum of the three cube roots of unity is zero.

Three cube roots of unity are 1, $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Let us take, $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ then $\omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ [since one imaginary cube root of unity is the square of the other]

Now, $1 + \omega + \omega^2 = 1 - \frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{1}{2} - \frac{i\sqrt{3}}{2} = 0$

$\therefore 1 + \omega + \omega^2 = 0 \Rightarrow$ the sum of the three cube roots of unity is zero.

**NOTE :** Since $1 + \omega + \omega^2 = 0$, $\therefore \omega + \omega^2 = -1$, $1 + \omega = -\omega^2$, $1 + \omega^2 = -\omega$

**(iii)** The product of two imaginary cube roots of unity is 1.

Two complex cube roots of unity are $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$. Let $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ then $\omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ ...... (1)

[since one imaginary cube root of unity is the square of the other]

$\therefore \omega.\omega^2 = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \left(-\frac{1}{2}\right)^2 - \left(\frac{i\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

Hence, the product of two imaginary cube roots is 1.

**NOTE :** Since $\omega.\omega^2 = 1 \Rightarrow \omega = \frac{1}{\omega^2} \Rightarrow \omega^2 = \frac{1}{\omega}$, shows that, two imaginary cube roots are reciprocal to each other and $\omega^3 = 1$. Hence, any positive or negative integral power of $\omega$ is equal to either 1 or $\omega$ or $\omega^2$.

## PROBLEM SET – II

[Problems with '*' marks are solved at the end of the problem set]

**11.** Show that :

*(i) $\frac{1}{1+2\omega}-\frac{1}{1+\omega}+\frac{1}{2+\omega}=0$

(ii) $(1-\omega)(1-\omega^2)(1-\omega^5)(1-\omega^{10})=9$

*(iii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)=9$

(iv) $(1+\omega-\omega^2)(1-\omega+\omega^2)=4$ (v) $(1+\omega-\omega^2)^3+(1-\omega+\omega^2)^3=-16$

*(vi) $(1+\omega-\omega^2)^5+(1-\omega+\omega^2)^5=32$

**12.** Prove that :

(i) $(3+3\omega+5\omega^2)^6=(3+5\omega+3\omega^2)^6=64$ *(ii) $\frac{x\omega^2+y\omega+z}{x\omega+y+z\omega^2}=\omega$

(iii) $(x+y)^2+(x\omega+y\omega^2)^2+(x\omega^2+y\omega)^2=6xy$

(iv) $(x+y\omega+z\omega^2)^2+(x\omega+y\omega^2+z)^2+(x\omega^2+y+z\omega)^2=0$

*(v) $(x+y\omega+z\omega^2)^4+(x\omega+y\omega^2+z)^4+(x\omega^2+y+z\omega)^4=0$

*(vi) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=(2a-b-c)(2b-c-a)(2c-a-b)$

*(vii) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=27abc$, if $a+b+c=0$ **[WBSC – 99, 04]**

*(viii) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)=a^2+b^2+c^2-bc-ca-ab$ **[WBSC – 99, 02]**

**13.** *(i) If $\omega$ be a cube root of unity, prove that, $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})=16$ **[WBSC – 91]**

(ii) If $x=a+b$, $y=a\omega+b\omega^2$, $z=a\omega^2+b\omega$, then show that $xyz=a^3+b^3$, where $\omega$ is a complex cube root of 1.

(iii) If $x=a+b$, $y=a\alpha+b\beta$, $z=a\beta+b\alpha$; $\alpha$ and $\beta$ are complex cube roots of unity. show that $xyz=a^3+b^3$.

*(iv) If $x=a+b\omega+c\omega^2$, $y=a\omega+b\omega^2+c$, $z=a\omega^2+b+c\omega$, prove that, $\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}=3$ **[WBSC – 96, 98]**

*(v) If $x=a+b$, $y=a+\omega b$, $z=a+\omega^2 b$, where $\omega$ is a complex cube root of 1 then prove that $x^3+y^3+z^3=3(a^3+b^3)$ **[WBSC – 09]**

*(vi) If $\omega$ be an imaginary cube root of unity and if

$(a+b\omega+c\omega^2)^2+(a\omega+b+c\omega^2)^2+(a\omega+b\omega^2+c)^2=0$ prove that, $a=c$ or $b=\frac{a+c}{2}$ **[WBSC – 87]**

**14.** *(i) If $\alpha=\frac{1}{2}(-1+\sqrt{-3})$, $\beta=\frac{1}{2}(-1-\sqrt{-3})$ prove that $\alpha^2+\alpha\beta+\beta^2=0$

(ii) If $\alpha$, $\beta$ are complex cube roots of 1 then prove that $\alpha^4.\beta^4-\alpha^{-1}.\beta^{-1}=0$

*(iii) If $\alpha$, $\beta$ are complex cube roots of 1 then prove that $\alpha^4+\beta^4+\alpha^{-1}.\beta^{-1}=0$

**15.** (i) For what real value of x and y, $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

*(ii) If $\sqrt[3]{x+iy}=a+ib$ then show that, (a) $\frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$ **[HS – 97, 00]** (b) $\sqrt[3]{x-iy}=a-ib$

*(iii) If $x+iy=\sqrt{\frac{a+ib}{c+id}}$, then show that $(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$ **[WBSC – 93]**

(iv) If $\frac{a+ib}{c+id}=x+iy$ prove that, (a) $\frac{a^2+b^2}{c^2+d^2}=x^2+y^2$ (b) $\frac{a-ib}{c-id}=x-iy$

**16.** *(i) Show that a real value of x will satisfy the equation $\frac{1-ix}{1+ix}=a-ib$ if $a^2+b^2=1$ **[WBSC – 05]**

*(ii) If a, b are real and $a^2+b^2=1$, show that the equation $\frac{\sqrt{1+x}-i\sqrt{1-x}}{\sqrt{1+x}+i\sqrt{1-x}}=a-ib$ is satisfied by a real value of x.

**17.** (i) If $x+iy=\frac{3}{2+\cos\theta+i\sin\theta}$, prove that, $x^2+y^2=4x-3$.

*(ii) If $x+iy=\frac{2}{3+\cos\theta+i\sin\theta}$ prove that, $2x^2+2y^2=3x-1$.

**18.** *(i) If $\omega=\frac{-1+\sqrt{-3}}{2}$, find the value of $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})$

*(ii) Evaluate $\left(\frac{1+\sqrt{-3}}{2}\right)^6+\left(\frac{1-\sqrt{-3}}{2}\right)^9$

(iii) Show that $\left(\frac{-1+\sqrt{-3}}{2}\right)^{19}+\left(\frac{-1-\sqrt{-3}}{2}\right)^{19}=-1$

**19.** (i) If $z=x+iy$ and $|2z+1|=|z-2i|$, then prove that, $3(x^2+y^2)+4(x+y)=3$.

*(ii) If $z=x+iy$ and $|z+6|=|2z+3|$, then prove that, $x^2+y^2=9$.

*(iii) If x, y, b are real, $z=x+iy$ and $\frac{z-i}{z-1}=ib$ show that, $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$

## ANSWERS

**1.** (a) (i) i (ii) $i$ (iii) 4 (b) (i) $\frac{4+\sqrt{3}i}{19}$ (ii) $-\frac{47+12\sqrt{15}}{7}$ (iii) $\frac{7}{5}+i\frac{\sqrt{6}}{5}$ (c) (i) $-2+i(-2)$ (ii) $\frac{1-i}{2}$ (iii) $-i$ or $0+i(-1)$ (iv) 2 (v) $i$ (vi) – i (vii) – 2 (d) (i) $\frac{3}{2}+i\left(-\frac{1}{2}\right)$ (ii) $\frac{32}{85}+i\left(-\frac{26}{85}\right)$ (iii) $\frac{17-9i}{10}$ (e) (i) $2i$ (ii) 2 (iii) $0+i\left(\frac{4ab}{a^2+b^2}\right)$ (iv) $-i\left(\frac{6ab}{a^2+4b^2}\right)$ **2.** (i) $\frac{4}{7},\frac{\sqrt{6}}{14}$ (ii) $\frac{5}{13}$ (iii) $-\frac{3}{5},-\frac{4}{5}$ (iv) $x=-2, y=-3$ **3.** (i) $-\frac{9i}{46}$ (ii) $-\frac{3}{23}$

**4.** (i) $-\frac{11}{7}$ (ii) 12 (iii) 9 (iv) – 35 (vi) 1 **5.** (i) 1 (ii) 1 (iii) 1 (iv) $\frac{\sqrt{2}}{2}$ (v) $4\sqrt{3}$ (vi) $\frac{\sqrt{2}}{2}$ (vii) $a^2+b^2$ (viii) $\frac{25}{13}$ (ix) $\sqrt{\frac{x^2+y^2}{a^2+b^2}}$ (x) $\frac{1}{\sqrt{5}}$ **6.** (i) $-\frac{\pi}{6}$ (ii) $-\frac{\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) $-\frac{\pi}{4}$ (v) $\tan^{-1}\left(\frac{2ab}{b^2-a^2}\right)$ (vi) $\tan^{-1}\left(\frac{16}{63}\right)$

**7.** (i) $\pm\{1-2i\}$ (ii) $\pm(5-3i)$ (iii) $\pm(1-3i)$ (iv) $\pm\frac{1+i}{\sqrt{2}}$ (v) $\pm\frac{1+i}{\sqrt{2}}$ (vi) $\pm\frac{1-i}{\sqrt{2}}$ (vii) $\pm(5-i)$ (viii) $\pm\frac{1+i\sqrt{3}}{\sqrt{2}}$ (ix) $\pm(\sqrt{3}+i\sqrt{2})$ (x) $\pm(3+\sqrt{-5})$ (xi) $\pm(5-6\sqrt{-1})$ (xii) $\pm(3-4i)$ (xiii) $\pm(2+i)$ **10.(i)** (a) $(ac-bd)^2+(ad+bc)^2$ (b) $(xyz-xbc-yca-zab)^2+(ayz+bzx+cxy-abc)^2$ **10.(ii)** (a) $(x-y\omega)(x-y\omega^2)$ (b) $(x+y)(x+y\omega)(x+y\omega^2)$ (c) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ **15.(i)** 3, –1 **18.(i)** 16 **(ii)** 0.

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1.** (a) (ii) $(-i)^{4n+3} = (-i)^{4n}.(-i)^3 = \{(-i)^2\}^{2n}.(-i^3) = (-1)^{2n}.i = i$ **(Ans)**

**1.** (a) (iii) $\left(\frac{1+i}{1-i}\right)^n = \left\{\frac{(1+i)^2}{(1+i)(1-i)}\right\}^n = \left\{\frac{1+2i+i^2}{1-i^2}\right\}^n = \left\{\frac{1+2i-1}{1+1}\right\}^n = \left(\frac{2i}{2}\right)^n = i^n$

Now, for $n = 1,\ i^n = i$ | $n = 2,\ i^n = i^2 = -1$

$n = 3,\ i^n = i^3 = i^2.i = -1.i = -i$ | $n = 4,\ i^n = i^4 = i^2.i^2 = (-1).(-1) = 1$

Therefore, $n = 4$ **(Ans)**

(b) (ii) $\frac{2\sqrt{-5}+3\sqrt{-3}}{2\sqrt{-5}-3\sqrt{-3}} = \frac{2i\sqrt5+3i\sqrt3}{2i\sqrt5-3i\sqrt3} = \frac{i(2\sqrt5+3\sqrt3)}{i(2\sqrt5-3\sqrt3)} = \frac{(2\sqrt5+3\sqrt3)(2\sqrt5+3\sqrt3)}{(2\sqrt5-3\sqrt3)(2\sqrt5+3\sqrt3)} = \frac{20+27+12\sqrt{15}}{20-27} = -\frac{1}{7}\left(47+12\sqrt{15}\right)$ **(Ans)**

(c) (iii) $\left(\frac{1+i}{1-i}\right)^3 = \left\{\frac{(1+i)^2}{(1+i)(1-i)}\right\}^3 = \left(\frac{1+2i+i^2}{1-i^2}\right)^3 = \left(\frac{1+2i-1}{1+1}\right)^3 = \left(\frac{2i}{2}\right)^3 = -i$ **(Ans)**

(c) (iv) $(1-i)\left(1-\frac{1}{i}\right) = (1-i)\left(1+\frac{i^2}{i}\right),\ [\because i^2=-1] = (1-i)(1+i) = 1-i^2 = 1+1 = 2$ **(Ans)**

(d) (ii) $\frac{i}{2+i}+\frac{3}{1+4i} = \frac{i(2-i)}{(2+i)(2-i)}+\frac{3(1-4i)}{(1+4i)(1-4i)} = \frac{2i-i^2}{4-i^2}+\frac{3-12i}{1-16i^2}$

$= \frac{2i+1}{4+1}+\frac{3-12i}{1+16} = \frac{2i+1}{5}+\frac{3-12i}{17} = \frac{34i+17+15-60i}{85} = \frac{32-26i}{85} = \frac{32}{85}+i\left(-\frac{26}{85}\right)$ **(Ans)**

(e) (iv) $\frac{2a+ib}{a+2ib}-\frac{2a-ib}{a-2ib} = \frac{(2a+ib)(a-2ib)-(2a-ib)(a+2ib)}{(a+2ib)(a-2ib)} = \frac{2a(a-2ib-a-2ib)+ib(a-2ib+a+2ib)}{a^2-4i^2b^2}$

$= \frac{2a(-4ib)+ib(2a)}{a^2+4b^2} = \frac{-8abi+2abi}{a^2+4b^2} = \frac{-6abi}{a^2+4b^2} = 0+i\left(\frac{-6abi}{a^2+4b^2}\right)$ **(Ans)**

**2.** (i) $\frac{\sqrt3+i\sqrt2}{2\sqrt3+i\sqrt2} = x+iy$ or, $\frac{(\sqrt3+i\sqrt2)(2\sqrt3-i\sqrt2)}{(2\sqrt3+i\sqrt2)(2\sqrt3-i\sqrt2)} = x+iy$ or, $\frac{6+2i\sqrt6-i\sqrt6+2}{12+2} = x+iy$, $[\because i^2=-1]$

or, $\frac{8+i\sqrt6}{14} = x+iy$ or, $\frac47+i\frac{\sqrt6}{14} = x+iy$

$\therefore x = \frac47,\ y = \frac{\sqrt6}{14}$ [Equating real and imaginary parts] **(Ans)**

**2.** (iv) $\because$ $x+3i$ and $-2+iy$ are conjugate to each other,

$\therefore x+3i = -2-iy \Rightarrow x = -2,\ y = -3$ **(Ans)**

**4.** (iii) $x = -2+\sqrt{-3}$ or, $(x+2)^2 = -3$ or, $x^2+4x+7 = 0$ ----------(1)

Now, $x^4+8x^3+24x^2+32x+16 = x^2(x^2+4x+7)+4x^3+17x^2+32x+16$

$= x^2(x^2+4x+7)+4x(x^2+4x+7)+(x^2+4x+7)+9 = 0+0+0+9 = 9$ **(Ans)** [by (1)]

**4.** (iv) $x = 2-i\sqrt3$ or, $(x-2)^2 = \left(-i\sqrt3\right)^2$ [squaring both sides] or, $x^2-4x+4 = -3$ or, $x^2-4x+7 = 0$ ----------(1)

Now, $2x^4-5x^3-3x^2+41x+k = 0$ or, $2x^2(x^2-4x+7)+3x^3-17x^2+41x+k = 0$

or, $2x^2(x^2-4x+7)+3x(x^2-4x+7)-5x^2+20x+k = 0$

or, $2x^2(x^2-4x+7)+3x(x^2-4x+7)-5(x^2-4x+7)+35+k = 0$

or, $0+0-0+35+k = 0$ $\therefore k = -35$ **(Ans)**

**4.** (vi) $x\sqrt{2} = 1+\sqrt{-1} = 1+i$ or, $x = \frac{1+i}{\sqrt{2}}$ or, $x^2 = \frac{1+i^2+2i}{2} = \frac{1-1+2i}{2} = i$

$\therefore x^6 + x^4 + x^2 + 2 = i^3 + i^2 + i + 2 = -i - 1 + i + 2 = 1$ **(Ans)**

**5.** (i) Let $z = \frac{3+4i}{4-3i} = \frac{(3+4i)(4+3i)}{(4-3i)(4+3i)} = \frac{12+16i+9i+12i^2}{16-9i^2} = \frac{12+25i-12}{16+9}$, $[\because i^2 = -1] = \frac{25i}{25} = i = 0 + i.1$

$\therefore$ Mod (z) $= \sqrt{0^2+1^2} = 1$ **(Ans)**

**5.** (iii) Let $z = \frac{8+i}{1+8i} = \frac{(8+i)(1-8i)}{(1+8i)(1-8i)} = \frac{8+i-64i-8i^2}{1-64i^2} = \frac{8-63i+8}{1+64}$, $[\because i^2 = -1] = \frac{16-63i}{65} = \frac{16}{65} + i\left(-\frac{63}{65}\right)$

$\therefore$ Mod (z) $= \sqrt{\left(\frac{16}{65}\right)^2 + \left(-\frac{63}{65}\right)^2} = \sqrt{\frac{256+3969}{(65)^2}} = \sqrt{\frac{(65)^2}{(65)^2}} = 1$ **(Ans)**

**5.** (iv) Let $z = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{1-i^2} = \frac{1-i}{2}$, $[\because i^2 = -1]$ or, $z = \frac{1}{2} + i\left(-\frac{1}{2}\right)$

$\therefore$ Mod (z) $= \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

**5.** (v) Let $z = \sqrt{12} + 6\left(\frac{1-i}{1+i}\right) = \sqrt{12} + \frac{6(1-i)^2}{(1+i)(1-i)} = \sqrt{12} + 6\left(\frac{1-2i+i^2}{1-i^2}\right)$

$= \sqrt{12} + 6\left(\frac{1-2i-1}{1+1}\right)$, $[\because i^2 = -1] = \sqrt{12} + 6(-i) = \sqrt{12} + i(-6)$

$\therefore$ Mod (z) $= \sqrt{(\sqrt{12})^2 + (-6)^2} = \sqrt{12+36} = \sqrt{48} = 4\sqrt{3}$ **(Ans)**

**5.** (vii) Let $z = (a - ib)^2 = a^2 - 2aib + i^2b^2 = (a^2 - b^2) + i(-2ab)$

$\therefore$ Mod (z) $= \sqrt{(a^2-b^2)^2 + (-2ab)^2} = \sqrt{(a^2-b^2)^2 + 4a^2b^2} = \sqrt{(a^2+b^2)^2} = a^2 + b^2$ **(Ans)**

**6.** (i) Solution : In the z – plane the point $z = \sqrt{3} - i$ lies in the fourth quadrant

Let amp (z) = θ, Therefore $\tan\theta = \frac{-1}{\sqrt{3}}$ where $-\frac{\pi}{2} < \theta < 0$

or, $\tan\theta = \tan\left(-\frac{\pi}{6}\right)$ or, $\theta = -\frac{\pi}{6}$ or, amp(z) $= -\frac{\pi}{6}$ **(Ans)**

**6.** (ii) Solution : Let $z = \sqrt{12} + 6\left(\frac{1-i}{1+i}\right) = \sqrt{12} + \frac{6(1-i)^2}{(1+i)(1-i)} = \sqrt{12} + 6\left(\frac{1-2i+i^2}{1-i^2}\right)$

$= \sqrt{12} + 6\left(\frac{1-2i-1}{1+1}\right)$, $[\because i^2 = -1] = \sqrt{12} + 6(-i) = \sqrt{12} + i(-6)$

In the z- plane the point $z = \sqrt{12} - 6i$ lies in the 4th quadrant.

Let amp(z) = θ, Therefore, $\tan\theta = \frac{-6}{\sqrt{12}}$ where $-\frac{\pi}{2} < \theta < 0$

or, $\tan\theta = \frac{-\sqrt{3}.\sqrt{12}}{\sqrt{12}} = -\sqrt{3}$ or, $\tan\theta = \tan\left(-\frac{\pi}{3}\right)$ or, $\theta = -\frac{\pi}{3}$ $\therefore$ the required amplitude is $-\frac{\pi}{3}$ **(Ans)**

**6.** (iii) **Solution :** Let $z = \frac{i}{1-i} = \frac{i(1+i)}{(1+i)(1-i)} = \frac{i+i^2}{1-i^2} = \frac{i-1}{2} = -\frac{1}{2} + i\frac{1}{2}$

$\therefore$ in the z – plane the point $z = -\frac{1}{2} + i\frac{1}{2}$ lies in the second quadrant.

Let amp(z) = θ, then $\tan\theta = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$ where $\frac{\pi}{2} < \theta \le \pi$ or, $\tan\theta = -1 = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$ or, $\theta = \frac{3\pi}{4}$

$\therefore$ the required amplitude $\frac{3\pi}{4}$ **(Ans)**

**6.** (iv) Let $z = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{1-i^2} = \frac{1-i}{2} = \frac{1}{2} + i\left(-\frac{1}{2}\right)$

$\therefore$ In the z – plane, $z = \frac{1}{2} + i\left(-\frac{1}{2}\right)$ lies in the fourth quadrant.

Let amp(z) = θ, $\tan\theta = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$ where $-\frac{\pi}{2} < \theta < 0 = \tan\left(-\frac{\pi}{4}\right)$

$\therefore$ amp(z) $= -\frac{\pi}{4}$ **(Ans)**

**7.** (i) $\frac{7-24i}{3+4i} = \frac{(7-24i)(3-4i)}{(3+4i)(3-4i)} = \frac{21-72i-28i+96i^2}{9-4^2i^2} = \frac{21-100i-96}{9+16}, \left[\because i^2 = -1\right] = \frac{-75-100i}{25}$

$= -3 - 4i = 1 - 2.1.2i - 4 = 1 - 2.1.2i + (2i)^2 = (1-2i)^2$

$\therefore$ the required square roots are $= \pm(1-2i)$ **(Ans)**

**7.** (iv) $i = \frac{1}{2}(2i) = \frac{1}{2}(1+2i-1) = \frac{1}{2}\left(1+2i+i^2\right) = \frac{1}{2}(1+i)^2$

$\therefore$ the required square roots are $\pm\frac{1}{\sqrt{2}}(1+i)$ **(Ans)**

**7.** (xii) $-7 - 24i = -7 - 2.3.4i = 9 - 2.3.4i + (4i)^2 = (3-4i)^2$

$\therefore$ the requried square roots are $= \pm(3-4i)$ **(Ans)**

**Alternative Method :**

Let $\sqrt{-7-24i} = x - iy$ ------------(1) (x, y are real)

or, $-7 - 24i = x^2 + i^2y^2 - 2ixy$ [squaring both sides] or, $-7 - 24i = (x^2 - y^2) - 2ixy$

Equating real and imaginary parts from both sides we get

$x^2 - y^2 = -7$ ------------(2), $2xy = 24$ -----------------(3)

Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = (-7)^2 + 576 = 49 + 576 = 625$

$\therefore x^2 + y^2 = 25$ ------------- (4)

Solving (2) and (4) we get $x^2 = 9$, $y^2 = 16$ or, $x = \pm 3$, $y = \pm 4$

Now, from (3), xy = 12 $\therefore$ either x = 3, y = 4 or, x = – 3, y = – 4

$\therefore$ the required square roots are $\pm(3-4i)$ **(Ans)**

**7.** (xiii) $3 + 4i = 4 + 4i - 1 = 2^2 + 2.2.i + i^2 = (2+i)^2$

$\therefore$ a square root is $= \pm(2+i)$ **(Ans)**

**8.** (i) $\sqrt{i} + \sqrt{-i} = \frac{1}{\sqrt{2}}\left(\sqrt{2i} + \sqrt{-2i}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{1+2i+i^2} + \sqrt{1-2i+i^2}\right)$

$= \frac{1}{\sqrt{2}}\left(\sqrt{(1+i)^2} + \sqrt{(1-i)^2}\right) = \frac{1}{\sqrt{2}}(1+i+1-i) = \frac{2}{\sqrt{2}} = \sqrt{2}$ **(Proved)**

**9.** (i) Let $z_1 = r_1\cos\theta_1 + i.r_1\sin\theta_1$, $z_2 = r_2\cos\theta_2 + i.r_2\sin\theta_2$ $\therefore |z_1| = \sqrt{r_1^2\cos^2\theta_1 + r_1^2\sin^2\theta_1} = r_1$

and $\text{amp}(z_1) = \tan^{-1}\left(\frac{r_1\sin\theta_1}{r_1\cos\theta_1}\right) = \tan^{-1}(\tan\theta_1) = \theta_1$

Similarly, $|z_2| = r_2$, $\text{amp}(z_2) = \theta_2$ $\therefore |z_1| = |z_2| \Rightarrow r_1 = r_2$ ----------- (1)

and amp $z_1$ – amp $z_2 = \pi$ or, $\theta_1 - \theta_2 = \pi$ $\therefore \theta_1 = \pi + \theta_2$

$\therefore z_1 = r_1\cos\theta_1 + i.r_1\sin\theta_1 = r_1(\cos\theta_1 + i.\sin\theta_1) = r_2\{\cos(\pi + \theta_2) + i.\sin(\pi + \theta_2)\}$ [from (1) and (2)]

$= r_2(-\cos\theta_2 - i.\sin\theta_2) = -r_2(\cos\theta_2 + i.\sin\theta_2) = -z_2$

$\therefore z_1 + z_2 = 0$ **(Proved)**

**10.** (i)(b) $(x^2 + a^2)(y^2 + b^2)(z^2 + c^2) = (x + ai)(x - ai)(y + ib)(y - ib)(z + ic)(z - ic)$ [where $i^2 = -1$]

$= \{(x + ia)(y + ib)(z + ic)\}\{(x - ia)(y - ib)(z - ic)\} = xyz\left(1+i\frac{a}{x}\right)\left(1+i\frac{b}{y}\right)\left(1+i\frac{c}{z}\right)\times xyz\left(1-i\frac{a}{x}\right)\left(1-i\frac{b}{y}\right)\left(1-i\frac{c}{z}\right)$

$= xyz\left\{1+i\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)-\left(\frac{ab}{xy}+\frac{bc}{yz}+\frac{ca}{zx}\right)-i\frac{abc}{xyz}\right\}\times xyz\left\{1-i\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)-\left(\frac{ab}{xy}+\frac{bc}{yz}+\frac{ca}{zx}\right)+i\frac{abc}{xyz}\right\}$

$= \{(xyz - abz - bcx - cay) + i(ayz + bzx + cxy - abc)\}\times\{(xyz - abz - bcx - cay) - i(ayz + bzx + cxy - abc)\}$

$= (xyz - abz - bcx - cay)^2 + (ayz + bzx + cxy - abc)^2$ [$\because (a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2$ ($\because i^2 = -1$)]

is the required sum of two squares. **(Ans)**

**10.** (ii)(a) **Solution :**

$x^2 + xy + y^2 = x^2 - (\omega + \omega^2)xy + y^2$ [$\because \omega + \omega^2 = -1$]

$= x^2 - \omega xy - \omega^2 xy + y^2 = x(x - y\omega) - \omega^2 y(x - \omega y)$ [$\because \omega^3 = 1$] $= (x - y\omega)(x - y\omega^2)$ **(Ans)**

**10** (ii)(c) **Solution :**

$a^3 + b^3 + c^3 - 3abc = (a+b)^3 - 3ab(a + b) + c^3 - 3abc$

$= (a + b)^3 + c^3 - 3ab(a + b) - 3abc = (a + b + c)^3 - 3(a + b).c(a + b + c) - 3ab(a + b + c)$

$= (a + b + c)\{(a + b + c)^2 - 3c(a + b) - 3ab\}$

$= (a + b + c)\{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3ca - 3bc - 3ab\} = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= (a + b + c)\{a^2 + b^2 + c^2 + (\omega + \omega^2)ab + (\omega + \omega^2)bc + (\omega + \omega^2)ca\}$ [$\because 1 + \omega + \omega^2 = 0 \therefore \omega + \omega^2 = -1$]

$= (a + b + c)\{(a^2 + ab\omega + ac\omega^2) + (b^2 + ba\omega^2 + bc\omega) + (c^2 + cb\omega^2 + ca\omega)\}$

$= (a + b + c)\{a(a + b\omega + c\omega^2) + b\omega^2(a + b\omega + c\omega^2) + c\omega(a + b\omega + c\omega^2)\}$

$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$ **(Ans)**

**11.** (i) $\frac{1}{1+2\omega}-\frac{1}{1+\omega}+\frac{1}{2+\omega} = \frac{1}{-\omega-\omega^2+2\omega}-\frac{1}{-\omega^2}+\frac{1}{1+1+\omega} = \frac{1}{\omega-\omega^2}+\frac{1}{\omega^2}+\frac{1}{1-\omega^2}$ $\left[\because 1+\omega=-\omega^2\right]$

$= \frac{\omega}{\omega^2-1}+\omega+\frac{1}{1-\omega^2}$ $\left[\because \omega^3=1\right]$ $= \frac{\omega}{\omega^2-1}-\frac{1}{\omega^2-1}+\omega$

$= \frac{\omega-1}{\omega^2-1}+\omega = \frac{1}{\omega+1}+\omega = \frac{1}{-\omega^2}+\omega = -\omega+\omega = 0$ **(Proved)**

**11.** (iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$ [$\because \omega^3 = 1, \therefore \omega^4 = \omega^3\omega = \omega$ and $\omega^8 = (\omega^3)^2.\omega^2 = \omega^2$]

$= \{(1 - \omega)(1 - \omega^2)\}^2 = \{1 - (\omega + \omega^2) + \omega^3\}^2$

$= (1 + 1 + 1)^2 = 9$ **(Proved)**

**11.** (vi) $(1 + \omega - \omega^2)^5 + (1 - \omega + \omega^2)^5 = (-\omega^2 - \omega^2)^5 + (-\omega - \omega)^5$ [$\because 1 + \omega + \omega^2 = 0,\ 1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$]

$= (-2\omega^2)^5 + (-2\omega)^5 = -2^5(\omega^{10} + \omega^5) = -32(\omega + \omega^2)$ [$\because \omega^3 = 1$] $= -32(-1) = 32$ **(Proved)**

**12.** (ii) $\dfrac{x\omega^2 + y\omega + z}{x\omega + y + z\omega^2} = \dfrac{\omega(x\omega^2 + y\omega + z)}{x\omega^2 + y\omega + z\omega^3}$ [multiplying numerator and denominator by $\omega$]

$= \dfrac{\omega(x\omega^2 + y\omega + z)}{x\omega^2 + y\omega + z}$ [$\because \omega^3 = 1$] $= \omega$ **(Proved)**

**12.** (v) $(x + y\omega + z\omega^2)^4 + (x\omega + y\omega^2 + z)^4 + (x\omega^2 + y + z\omega)^4$

$= (x + y\omega + z\omega^2)^4 + (x\omega + y\omega^2 + z\omega^3)^4 + (x\omega^2 + y\omega^3 + z\omega^4)^4$ [$\because \omega^3 = 1$]

$= (x + y\omega + z\omega^2)^4 + \omega^4(x + y\omega + z\omega^2)^4 + \omega^8(x + y\omega + z\omega^2)^4 = (x + y\omega + z\omega^2)^4(1 + \omega^4 + \omega^8)$

$= (x + y\omega + z\omega^2)^4(1 + \omega + \omega^2)$ [$\because \omega^3 = 1$] $= (x + y\omega + z\omega^2)^4 \times 0$ ($\because 1 + \omega + \omega^2 = 0$) $= 0$ **(Proved)**

**12.** (vi) **Solution :**

Let $x = a + b\omega + c\omega^2,\ y = a + b\omega^2 + c\omega$ -----------(1)

Now, $x^3 + y^3 = (x + y)(x^2 - xy + y^2) = (x + y)\{x^2 + (\omega + \omega^2)xy + y^2\}$ [$\therefore \omega + \omega^2 = -1$]

$= (x + y)\{x^2 + \omega xy + \omega^2 xy + y^2\} = (x + y)\{x(x + \omega y) + \omega^2 y(x + y\omega)\}$ [$\because \omega^3 = 1$]

or, $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$ -----------(2)

or, $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$

$= (a + b\omega + c\omega^2 + a + b\omega^2 + c\omega) \times (a + b\omega + c\omega^2 + a\omega + b\omega^3 + c\omega^2) \times (a + b\omega + c\omega^2 + a\omega^2 + b\omega^4 + c\omega^3)$

[Putting the values of x and y from (1) in (2) ]

or, $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$

$= \{2a + b(\omega + \omega^2) + c(\omega + \omega^2)\} \times \{2c\omega^2 + a(1 + \omega) + b(1 + \omega)\} \times \{2b\omega + a(1 + \omega^2) + c(1 + \omega^2)\}$

[$\because \omega^3 = 1 \because \omega^4 = \omega^3.\omega = \omega$]

$= (2a - b - c) \times (2c\omega^2 - a\omega^2 - b\omega^2) \times (2b\omega - a\omega - c\omega)$ [$\because 1 + \omega + \omega^2 = 0$]

$= \omega^2.\omega\,(2a - b - c)(2c - a - b)(2b - a - c)$

$\therefore (a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = (2a - b - c)(2b - c - a)(2c - a - b)$ **(Proved)**

**12.**(vii) Let $x = a + b\omega + c\omega^2,\ y = a + b\omega^2 + c\omega$ -----------(1)

Now, $x^3 + y^3 = (x + y)(x^2 - xy + y^2) = (x + y)\{x^2 + (\omega + \omega^2)xy + y^2\}$ [$\therefore \omega + \omega^2 = -1$]

$= (x + y)\{x^2 + \omega xy + \omega^2 xy + y^2\} = (x + y)\{x(x + \omega y) + \omega^2 y(x + y\omega)\}$ [$\because \omega^3 = 1$]

or, $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$ -----------(2)

or, $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$

$= (a + b\omega + c\omega^2 + a + b\omega^2 + c\omega) \times (a + b\omega + c\omega^2 + a\omega + b\omega^3 + c\omega^2) \times (a + b\omega + c\omega^2 + a\omega^2 + b\omega^4 + c\omega^3)$

[Putting the values of x and y from (1) in (2) ]

or, $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$

$= \{2a + b(\omega + \omega^2) + c(\omega + \omega^2)\} \times \{2c\omega^2 + a(1 + \omega) + b(1 + \omega)\} \times \{2b\omega + a(1 + \omega^2) + c(1 + \omega^2)\}$

$[\because \omega^3 = 1 \because \omega^4 = \omega^3.\omega = \omega]$

$= (2a - b - c) \times (2c\omega^2 - a\omega^2 - b\omega^2) \times (2b\omega - a\omega - c\omega)$ $[\because 1 + \omega + \omega^2 = 0]$

$= \omega^2.\omega\,(2a - b - c)(2c - a - b)(2b - a - c)$

$\therefore (a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = (2a + a)(2b + b)(2c + c) = 27abc$

[ Since $a + b + c = 0 \therefore -b - c = a, -c - a = b$ and $-a - b = c$ ] **(Proved)**

**12.** (viii) $(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^2 + ab\omega^2 + ac\omega + ab\omega + b^2\omega^3 + bc\omega^2 + ca\omega^2 + bc\omega^4 + c^2\omega^3$

$= a^2 + b^2 + c^2 + ab(\omega^2 + \omega) + ac(\omega + \omega^2) + bc(\omega^2 + \omega^4)$

$= a^2 + b^2 + c^2 - ab - bc - ca$ **(Proved)** $[\because \omega^3 = 1 \;\&\; \omega + \omega^2 = -1]$

**13. (i) Solution :** L.H.S. $= (1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)(1 - \omega^8 + \omega^{16})$

$= (1 - \omega + \omega^2)(1 - \omega^2 + \omega)(1 - \omega + \omega^2)(1 - \omega^2 + \omega)$ $(\because \omega^3 = 1)$

$= (1 - \omega + \omega^2)^2 (1 - \omega^2 + \omega)^2 = (1 + \omega + \omega^2 - 2\omega)^2 (1 + \omega + \omega^2 - 2\omega^2)^2$

$= (-2\omega)^2(-2\omega^2)^2$ $[\because 1 + \omega + \omega^2 = 0] = 4\omega^2.4\omega^4 = 16\omega^6 = 16$ $[\because \omega^3 = 1]$ = R.H.S. **(Proved)**

**13. (iv) Solution :** Given, $x = a + b\omega + c\omega^2$, $y = a\omega + b\omega^2 + c$, $z = a\omega^2 + b + c\omega$

$\therefore x + y + z = a(1 + \omega + \omega^2) + b(1 + \omega + \omega^2) + c(1 + \omega + \omega^2)$

$= (a + b + c)(1 + \omega + \omega^2) = 0$ $(\because 1 + \omega + \omega^2 = 0)$

Now, we know $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$ $[\because x + y + z = 0]$

$\therefore x^3 + y^3 + z^3 = 3xyz$ or, $\frac{x^3}{xyz} + \frac{y^3}{xyz} + \frac{z^3}{xyz} = 3$ or, $\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3$

**13.** (v) $x = a + b$, $y = a + b\omega$, $z = a + b\omega^2$

$\therefore x^3 + y^3 + z^3 = (a + b)^3 + (a + b\omega)^3 + (a + b\omega^2)^3$

$= a^3 + 3a^2b + 3ab^2 + b^3 + a^3 + 3a^2b\omega + 3ab^2\omega^2 + b^3\omega^3 + a^3 + 3a^2b\omega^2 + 3ab^2\omega^4 + b^3\omega^6$

$= 3a^3 + b^3(1 + \omega^3 + \omega^6) + 3a^2b(1 + \omega + \omega^2) + 3ab^2(1 + \omega^2 + \omega^4)$

$= 3a^3 + 3b^3 + 3a^2b \times 0 + 3ab^2 \times 0$ $[\because \omega^3 = 1$ and $1 + \omega + \omega^2 = 0]$

$\therefore x^3 + y^3 + z^3 = 3a^3 + 3b^3 = 3(a^3 + b^3)$ **(Proved)**

**13.** (vi) $(a + b\omega + c\omega^2)^2 + (a\omega + b + c\omega^2)^2 + (a\omega + b\omega^2 + c)^2 = 0$

or, $(a + b\omega + c\omega^2)^2 + (a\omega + b + c\omega^2)^2 + \omega^2(a + b\omega + c\omega^2)^2 = 0$ $[\therefore \omega^3 = 1]$

or, $(1 + \omega^2)(a + b\omega + c\omega^2)^2 + (a\omega + b + c\omega^2)^2 = 0$ or, $-\omega(a + b\omega + c\omega^2)^2 + (a\omega + b + c\omega^2)^2 = 0$

or, $-\omega^3(a\omega^2 + b + c\omega)^2 + (a\omega + b + c\omega^2)^2 = 0$ or, $(a\omega^2 + b + c\omega)^2 = (a\omega + b + c\omega^2)^2$

$\therefore a\omega^2 + b + c\omega = \pm(a\omega + b + c\omega^2)$

or, $a\omega^2 + b + c\omega = a\omega + b + c\omega^2$ [taking + ve sign]

or, $a\omega^2 + c\omega = a\omega + c\omega^2$ or, $a(\omega^2 - \omega) - c(\omega^2 - \omega) = 0$ or, $a - c = 0 \therefore a = c$

Again taking –ve sign

$a\omega^2 + b + c\omega = -a\omega - b - c\omega^2$ or, $a(\omega + \omega^2) + 2b + c(\omega + \omega^2) = 0$ or, $-a + 2b - c = 0$ $[\therefore \omega + \omega^2 = -1]$

or, $2b = a + c \therefore b = \frac{a+c}{2}$ $\therefore a = c$ or, $b = \frac{a+c}{2}$ **(Proved)**

**14.** (i) $\because \alpha = \frac{1}{2}\left(-1 + \sqrt{-3}\right),\ \beta = \frac{1}{2}\left(-1 - \sqrt{-3}\right)$

$\therefore \alpha + \beta = \frac{1}{2}\left(-1 + \sqrt{-3} - 1 - \sqrt{-3}\right) = \frac{1}{2}(-2) = -1$

and $\alpha\beta = \frac{1}{4}\left(-1 + \sqrt{-3}\right)\left(-1 - \sqrt{-3}\right) = \frac{1}{4}\left\{(-1)^2 - \left(\sqrt{-3}\right)^2\right\} = \frac{1}{4}(1 + 3) = 1$

$\therefore \alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta = (\alpha + \beta)^2 - \alpha\beta = (-1)^2 - 1 = 1 - 1 = 0$ **(Proved)**

**14.** (iii) $\because \alpha, \beta$ are complex cube roots of 1 $\therefore \alpha = \beta^2$ and $\beta = \alpha^2$ and $\alpha\beta = 1$

$\therefore \alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \left(\alpha^2\right)^2 + \left(\beta^2\right)^2 + \frac{1}{\alpha\beta} = \beta^2 + \alpha^2 + 1$

$= \alpha + \alpha^2 + 1 = 0$ $[\because \alpha + \alpha^2 + 1 = 0]$ **(Proved)**

**15.** (ii) $\sqrt[3]{x + iy} = a + ib$ or $x + iy = (a + ib)^3$ [cubing both sides]

or, $x + iy = a^3 + 3a^2ib + 3ai^2b^2 + i^3b^3 = a^3 + 3a^2bi - 3ab^2 - ib^3 = (a^3 - 3ab^2) + i(3a^2b - b^3)$

Equating real and imarginary parts from both sides we get,

$x = a^3 - 3ab^2,\ y = 3a^2b - b^3$

$\therefore \frac{x}{a} + \frac{y}{b} = \frac{a^3 - 3ab^2}{a} + \frac{3a^2b - b^3}{b} = a^2 - 3b^2 + 3a^2 - b^2 = 4(a^2 - b^2)$ **(Proved (a))**

$x - iy = a^3 - 3ab^2 - i(3a^2b - b^3) = a^3 - 3a^2ib - 3ab^2 + ib^3$

$= a^3 - 3a^2.ib + 3a(ib)^2 - (ib)^3$ $[\because i^2 = -1] = (a - ib)^3$

$\therefore \sqrt[3]{x - iy} = a - ib$ **(Proved (b))** [taking cube root on both sides]

**15.** (iii) $x + iy = \sqrt{\frac{a + ib}{c + id}}$

or, $(x + iy)^2 = \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)}$ or, $x^2 - y^2 + 2ixy = \frac{ac + ibc - iad - i^2bd}{c^2 - i^2d^2}$

or, $\left(x^2 - y^2\right) + 2ixy = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$, $\left[\because i^2 = -1\right]$ or, $\left(x^2 - y^2\right) + 2ixy = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}$

$\therefore x^2 - y^2 = \frac{ac + bd}{c^2 + d^2}$, $2xy = \frac{bc - ad}{c^2 + d^2}$ [Equating real and imaginary parts from both sides]

$$\therefore\ \left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+4x^2y^2=\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{bc-ad}{c^2+d^2}\right)^2$$

$$=\frac{(ac+bd)^2+(bc-ad)^2}{\left(c^2+d^2\right)^2}=\frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{\left(c^2+d^2\right)^2}=\frac{c^2\left(a^2+b^2\right)+d^2\left(a^2+b^2\right)}{\left(c^2+d^2\right)^2}=\frac{\left(a^2+b^2\right)\left(c^2+d^2\right)}{\left(c^2+d^2\right)^2}$$

$\therefore\ \left(x^2+y^2\right)^2=\dfrac{a^2+b^2}{c^2+d^2}$ **(Proved)**

**16.** (i) $\dfrac{1-ix}{1+ix}=a-ib$ or, $\dfrac{2ix}{2}=\dfrac{1-a+ib}{1+a-ib}$ [by componendo and dividendo we get]

or, $ix=\dfrac{1-a+ib}{1+a-ib}$ or, $x=\dfrac{b+i(a-1)}{a+1-ib}=\dfrac{\{b+i(a-1)\}\{(a+1)+ib\}}{(a+1-ib)(a+1+ib)}$

$$=\frac{b(a+1)-b(a-1)+i\{(a+1)(a-1)+b^2\}}{(a+1)^2+b^2},\ \left[\because i^2=-1\right]=\frac{b(a+1-a+1)+i\left(a^2-1+b^2\right)}{(a+1)^2+b^2}=\frac{2b+i\left(a^2+b^2-1\right)}{(a+1)^2+b^2}$$

$\therefore\ x=\dfrac{2b}{(a+1)^2+b^2}+i\,\dfrac{a^2+b^2-1}{(a+1)^2+b^2}$

$\therefore$ for real x, $\dfrac{a^2+b^2-1}{(a+1)^2+b^2}=0$ or, $a^2+b^2=1$ Hence **proved**.

**16.** (ii) $\dfrac{\sqrt{1+x}-i\sqrt{1-x}}{\sqrt{1+x}+i\sqrt{1-x}}=a-ib$ or, $\dfrac{2\sqrt{1+x}}{2i\sqrt{1-x}}=\dfrac{1+a-ib}{1-a+ib}$ (by comp. and div.)

or, $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}=\dfrac{i(1+a-ib)(1-a-ib)}{(1-a+ib)(1-a-ib)}=\dfrac{i\{(1+a)(1-a)-b^2\}-i^2b(1-a+1+a)}{(1-a)^2+b^2}$ $[\because i^2=-1]=\dfrac{\left(1-a^2-b^2\right)i+2b}{(1-a)^2+b^2}$

If $1-a^2-b^2=0$, then $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}=\dfrac{2b}{(1-a)^2+b^2}$ or, $\dfrac{1+x}{1-x}=\dfrac{4b^2}{\{(1-a)^2+b^2\}^2}$

or, $\dfrac{2x}{2}=\dfrac{4b^2-\{(1-a)^2+b^2\}^2}{4b^2+\{(1-a)^2+b^2\}^2}$ [by componendo and dividendo] or, $x=\dfrac{4b^2-\{(1-a)^2+b^2\}^2}{4b^2+\{(1-a)^2+b^2\}^2}$ which is real.

Hence, for real x, $a^2+b^2=1$ **(Proved)**

**17.** (ii) $x+iy=\dfrac{2}{3+\cos\theta+i\sin\theta}$

or, $x+iy=\dfrac{2(3+\cos\theta-i\sin\theta)}{(3+\cos\theta+i\sin\theta)(3+\cos\theta-i\sin\theta)}=\dfrac{2(3+\cos\theta)-2i\sin\theta}{(3+\cos\theta)^2+\sin^2\theta}=\dfrac{2(3+\cos\theta)-2i\sin\theta}{9+6\cos\theta+\cos^2\theta+\sin^2\theta}$

$$=\frac{2(3+\cos\theta)-2i\sin\theta}{10+6\cos\theta}=\frac{3+\cos\theta}{5+3\cos\theta}+i\left(\frac{-\sin\theta}{5+3\cos\theta}\right)$$

$\therefore\ x=\dfrac{3+\cos\theta}{5+3\cos\theta},\quad y=\dfrac{-\sin\theta}{5+3\cos\theta}$

$\therefore$ L. H. S $=2(x^2+y^2)=2\left\{\left(\dfrac{3+\cos\theta}{5+3\cos\theta}\right)^2+\left(\dfrac{-\sin\theta}{5+3\cos\theta}\right)^2\right\}=2\left\{\dfrac{9+6\cos\theta+\cos^2\theta+\sin^2\theta}{(5+3\cos\theta)^2}\right\}$

$$=2\,\frac{10+6\cos\theta}{(5+3\cos\theta)^2}=\frac{4(5+3\cos\theta)}{(5+3\cos\theta)^2}=\frac{4}{5+3\cos\theta}$$

R. H. S $=3x-1=3\left(\dfrac{3+\cos\theta}{5+3\cos\theta}\right)-1=\dfrac{9+3\cos\theta-5-3\cos\theta}{5+3\cos\theta}=\dfrac{4}{5+3\cos\theta}$

or, $2(x^2+y^2)^2=3x-1$ **(Proved)**

**18.** **(i) Given** $\omega=\dfrac{-1+\sqrt{-3}}{2}=\dfrac{-1+\sqrt{3}\,i}{2}$, $i^2=-1$

$\therefore\ \omega^2=\dfrac{\left(-1+\sqrt{3}\,i\right)^2}{4}=\dfrac{1-3-2\sqrt{3}\,i}{4}=\dfrac{-2-2\sqrt{3}\,i}{4}=\dfrac{-1-\sqrt{3}\,i}{4}$

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$\therefore\ \omega+\omega^2=\frac{-1+\sqrt{3}\,i}{2}+\frac{-1-\sqrt{3}\,i}{2}=\frac{-1+\sqrt{3}\,i-1-\sqrt{3}\,i}{2}=-1\quad\therefore\ 1+\omega+\omega^2=0$

and $\omega.\omega^2=\left(\frac{-1+\sqrt{3}\,i}{2}\right)\left(\frac{-1-\sqrt{3}\,i}{2}\right)=\frac{(-1)^2-(\sqrt{3}\,i)^2}{4}=\frac{1+3}{4}=1\ \therefore\ \omega^3=1$

$\therefore (1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})$

$=(1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)(1-\omega^2+\omega)\quad(\because\ \omega^3=1)$

$=\{(1-\omega+\omega^2)(1+\omega-\omega^2)\}^2=\{(-\omega-\omega)(-\omega^2-\omega^2)\}\quad(\because\ 1+\omega+\omega^2=0)$

$=\{(-2\omega)\ (-2\omega^2)\}^2=(4\omega^3)^2=16.1^2=16$ **(Ans)**

**18.** (ii) Let $\omega=\frac{1+\sqrt{-3}}{2}=\frac{1+\sqrt{3}\,i}{2},\ i^2=-1\ \therefore\ \omega^2=\frac{1-3+2\sqrt{3}\,i}{4}=-\frac{1-\sqrt{3}\,i}{2}$

$\omega^3=-\left(\frac{1+\sqrt{3}\,i}{2}\right)\left(\frac{1-\sqrt{3}\,i}{2}\right)=-\frac{(1)^2-(\sqrt{3}\,i)^2}{4}=-\frac{1+3}{4}=-1$ and $\omega-\omega^2=\frac{1+\sqrt{3}\,i}{2}+\frac{1-\sqrt{3}\,i}{2}=\frac{2}{2}=1$

$\therefore\ \left(\frac{1+\sqrt{-3}}{2}\right)^6+\left(\frac{1-\sqrt{-3}}{2}\right)^9=\omega^6+(-\omega^2)^9=\omega^6-\omega^{18}=(\omega^3)^2-(\omega^3)^6=(-1)^2-(-1)^6=1-1=0$ **(Ans)**

**19.** (ii) $z=x+iy$

$\therefore\ |z+6|=|2z+3|$ or, $|x+iy+6|=|2x+2iy+3|$

or, $|(x+6)+iy|=|(2x+3)+2iy|$ or, $\sqrt{(x+6)^2+y^2}=\sqrt{(2x+3)^2+(2y)^2}$

or, $(x+6)^2+y^2=(2x+3)^2+(2y)^2$ or, $x^2+12x+36+y^2=4x^2+12x+9+4y^2$

or, $3x^2+3y^2=27$ or, $x^2+y^2=9$ **(Proved)**

**19.** (iii) Given, $z=x+iy\quad\therefore\ \frac{z-i}{z-1}=ib$ or, $\frac{x+iy-i}{x+iy-1}=ib$

or, $x+iy-i=ibx-by-ib$ or, $(x+by)+i(y-1-bx+b)=0$

$\therefore\ x+by=0$ or, $b=-\frac{x}{y}$ ---------- (1) and, $y-1-b(x-1)=0$ ---------- (2)

Eliminating b from (1) and (2) we get,

$y-1+\frac{x}{y}(x-1)=0$ or, $y^2-y+x^2-x=0$ or, $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$ **(Proved)**

## 2.9 De Moiver's theorem :

For all integral values of n, the value of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$ and for all fractional values of n, one of the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$.

## PROBLEM SET – III

[.....Problems with '*' marks are solved at the end of the problem set.....]

**20.** Express in the form x + iy.

(i) $\dfrac{(\cos\theta+i\sin\theta)^8}{(\sin\theta+i\cos\theta)^4}$ *(ii) $\dfrac{(\cos\theta+i\sin\theta)^6}{(\sin\theta+i\cos\theta)^5}$ (iii) $\dfrac{(\cos\theta+i\sin\theta)^9}{(\cos\theta-i\sin\theta)^4}$

*(iv) $\left(\dfrac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n$ (v) $\left(\dfrac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n$

**21.** Prove that

*(i) $(a+ib)^n+(a-ib)^n = 2(a^2+b^2)^{\frac{n}{2}}.\cos\left\{n\tan^{-1}\left(\frac{b}{a}\right)\right\}$

(ii) $(1+i)^n+(1-i)^n = 2^{\frac{n}{2}+1}.\cos\frac{n\pi}{4}$ (iii) $(\sqrt{3}+i)^n+(\sqrt{3}-i)^n = 2^{n+1}.\cos\frac{n\pi}{6}$

**22.** With the help of De Moiver's theorem,

expand *(i) $\cos 3\theta$ in powers of $\cos\theta$ *(ii) $\sin 3\theta$ in powers of $\sin\theta$

(iii) $\cos 5\theta$ in powers of $\cos\theta$.

### • Extraction of any assigned root of a complex number :

We know, $\cos\theta + i\sin\theta = \cos(2n\pi+\theta) + i\sin(2n\pi+\theta)$, where n is zero or any integer.

Then by De Moiver's theorem, the m-th roots of $\cos\theta + i\sin\theta$ are,

$$(\cos\theta+i\sin\theta)^{\frac{1}{m}} = \{\cos(2n\pi+\theta)+i\sin(2n\pi+\theta)\}^{\frac{1}{m}} = \cos\left(\frac{2n\pi+\theta}{m}\right)+i\sin\left(\frac{2n\pi+\theta}{m}\right);\ n = 0, 1, 2, \ldots\ldots, (m-1).$$

If values of n greater than (m – 1), that is n = m, m + 1, m + 2, ...................., then we would get the same values already obtained by putting n = 0, 1, 2, 3, ....... ( m – 1) repeated over and over again.

**For example :** Let, $z = (1)^{\frac{1}{3}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{3}} = \cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}$ ; n = 0, 1, 2 [by De Moiver's theorem]

For, n = 0; $z = \cos 0 + i\sin 0 = 1$ For, n = 1, $z = \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$

For n = 2, $z = \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3} = \cos\left(\pi+\frac{\pi}{3}\right)+i\sin\left(\pi+\frac{\pi}{3}\right) = -\cos\frac{\pi}{3}-i\sin\frac{\pi}{3} = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

$\therefore (1)^{\frac{1}{3}} = 1,\ -\frac{1}{2}+i\frac{\sqrt{3}}{2},\ -\frac{1}{2}-i\frac{\sqrt{3}}{2}$ known as **imaginary cube roots of unity.**

# PROBLEM SET – IV

[.....Problems with '*' marks are solved at the end of the problem set.....]

**23.** Find the values of the following.

*(i) $(-1)^{\frac{1}{3}}$ (ii) $(i)^{\frac{3}{4}}$ (iii) $(-i)^{\frac{3}{5}}$ *(iv) $(\sqrt{3}-i)^{\frac{1}{7}}$ (v) $(1+i)^{\frac{2}{3}}$ (vi) $(1+\sqrt{3}i)^{\frac{3}{4}}$

**24.** Solve :

(i) $x^5 = 1$. (ii) $x^7 = 1$ *(iii) $x^3 + 8 = 0$.

**25.** (i) Find the continued product of all the values of $(1+i)^{\frac{1}{5}}$.

*(ii) Show that the product of all the values of $(1+i\sqrt{3})^{\frac{3}{4}}$ is 8.

***26.** If $x = \cos\theta + i\sin\theta$ and $1+\sqrt{1-a^2} = na$, prove that, $1 + a\cos\theta = \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$.

**27.** If $x = \cos\theta + i\sin\theta$, show that,(i) $x^m + \frac{1}{x^m} = 2\cos m\theta$ (ii) $x^m - \frac{1}{x^m} = 2i\sin m\theta$

**28.** If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$ then prove that,

*(i) $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \phi)$ (ii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

(iii) $xy + \frac{1}{xy} = 2\cos(\theta + \phi)$ *(iv) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$ **[WBSC – 03, 06, 08]**

**29.** If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\phi$, $z + \frac{1}{z} = 2\cos\psi$, show that $xyz + \frac{1}{xyz} = 2\cos(\theta + \phi + \psi)$

**30.** If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, prove that

(i) $\sum\cos 3\alpha = 3\cos(\alpha+\beta+\gamma)$ (ii) $\sum\sin 3\alpha = 3\sin(\alpha+\beta+\gamma)$ (iii) $\sum\cos^2\alpha = \sum\sin^2\alpha = \frac{3}{2}$

***31.** If $x + \frac{1}{x} = 2\cos\alpha$, $y + \frac{1}{y} = 2\cos\beta$, $z + \frac{1}{z} = 2\cos\gamma$ prove that,

$\left.\begin{array}{l}\text{(i) } \sum\cos 4\alpha = 2\sum\cos 2(\beta+\gamma) \text{ (ii) } \sum\sin 4\alpha = 2\sum\sin 2(\beta+\gamma)\end{array}\right\}$ if, $x + y + z = 0$

(iii) $\sum\cos(\beta - \gamma) = -1$, if $x + y + z = xyz$

**32.** Show that, $1 - {}^nC_2 + {}^nC_4 - \ldots\ldots = 2^{\frac{n}{2}}\cdot\cos\frac{n\pi}{4}$ and ${}^nC_1 - {}^nC_3 + {}^nC_5 - \ldots\ldots = 2^{\frac{n}{2}}\cdot\sin\frac{n\pi}{4}$

**33.** If $\alpha$, $\beta$ be the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$, find the equation whose roots are $\alpha^n$ and $\beta^n$.

***34.** Find the equation whose roots are the n-th powers of those of the equation $x^2 - 2x + 4 = 0$. Show that the sum of the n-th powers of the roots is

## ANSWERS

**20.** **(i)** $\cos 12\theta + i\sin 12\theta$ (ii) $\sin 11\theta - i\cos 11\theta$ (iii) $\cos 13\theta + i\sin 13\theta$

(iv) $\cos n\theta + i\sin n\theta$ (v) $\cos\left(\frac{n\pi}{2} - n\phi\right) + i\sin\left(\frac{n\pi}{2} - n\phi\right)$

**22.** **(i)** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

**23.** (i) $-1$, $\frac{1}{2}\left(1 \pm i\sqrt{3}\right)$ (ii) $\cos\frac{1}{4}\left(2n\pi + \frac{3\pi}{2}\right) + i\sin\frac{1}{4}\left(2n\pi + \frac{3\pi}{2}\right)$; $n = 0, 1, 2, 3$.

(iii) $\cos\frac{1}{5}\left(2n\pi + \frac{9\pi}{2}\right) + i\sin\frac{1}{5}\left(2n\pi + \frac{9\pi}{2}\right)$; $n = 0, 1, 2, 3, 4$.

(iv) $2^{\frac{1}{7}}\left\{\cos\frac{1}{7}\left(2n\pi + \frac{\pi}{6}\right) - i\sin\frac{1}{7}\left(2n\pi + \frac{\pi}{6}\right)\right\}$; $n = 0, 1, 2, 3, 4, 5, 6$.

(v) $2^{\frac{1}{3}}\left\{\cos\left(\frac{4n\pi}{3} + \frac{\pi}{6}\right) + i\sin\left(\frac{4n\pi}{3} + \frac{\pi}{6}\right)\right\}$; $n = 0, 1, 2$.

(vi) $2^{\frac{3}{4}}\left[\cos\frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right) + i\sin\frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right)\right]$; $n = 0, 1, 2, 3$.

**24.** (i) $\cos\frac{2n\pi}{5} + i\sin\frac{2n\pi}{5}$; $n = 0, 1, 2, 3, 4$. (ii) $\cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}$; $n = 0, 1, 2, 3, 4, 5, 6$.

(iii) $2\left[\cos\left(\frac{2n\pi + \pi}{3}\right) + i\sin\left(\frac{2n\pi + \pi}{3}\right)\right]$; $n = 0, 1, 2$. **25.** (i) $1 + i$.

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**20. (ii) Solution :** Given expression

$$\frac{(\cos\theta + i\sin\theta)^6}{(\sin\theta + i\cos\theta)^5} = \frac{(\cos\theta + i\sin\theta)^6}{i^5\left(\cos\theta + \frac{1}{i}\sin\theta\right)^5} = \frac{(\cos\theta + i\sin\theta)^6}{i(\cos\theta - i\sin\theta)^5} = \frac{(\cos\theta + i\sin\theta)^6.(\cos\theta + i\sin\theta)^5}{i\{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)\}^5} = \frac{(\cos\theta + i\sin\theta)^{11}}{i\left(\cos^2\theta + \sin^2\theta\right)^5}$$

$= -i(\cos 11\theta + i\sin 11\theta) = \sin 11\theta - i\cos 11\theta$ **(Ans)**

**20. (iv) Solution :** Given expression :

$$\left(\frac{1 + \cos\theta + i\sin\theta}{1 + \cos\theta - i\sin\theta}\right)^n = \left(\frac{2\cos^2\frac{\theta}{2} + i.2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} - i.2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right)^n = \left(\frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}\right)^n$$

$$\left[\frac{\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}\right]^n = \frac{\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{2n}}{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)^n} = \frac{\cos n\theta + i\sin n\theta}{1} = \cos n\theta + i\sin n\theta \textbf{ (Ans)}$$

**21. (i) Solution :** Let $a = r\cos\theta$, $b = r\sin\theta$.

$\therefore \frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \quad \therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)$ and $r^2 = a^2 + b^2 \therefore r = \left(a^2 + b^2\right)^{\frac{1}{2}}$

$\therefore$ L.H.S. $= (a + ib)^n + (a - ib)^n = (r\cos\theta + ir\sin\theta)^n + (r\cos\theta - ir\sin\theta)^n$

N.Tute

$= r^n[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n] = r^n[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta] = 2r^n \cdot \cos n\theta$

$= 2(a^2 + b^2)^{\frac{n}{2}} . \cos\left\{n. \tan^{-1}\left(\frac{b}{a}\right)\right\}$ **(Proved)**

**22. Solution :** we know, $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta \cdot i\sin\theta + 3\cos\theta \cdot i^2\sin^2\theta + i^3\sin^3\theta = \cos^3\theta + i3(1 - \sin^2\theta)\cdot\sin\theta - 3\cos\theta(1 - \cos^2\theta) - i\sin^3\theta$

$= (\cos^3\theta - 3\cos\theta + 3\cos^3\theta) + i(3\sin\theta - 3\sin^3\theta - \sin^3\theta)$

$\therefore \cos 3\theta + i\sin 3\theta = (4\cos^3\theta - 3\cos\theta) + i(3\sin\theta - 4\sin^3\theta)$

Equating real and imaginary parts from both sides we get,

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **(Ans)**

**23. (i) Solution :** Let $z = (-1)^{\frac{1}{3}} = (\cos\pi + i\sin\pi)^{\frac{1}{3}} = \{\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)\}^{\frac{1}{3}}$

$\therefore z = \cos\frac{(2n+1)\pi}{3} + i\sin\frac{(2n+1)\pi}{3}$, $n = 0, 1, 2$

For, $n = 0, z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i.\frac{\sqrt{3}}{2}$

For, $n = 1, z = \cos\pi + i\sin\pi = -1$

For, $n = 2, z = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right) = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\therefore (-1)^{\frac{1}{3}} = -1, \ \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ **(Ans)**

**23. (iv) Solution :** Let $z = (\sqrt{3} - i)^{\frac{1}{7}} = \left\{2\left(\frac{\sqrt{3}}{2} - i.\frac{1}{2}\right)\right\}^{\frac{1}{7}}$

or, $z = 2^{\frac{1}{7}}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{\frac{1}{7}} = 2^{\frac{1}{7}}\left\{\cos\left(2n\pi + \frac{\pi}{6}\right) - i\sin\left(2n\pi + \frac{\pi}{6}\right)\right\}^{\frac{1}{7}}$

$\therefore z = 2^{\frac{1}{7}}\left\{\cos\frac{1}{7}\left(2n + \frac{1}{6}\right)\pi - i\sin\frac{1}{7}\left(2n + \frac{1}{6}\right)\pi\right\}$ where, $n = 0, 1, 2, 3, 4, 5, 6$ **(Ans)**

**24. (iii) Solution :** $x^3 + 8 = 0$

or, $x^3 = 8 \times (-1) = 8 \times (\cos\pi + i\sin\pi)$

$\therefore x = 8^{\frac{1}{3}}(\cos\pi + i\sin\pi)^{\frac{1}{3}} = 2\{\cos(2n + 1)\pi + i\sin(2n + 1)\pi\}^{\frac{1}{3}}$

$= 2\left\{\cos\frac{(2n+1)\pi}{3} + i\sin\frac{(2n+1)\pi}{3}\right\}$, $n = 0, 1, 2$ **(Ans)**

**25. (ii) Solution :**

Let $z = (1 + i\sqrt{3})^{\frac{3}{4}} = \left\{2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right\}^{\frac{3}{4}} = 2^{\frac{3}{4}}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = 2^{\frac{3}{4}}\left\{\cos\left(2n\pi + \frac{\pi}{3}\right) + i\sin\left(2n\pi + \frac{\pi}{3}\right)\right\}^{\frac{3}{4}}$

$= 2^{\frac{3}{4}}\left[\cos\frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right) + i\sin\frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right)\right]$ $n = 0, 1, 2, 3.$

For, $n = 0, \ z = 2^{\frac{3}{4}}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = 2^{\frac{3}{4}}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

For, n = 1,

$$z = 2^{\frac{3}{4}}\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = 2^{\frac{3}{4}}\left[\cos\left(2\pi - \frac{\pi}{4}\right) + i\sin\left(2\pi - \frac{\pi}{4}\right)\right] = 2^{\frac{3}{4}}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] = 2^{\frac{3}{4}}\left(\frac{1}{\sqrt{2}} - i.\frac{1}{\sqrt{2}}\right)$$

For, n = 2,

$$z = 2^{\frac{3}{4}}\left[\cos\frac{13\pi}{4} + i\sin\frac{13\pi}{4}\right] = 2^{\frac{3}{4}}\left[\cos\left(3\pi + \frac{\pi}{4}\right) + i\sin\left(3\pi + \frac{\pi}{4}\right)\right] = 2^{\frac{3}{4}}\left[-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] = -2^{\frac{3}{4}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

For, n = 3,

$$z = 2^{\frac{3}{4}}\left[\cos\frac{19\pi}{4} + i\sin\frac{19\pi}{4}\right] = 2^{\frac{3}{4}}\left[\cos\left(5\pi - \frac{\pi}{4}\right) + i\sin\left(5\pi - \frac{\pi}{4}\right)\right] = 2^{\frac{3}{4}}\left[-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = -2^{\frac{3}{4}}\left(\frac{1}{\sqrt{2}} - i.\frac{1}{\sqrt{2}}\right)$$

Therefore, the product of all the four values

$$= \left\{2^{\frac{3}{4}}.\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)\right\} \times \left\{2^{\frac{3}{4}}.\left(\frac{1}{\sqrt{2}} - i.\frac{1}{\sqrt{2}}\right)\right\} \times \left\{-2^{\frac{3}{4}}.\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)\right\} \times \left\{-2^{\frac{3}{4}}.\left(\frac{1}{\sqrt{2}} - i.\frac{1}{\sqrt{2}}\right)\right\}$$

$$= 2^{4\times\frac{3}{4}}\left\{\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - i.\frac{1}{\sqrt{2}}\right)\right\}^2 = 2^3\left(\frac{1}{2} + \frac{1}{2}\right)^2 = 8 \times 1^2 = 8 \textbf{ (Proved)}$$

**26. Solution :** Given, $1 + \sqrt{1 - a^2} = na$

or, $\sqrt{1 - a^2} = na - 1$ or, $1 - a^2 = n^2a^2 + 1 - 2na$ or, $n^2a^2 + a^2 = 2na$ or, $n^2 + 1 = \frac{2n}{a}$ ................(1

Now, $\frac{a}{2n}(1 + nx)\left(1 + \frac{n}{x}\right) = \frac{a}{2n}\left\{1 + n^2 + n\left(x + \frac{1}{x}\right)\right\} = \frac{a}{2n}\left\{\left(1 + n^2\right) + n(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)\right\}$

$$\left[\because x = \cos\theta + i\sin\theta \ \therefore \frac{1}{x} = x^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta\right]$$

$= \frac{a}{2n}\left\{\left(n^2 + 1\right) + 2n.\cos\theta\right\} = \frac{a}{2n}\left\{\frac{2n}{a} + 2n\cos\theta\right\}$ [from (1)] $= 1 + a\cos\theta$

$\therefore 1 + a\cos\theta = \frac{a}{2n}(1 + nx)\left(1 + \frac{n}{x}\right)$ **(Proved)**

**28. Solution :** Given $2\cos\theta = x + \frac{1}{x}$, or, $2x\cos\theta = x^2 + 1$

or, $x^2 - 2x\cos\theta + 1 = 0$ $\therefore x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\times1\times1}}{2\times1}$ or, $x = \frac{2\cos\theta \pm 2\sqrt{-\left(1-\cos^2\theta\right)}}{2} = \cos\theta \pm i\sin\theta$

Taking +ve sign only in place of ±, we get $x = \cos\theta + i\sin\theta$,

Similarly, $y = \cos\phi + i\sin\phi$

(i) Now, $\frac{x}{y} + \frac{y}{x} = x\cdot y^{-1} + y\cdot x^{-1} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)^{-1} + (\cos\theta + i\sin\theta)^{-1}\cdot(\cos\phi + i\sin\phi)$

$= (\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi) + (\cos\theta - i\sin\theta)(\cos\phi + i\sin\phi)$ [By De Moivre's Theorem]

$= (\cos\theta\cdot\cos\phi + \sin\theta\cdot\sin\phi) + i(\sin\theta\cdot\cos\phi - \cos\theta\cdot\sin\phi) + (\cos\theta\cdot\cos\phi + \sin\theta\cdot\sin\phi) - i(\sin\theta\cdot\cos\phi - \cos\theta\cdot\sin\phi)$

$= \cos(\theta - \phi) + i\sin(\theta - \phi) + \cos(\theta - \phi) - i\sin(\theta - \phi) = 2\cos(\theta - \phi)$

$\therefore \frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \phi)$ **(Proved)**

(iv) Now $x^m \cdot y^n = (\cos\theta + i\sin\theta)^m \cdot (\cos\phi + i\sin\phi)^n$

$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$ [By De Moivre's Theorem] $= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$

$\therefore\ x^{-m} \cdot y^{-n} = [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$

$= \cos(m\theta + n\phi) - i\sin(mq + n\phi)$ [By De Moivre's Theorem]

$\therefore\ x^m y^n + x^{-m}y^{-n} = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$

or, $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$ **(Proved)**

**31. Solution :** Given $x + \dfrac{1}{x} = 2\cos\alpha$ or, $x^2 + 1 = 2x\cos\alpha$ or, $x^2 - 2x\cos\alpha + 1 = 0$

$\therefore\ x = \dfrac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4 \times 1 \times 1}}{2 \times 1}$ or, $x = \dfrac{2\cos\alpha \pm 2\sqrt{-1\left(1 - \cos^2\alpha\right)}}{2} = \cos\alpha \pm i\sin\alpha$

$\therefore\ x = \cos\alpha + i\sin\alpha$ [taking +ve sign in place of $\pm$]

Similarly, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$.

(i) Now, $x + y + z = 0 \Rightarrow x + y = -z$ or, $(x + y)^2 = (-z)^2 \Rightarrow x^2 + 2xy + y^2 = z^2$

$\Rightarrow x^2 + y^2 - z^2 = -2xy \Rightarrow (x^2 + y^2 - z^2)^2 = 4x^2y^2 \Rightarrow x^4 + y^4 + z^4 + 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = 4x^2y^2$

or, $x^4 + y^4 + z^4 = 2(x^2y^2 + y^2z^2 + z^2x^2)$

or, $(\cos\alpha + i\sin\alpha)^4 + (\cos\beta + i\sin\beta)^4 + (\cos\gamma + i\sin\gamma)^4$

$= 2[(\cos\alpha + i\sin\alpha)^2(\cos\beta + i\sin\beta)^2 + (\cos\beta + i\sin\beta)^2(\cos\gamma + i\sin\gamma)^2 + (\cos\gamma + i\sin\gamma)^2(\cos\alpha + i\sin\alpha)^2]$

or, $\cos 4\alpha + i\sin 4\alpha + \cos 4\beta + i\sin 4\beta + \cos 4\gamma + i\sin 4\gamma$

$= 2[(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta) + (\cos 2\beta + i\sin 2\beta)(\cos 2\gamma + i\sin 2\gamma) + (\cos 2\gamma + i\sin 2\gamma)(\cos 2\alpha + i\sin 2\alpha)]$

[By De Moivre's Theorem]

or, $\sum\cos 4\alpha + i\sum\sin 4\alpha$

$= 2[\cos 2(\alpha + \beta) + i\sin 2(\alpha + \beta) + \cos 2(\beta + \gamma) + i\sin 2(\beta + \gamma) + \cos 2(\gamma + \alpha) + i\sin 2(\gamma + \alpha)]$

$= 2\sum\cos 2(\beta + \gamma) + i2\sum\sin 2(\beta + \gamma)$

Equating real and imaginary parts from both sides we get,

$\sum\cos 4\alpha = 2\sum\cos 2(\beta + \gamma)$ and $\sum\sin 4\alpha = 2\sum\sin 2(\beta + \gamma)$ **(Proved)**

(iii) Given, $x + y + z = xyz$

or, $\cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$

$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma) = \cos\sum\alpha + i\sin\sum\alpha$

or, $\sum\cos\alpha + i\sum\sin\alpha = \cos\sum\alpha + i\sin\sum\alpha \Rightarrow \sum\cos\alpha = \cos\sum\alpha,\ \sum\sin\alpha = \sin\sum\alpha$ .......................... (1)

Now, $\sum \cos(\beta - \gamma) = \frac{1}{2}[\sum 2\cos\beta\cos\gamma + \sum 2\sin\beta\,\sin\gamma]$

$= \frac{1}{2}[\sum \cos^2\alpha + \sum \sin^2\alpha + \sum 2\cos\beta\,\cos\gamma + \sum 2\sin\beta\,\sin\gamma - 3]$

$= \frac{1}{2}[(\sum \cos^2\alpha + \sum 2\cos\beta\sin\gamma) + (\sum \sin^2\alpha + \sum 2\sin\beta\,\sin\gamma) - 3] = \frac{1}{2}[(\sum \cos\alpha)^2 + (\sum \sin\alpha)^2 - 3]$

$= \frac{1}{2}[\cos^2\sum\alpha + \sin^2\sum\alpha - 3]$ [by (1)] $= \frac{1}{2}[1 - 3] = \frac{1}{2}(-2) = -1$ **(Proved)**

**34. Solution :** Given equation is $x^2 - 2x + 4 = 0$

$$\therefore\ x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2 \times 1} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm i\sqrt{3} = 2\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$$

Let, $\alpha$, $\beta$ be the roots of the given equation.

$$\therefore\ \alpha = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right),\quad \beta = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$\therefore\ \alpha^n = 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$$

Similarly, $\beta^n = 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$

$\therefore$ Sum of the nth powers of the roots,

i.e., $\alpha^n + \beta^n = 2^n\left\{\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right\} = 2^n \cdot 2\cos\frac{n\pi}{3} = 2^{n+1}\cdot\cos\frac{n\pi}{3}$ **(Proved)**

and $\alpha^n \cdot \beta^n = (2^n)^2\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right) = 2^{2n}\left(\cos^2\frac{n\pi}{3} + \sin^2\frac{n\pi}{3}\right) = 2^{2n}$

Hence, the required equation is

$x^2 - (\alpha^n + \beta^n)x + \alpha^n\cdot\beta^n = 0$ or, $x^2 - 2^{n+1}\cdot\cos\frac{n\pi}{3}\cdot x + 2^{2n} = 0$ **(Ans)**

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# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. If x, y are real and $x + 3i$ and $-2 + iy$ are conjugate to each other then the value of x and y are: (a) $x = -2, y = -3$, (b) $x = 2, y = -3$, (c) $x = -2, y = 3$, (d) $x = 2, y = 3$. **[WBSC – 03]**
2. Modulus of $\frac{3+4i}{4-3i}$ is – (a) i (b) 1 (c) 7 (d) none of these. **[WBSC – 07]**
3. Mod of $\frac{8+i}{1+8i}$ is – (a) 1 (b) $\sqrt{2}$ (c) $\pm1$ (d) none of these. **[WBSC – 09]**
4. The modulus of $\frac{7-24i}{3+4i}$ is (a) 1 (b) 25 (c) 5 (d) none of these. **[WBSC – 11]**
5. The amplitude of $\sqrt{12}+6\left(\frac{1-i}{1+i}\right)$ is : (a) $\frac{\pi}{6}$, (b) $-\frac{\pi}{6}$, (c) $-\frac{\pi}{3}$, (d) none of these. **[WBSC – 04]**
6. Amplitude of the complex number $\frac{1}{1-i}$ is – (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\tan^{-1}2$. **[WBSC – 10]**
7. The value of $\sqrt{i}+\sqrt{-i}$, where $i=\sqrt{-1}$ is – (a) 2i (b) 0 (c) $\sqrt{2}$ (d) none is true. **[WBSC – 06, 08]**
8. If $i=\sqrt{-1}$, then the value of $\frac{1}{2}\left(\sqrt{i}+\sqrt{-i}\right)$ is equal to – (a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{-2}$ (d) $\frac{1}{\sqrt{-2}}$ **[WBSC – 07]**
9. A square root of $(3 + 4i)$ is – (a) $\sqrt{3}+i$, (b) $2 - i$, (c) $2 + i$, (d) none of these. **[WBSC – 05]**
10. The square root of $\frac{7-24i}{3+4i}$ are: (a) $\pm(1-i)$, (b) $\pm(1-2i)$, (c) $\pm(1+2i)$, (d) none of these. **[WBSC – 04]**
11. The smallest integer for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is : (a) n = 8, (b) n = 12, (c) n = 16, (d) none of these. **[WBSC – 05]**
12. The value of $\left(\frac{1+i}{1-i}\right)^4$ is – (a) –1 (b) 1 (c) i (d) –i **[WBSC – 10]**
13. If $i=\sqrt{-1}$, the value of $i^{197}$ is (a) i (b) –i (c) 1 (d) none of these. **[WBSC – 12]**
14. If 1, $\omega$, $\omega^2$ are the cube roots of unity, then the value of $3 + \omega^2 + \omega^4$ is – (a) 2 (b) 1 (c) 0 (d) 3 **[WBSC – 07]**
15. The value of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega)$ where $\omega$ is the complex cube root of unity is – (a) 2 (b) –4 (c) 4 (d) none of these. **[WBSC – 11]**
16. If $z=2+i\sqrt{3}$, then $z\bar{z}$ is – (a) 7 (b) 8 (c) $2-i\sqrt{3}$ (d) 13 **[WBSC – 09]**
17. If 1, $\omega$, $\omega^2$ are the cube roots of unity, then $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$ – (a) 4 (b) 8 (c) 16 (d) 32 **[WBSC – 09]**
18. 1, $\omega$, $\omega^2$ are cube roots of unity, then the value of $(1 + \omega)^3 - (1 + \omega^2)^3$ is – (a) 0 (b) 2 (c) –2 (d) $\omega$ **[WBSC – 08]**
19. If $z = 2 + 3i$, then the value of $z^3 - 5z^2 + 33z - 19$ is 6 – (a) True (b) False. **[WBSC – 08]**

## SUBJECTIVE TYPE

1. Find the value of $i^4 + i^5 + i^6 + i^7$, where $i = \sqrt{-1}$. [WBSC – 17]

2. If $\sqrt[3]{x+iy} = a+ib$ show that $\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$ [WBSC – 14, 17]

3. If $x+iy = \sqrt{\frac{a+ib}{c+id}}$, then show that $(x^2+y^2) = \frac{a^2+b^2}{c^2+d^2}$. [WBSC – 15]

4. If the complex number $z = x + iy$ is taken such that the amplitude of fraction $z-\frac{1}{z}+1$ is always $\frac{\pi}{4}$, prove that $x^2 + y^2 - 2y = 1$. [WBSC – 08]

5. If a, b are real and $a^2 + b^2 = 1$, then show that, the equation $\frac{1-ix}{1+ix} = a-ib$ is satisfied by a real value of x. [WBSC – 05]

6. If $\omega$ be an imaginary cube root of unity and $a + b + c = 0$, show that $(a + b\omega + c\,\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = 27abc$. [WBSC – 04]

7. If $x = a + b$, $y = a\omega^2 + b\omega$ and $z = a\omega + b\omega^2$, show that $x^3 + y^3 + z^3 = 3(a^3 + b^3)$ where $\omega$, $\omega^2$ are cube root of unity. [WBSC – 09]

8. If $x+\frac{1}{x} = 2\cos\theta$ and $y+\frac{1}{y} = 2\cos\phi$, using De Moiver's theorem, prove that $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta+n\phi)$ [WBSC – 03, 06, 08, 16, 18]

9. If $x+\frac{1}{x} = 2\cos\theta$ and $y+\frac{1}{y} = 2\cos\phi$, using De Moiver's theorem, prove that $x^3 y^4 + \frac{1}{x^3 y^4} = 2\cos(3\theta+4\phi)$ [WBSC – 19]

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Chapter 3

# QUADRATIC EQUATION

## 3.1 Introduction :

An equation of the form $ax^2 + bx + c = 0$, $(a \neq 0)$ is called a **quadratic equation** in the unknown quantity x. The values of x, which satisfy the equation, are called the **roots** of the quadratic equation. Here a, b and c are called the coefficient of $x^2$, coefficient of x and constant term respectively.

**Solution of Quadratic Equation $ax^2 + bx + c = 0$, $(a \neq 0)$**

$$ax^2 + bx + c = 0 \qquad \text{.....................} (1)$$

or, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0, \; a \neq 0$ or, $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$ or, $\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$

or, $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$ or, $\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = 0$

or, $(x - \alpha)(x - \beta) = 0 \qquad \text{---------------} (2)$

where, $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Equation (2) shows that, given equation satisfied only by $x = \alpha$, $x = \beta$ and hence it has only **two** roots.

## 3.2 Theorem (Statement only) :

Every quadratic equation has two and only two roots.

## 3.3 Relation between roots and coefficients :

Let $\alpha$ and $\beta$ be the roots of the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$

Then, $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Now $\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

and $\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{(-b)^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Therefore, sum of the roots $= -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

and product of the roots $= \frac{\text{constant term}}{\text{coefficient of } x^2}$

## 3.4 Formation of a Quadratic Equation whose roots are given :

Let $\alpha$ and $\beta$ be the roots of the quadratic equation $ax^2 + bx + c = 0$ then, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Now, $ax^2 + bx + c = 0$ or, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0,\ a \neq 0$ or, $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or, $x^2$ – **(sum of the roots)x + product of the roots = 0**

## 3.5 Theorem :

**(i) In a quadratic equation with real coefficients imaginary roots occur in conjugate pairs.**

i.e., if $\alpha + i\beta$ is a root of the quadratic equation $ax^2 + bx + c = 0$ then $\alpha - i\beta$ is also a root of the equation and vice-verse, where $\alpha$, $\beta$ are real and $i = \sqrt{-1}$.

**(ii) In a quadratic equation with real coefficients irrational roots occur in conjugate pairs.**

i.e., if $\alpha + \sqrt{\beta}$ is a root of the quadratic equation $ax^2 + bx + c = 0$ then $\alpha - \sqrt{\beta}$ is also a root of the equation.

## 3.6 Nature of the roots of a Quadratic Equation :

We consider the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ and let $\alpha$ and $\beta$ are the roots of it.

Then $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

The expression $(b^2 - 4ac)$ is called the **discriminant** of the quadratic equation, where a, b and c are **real** and **rational.**

| **a, b, c, (Rational or Irrational)** | **$D = b^2 - 4ac$ (Positive/Negative)** | **D (Perfect square or Not)** | **Nature of the roots** |
|---|---|---|---|
| a, b, c rational | $D = 0$ | | Real, Equal, Rational |
| a, b, c rational | $D < 0$ | | Imaginary, Different |
| a, b, c rational | $D > 0$ | Prefect square | Real, Unequal, Rational |
| a, b, c rational | $D > 0$ | Not perfect square | Real, Unequal, Irrational |
| Either a or b irrational | $D > 0$ | | Real, Unequal, Irrational |

### Note :

(i) For real roots $D \geq 0$

(ii) Quadratic equation with real coefficients cannot have one real and one imaginary roots; either both roots are real $(D > 0)$ or both roots are imaginary $(D < 0)$.

(iii) Quadratic equation with real coefficients cannot have one rational and one irrational roots; either both roots are rational (D is a perfect sqaure) or both roots are irrational (D is not a perfect sqaure)

## PROBLEM SET – I

[..... Problems with '*' marks are solved at the end of the problem set .....]

**1.** If $\alpha$, $\beta$ are the roots of the equation $ax^2 + bx + c = 0$, find the **value** of

*(i) $\frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$ (ii) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$ (iii) $\left(1 + \frac{\beta}{\alpha}\right)\left(1 + \frac{\alpha}{\beta}\right)$ (iv) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

**2.** If $\alpha$ and $\beta$ are the roots of $x^2 - px + q = 0$, find in terms of p and q, the **value** of

(i) $\alpha^2 + \alpha\beta + \beta^2$ (ii) $\alpha^4\beta^7 + \beta^4\alpha^7$ *(iii) $\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha}$ (iv) $\alpha^4 + \beta^4$

**3.** If $\alpha$ and $\beta$ are two roots of $x^2 + px + q = 0$, find in terms of p and q the **values** of the following :

(i) $\alpha^{-3} + \beta^{-3}$ (ii) $\alpha\beta^{-2} + \beta\alpha^{-2}$ (iii) $\alpha^3\beta^{-1} + \beta^3\alpha^{-1}$ *(iv) $\alpha^4 + \alpha^2\beta^2 + \beta^4$

**4.** *(i) If $\alpha$ and $\beta$ are two roots of the equation $x^2 + \alpha x + \beta = 0$, find the numerical **value** of $\alpha$ and $\beta$

(ii) If $\alpha$ and $\beta$ be the roots of the equation $x^2 + x + 1 = 0$, find the **value** of $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1}$.

(iii) If $\alpha$ and $\beta$ be the roots of the equation $5x^2 + 7x + 3 = 0$, find the **value** of $\frac{\alpha^3 + \beta^3}{\alpha^{-1} + \beta^{-1}}$.

**5.** *(i) If $\alpha$, $\beta$ be the roots of $3x^2 - 6x + 4 = 0$, find the **value** of $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta$. **[WBSC – 04]**

*(ii) If the roots of the equation $k^2x^2 + (kx + 1)(x + k) + 1 = 0$, $(k \neq 0, -1)$ are $\alpha$ and $\beta$, find the **value** of $\alpha^2\beta^2 + (\alpha\beta + 1)(\alpha + \beta) + 1$.

(iii) If $\alpha$, $\beta$ be the roots of the equation $ax^2 + bx + c = 0$, find the **value** of $\frac{a\alpha^2}{b\alpha + c} - \frac{a\beta^2}{b\beta + c}$.

(iv) If p and q are the roots of the equation $ax^2 + bx + c = 0$, find the **value** of $\frac{1}{(ap^2 + c)^2} + \frac{1}{(aq^2 + c)^2}$.

*(v) If $\alpha$, $\beta$ be the roots of the equation $px^2 + qx + r = 0$, find the value of $\frac{1}{(p\alpha + q)^3} + \frac{1}{(p\beta + q)^3}$.

**6.** *(i) If $\alpha$, $\beta$ be the roots of the equation $x^2 - ax + b = 0$ and $V_n = \alpha^n + \beta^n$, then prove that $V_{n+1} = aV_n - bV_{n-1}$. Hence evaluate $\alpha^3 + \beta^3$.

(ii) If $\alpha$, $\beta$ be the roots of the equation $ax^2 + bx + c = 0$ and $V_n = \alpha^n + \beta^n$, then prove that $aV_{n+1} + bV_n + cV_{n-1} = 0$. Hence **evaluate** $V_5$.

**7.** (i) If $\alpha$, $\beta$ and $\gamma$, $\delta$ be the roots of $x^2 - px + q = 0$ and $x^2 - rx + s = 0$ respectively, find the **value** of $(\alpha - \gamma)^2 + (\beta - \gamma)^2 + (\alpha - \delta)^2 + (\beta - \delta)^2$.

*(ii) If $\alpha$, $\beta$ and $\gamma$, $\delta$ be the roots of $x^2 + px + q = 0$ and $x^2 - rx + s = 0$ respectively, show that, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = (s - q)^2 + (r + p)(qr + ps)$. Deduce the condition that the equations have a **common root.**

*(iii) If $\alpha$, $\beta$ and $\gamma$, $\delta$ be the roots of $x^2 - bx + c = 0$ and $x^2 - px + q = 0$ respectively, show that, $(\alpha - \gamma)(\beta - \delta) + (\beta - \gamma)(\alpha - \delta) = 2(c + q) - bp$

(iv) If $\alpha$, $\beta$ and $\gamma$, $\delta$ be the roots of $x^2 + px - r = 0$ and $x^2 + px + r = 0$ respectively, show that, $(\alpha - \gamma)(\alpha - \delta) = (\beta - \gamma)(\beta - \delta) = 2r$.

*(v) If a, b are the roots of $x^2 + px + 1 = 0$ and c, d are the roots of $x^2 + qx + 1 = 0$, show that $q^2 - p^2 = (a - c)(b - c)(a + d)(b + d)$. **[WBSC – 92]**

**8.** (i) If the roots of $ax^2 + bx + c = 0$ be **reciprocals** to each other, show that $a = c$

(ii) Find the value of p so that the roots of the equation $3x^2 - 2(7 + 9p)x + (8 - 5p) = 0$ are **reciprocal** to one another.

*(iii) If the roots of $ax^2 + bx + c = 0$ are the reciprocals of those of $px^2 + qx + r = 0$, show that $a : b : c = r : q : p$. **[WBSC – 04]**

*(iv) If one root of $5x^2 + 13x + k = 0$ is reciprocals of the other, then find k. **[WBSC – 05]**

**9.** *(i) Show that the equations $(a - b)x^2 + (b - c)x + (c - a) = 0$ and $(b - c)x^2 + (c - a)x + (a - b) = 0$ have a **common root**.

(ii) If $a + b + c = 0$, show that the equation $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ have a **common root**.

(iii) Show that the equation $px^2 + qx + r = 0$ and $qx^2 + rx + p = 0$ will have a **common root** if $p + q + r = 0$ or, $p = q = r$.

*(iv) Show that the equation $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ will have a **common root** if $p + q + 1 = 0$ or, $p = q$. **[WBSC – 88]**

(v) If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$, have a **common root**, prove that, it is either $\frac{pq' - p'q}{q - q'}$ or, $\frac{q - q'}{p' - p}$.

**10.** (i) If roots of $ax^2 + x + b = 0$ are **equal**, show that, $(a + b)^2 = 1 + (a - b)^2$.

(ii) If one root of the equation $x^2 + bx + 8 = 0$ be 4 and the roots of the equation $x^2 + bx + c = 0$ are **equal**, find the value of c.

*(iii) Find the value of k, for which the roots of the equation $9x^2 - kx + 16 = 0$ are equal. **[WBSC – 03]**

(iv) For what value of m the roots of the equation $(m + 1)x^2 + 2(m + 3)x + m + 8 = 0$ are **equal** ?

(v) If the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ be **equal**, then prove that $2b = a + c$.

*(vi) If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are **equal** show that $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$.

(vii) If the roots of the equation $x^2 - 2(a + b)x + a(a + 2b + c) = 0$ be **equal**, prove that $b^2 = ac$.

(viii) Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + c^2 - ab = 0$ are **equal** if either $b = 0$ or, $a^3 + b^3 + c^3 - 3abc = 0$

*(ix) Prove that the equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ will have **equal** roots if $a=b=c$.

(x) Prove that if the roots of the equation $(a^2+b^2)x^2-2(ac+bd)x+(c^2+d^2)=0$ are **equal** then $ad=bc$.

[WBSC – 08]

**11.** (i) If one root of the equation $x^2-px+q=0$ be **twice** the other, prove that, $2p^2=9q$.

*(ii) If one root of the equation $x^2+(5a+2)x+(5a+2)=0$ is **five times** the other root, then find numerical value of a.

(iii) The sum of the roots of the equation $x^2-px+q=0$ be **three times** their difference, show that, $2p^2=9q$.

**12.** *(i) If the **ratio** of the roots of the equation $x^2-px+q=0$, be $1:2$, find the relation between p and q.

[WBSC – 07]

(ii) Show that the **ratio** r of one root of the equation $ax^2+bx+c=0$, to the other is given by the equation $acr^2+(2ac-b^2)r+ac=0$.

(iii) If the roots of the equation $ax^2+bx+c=0$ are in the **ratio** $3:4$, show that $12b^2=49ac$. Also if the **ratio** is $2:3$, show that $6b^2=25ac$.

*(iv) If the roots of the equation $px^2+rx+r=0$ are in the **ratio** $a:b$, prove that, $p(a+b)^2=rab$.

(v) The **ratio** of the roots of the equation $ax^2+bx+c=0$ is $r:1$. Prove that, $b^2r=ac(r+1)^2$ and hence find the condition so that the two roots may be **equal** to each other.

(vi) If the **ratio** of the roots of the equation $x^2-px+q=0$ be $a:b$, prove that, $p^2ab=q(a+b)^2$. Hence find the condition of **equal** roots of the given equation.

(vii) If the **ratio** of the roots of the equation $lx^2+nx+n=0$ be $p:q$ then prove that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$

(viii) If the **ratio** of the roots of the equation $ax^2+bx+c=0$ be $m:n$, then prove that $\frac{(m+n)^2}{mn}=\frac{b^2}{ac}$.

*(ix) If the roots of the equation $ax^2+bx+c=0$ are in the **ratio** $m:n$, show that, $\sqrt{\frac{m}{n}}+\sqrt{\frac{n}{m}}=\sqrt{\frac{b^2}{ac}}$

[WBSC – 90]

**13.** *(i) If the **ratio** of the roots of the equation $x^2-2px+q^2=0$ be equal to the **ratio** of the roots of the equation $x^2-2rx+s^2=0$, prove that, $p^2s^2=q^2r^2$. [WBSC – 97, 00]

(ii) If $\alpha$, $\beta$ and $\gamma$, $\delta$ be the roots of the equations $ax^2+bx+c=0$ and $px^2+qx+r=0$ respectively, prove that, $\frac{ac}{pr}=\frac{b^2}{q^2}$, if $\alpha\delta=\beta\gamma$.

**14.** *(i) If one root of the equation $ax^2 + bx + c = 0$ is the **square** of the other, prove that, $b^3 + ac^2 + a^2c = 3abc$.

[WBSC – 91, 03]

(ii) If one root of the equation $x^2 + rx - s = 0$ is the **square** of the other, prove that, $r^3 + s^2 + 3sr - s = 0$.

(iii) If one root of the equation $x^2 + px + q = 0$ be the **square** of the other, show that, $p^3 - q(3p - 1) + q^2 = 0$.

**15.** *(i) If $b^3 + a^2c + ac^2 = 3abc$, find the **relation** between the roots of the equation $ax^2 + bx + c = 0$. [WBSC – 93]

(ii) If $p^3 - q(3p - 1) + q^2 = 0$, find the **relation** between the roots of the equation $x^2 + px + q = 0$.

*(iii) If $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$, then prove that $\alpha^2 = 3\beta$, where $3b^3 + 9a^2c + ac^2 = 9abc$.

[WBSC – 06]

**16.** (i) Find m, given that the **difference** of the roots of the equation $2x^2 - 12x + m + 2 = 0$ is 2.

(ii) If the sum of the roots of $ax^2 + bx + c = 0$ be equal to the sum of their squares, then prove that $2ac = ab + b^2$.

**17.** (i) If the roots of the quadratics $x^2 + 2px + q = 0$ and $x^2 + 2qx + p = 0$, $(p \neq q)$ **differ by a constant**, show that, $p + q + 1 = 0$.

(ii) If the **difference** of the roots of $x^2 - px + q = 0$ be unity, show that $p^2 + 4q^2 = (1 + 2q)^2$.

*(iii) If the difference of the roots of the equation $x^2 - px + q = 0$ be the same as that of the equation $x^2 - qx + p = 0$, show that $p + q + 4 = 0$, unless $p = q$.

***18.** If $\alpha$ and $\beta$ be the roots of $ax^2 + 2bx + c = 0$ and $\alpha + \delta$, $\beta + \delta$ be those of $Ax^2 + 2Bx + C = 0$, prove that, $\frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2}$.

[WBSC – 03]

***19.** If the roots of the equation $ax^2 + bx + c = 0$ be of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$ show that, $(a + b + c)^2 = b^2 - 4ac$.

**20.** (i) If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, find the value of p and q.

[WBSC – 96, 00]

(ii) Find the quadratic equation with real coefficients which has $2 + i$ as a root, $i = \sqrt{-1}$.

(iii) Find the quadratic equation with rational coefficients which has $3 - \sqrt{2}$ as a root.

*(iv) Form the quadratic equation whose roots are **reciprocals** of the roots of $ax^2 + bx + c = 0$.

(v) Form the quadratic equation whose roots are **reciprocals** of the roots of $x^2 + 3x + 4 = 0$.

*(vi) Form the quadratic equation whose roots are the **squares** of the roots of $x^2 + 3x + 4 = 0$.

(vii) If $\alpha$ and $\beta$ be the roots of the equation $2x^2 + 6x + 3 = 0$, find the equation whose roots are $\alpha^2$ and $\beta^2$.

[WBSC – 07]

**21.** *(i) If the sum of the roots of a quadratic equation is 2 and the sum of their cubes is 27, find the equation.

(ii) Form the quadratic equation whose roots $\alpha$ and $\beta$ satisfy the relations $\alpha\beta = 768$ and $\alpha^2 + \beta^2 = 1600$.

**22.** (i) If the roots of the equation $ax^2 + bx + c = 0$ be $\alpha$ and $\beta$, find the equation whose roots are $2\alpha$ and $2\beta$.

(ii) If $\alpha$ and $\beta$ are the roots of the equation $x(2x - 1) = 1$, find the value of $\alpha^2 - \beta^2$ and form the equation whose roots are $2\alpha - 1$ and $2\beta - 1$.

*(iii) If $\alpha$, $\beta$ be the roots of the quadratic equation $x^2 - px + q = 0$, find the equation whose roots are $2\alpha - \beta$ and $2\beta - \alpha$.

**23.** (i) If the roots of the equation $ax^2 + bx + c = 0$ be $\alpha$ and $\beta$, find the equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$.

(ii) If $\alpha$, $\beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, find the quadratic equation whose roots are $\frac{1}{\alpha} + 1$ and $\frac{1}{\beta} + 1$.

*(iii) If $\alpha$ and $\beta$ be the roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha}{\beta} + 1$ and $\frac{\beta}{\alpha} + 1$.

(iv) If $\alpha$, $\beta$ be the roots of $ax^2 + bx + c = 0$, form an equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

*(v) If $\alpha$ and $\beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$. find the quadratic equation whose roots are $\frac{1}{\alpha + 1}$ and $\frac{1}{\beta + 1}$. **[WBSC – 07]**

**24.** (i) If $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\alpha^2$ and $\beta^2$.

(ii) If the roots of the equation $ax^2 - bx + a = 0$ be $\alpha$ and $\beta$ show that the equation with roots $\alpha^2 + 1$ and $\beta^2 + 1$ will be $a^2x^2 - b^2x + b^2 = 0$.

*(iii) If the roots of the equation $ax^2 + bx + c = 0$ be $\alpha$ and $\beta$, find the equation whose roots are $\alpha + \frac{\alpha^2}{\beta}$ and $\beta + \frac{\beta^2}{\alpha}$

**25.** (i) If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ where $p \neq q$, obtain the equation whose roots are $(3p - 2q)$ and $(3q - 2p)$.

*(ii) If $a^2 = 5a - 3$ and $b^2 = 5b - 3$, $(a \neq b)$, find the quadratic equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$. **[WBSC – 01]**

(iii) If $a^2 = 5a - 3$ and $b^2 = 5b - 3$, $(a \neq b)$, find the quadratic equation whose roots are $-\frac{a^2}{b}$ and $-\frac{b^2}{a}$.

**26.** (i) $-\frac{\alpha^2}{\beta}$ and $-\frac{\beta^2}{\alpha}$ are the roots of the equation $3x^2 - 18x + 2 = 0$. Find the equation whose roots are $\alpha$ and $\beta$($\alpha$ and $\beta$ are real).

(ii) If $\frac{p^2}{q}$ and $\frac{q^2}{p}$ are the roots of the equation $2x^2 + 7x - 4 = 0$, find the equation whose roots are p and q (p, q are real).

(iii) If $\alpha$ and $\beta$ be the roots of the equation $2x^2 + x + 1 = 0$, find the equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$.

*(iv) If $\alpha$ and $\beta$ be the roots of the equation $2x^2 + x - 1 = 0$, find the equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$. **[WBSC – 83]**

(v) If p and q are the roots of the equation $3x^2 + 6x + 2 = 0$, show that the equation whose roots are $-\frac{p^2}{q}$ and $-\frac{q^2}{p}$ is $3x^2 - 18x + 2 = 0$.

**27.** (i) If the roots of the equation $ax^2 + bx + c = 0$ be $\alpha$ and $\beta$, find the equation whose roots are $\frac{1}{a\alpha + b}$ and $\frac{1}{a\beta + b}$.

(ii) If the roots of the equation $ax^2 + bx + c = 0$ be $\alpha$ and $\beta$, find the equation whose roots are $(a\alpha + b)^{-3}$ and $(a\beta + b)^{-3}$.

**28.** (i) If $\alpha$, $\beta$ be the roots of the quadratic equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{1}{\alpha} + \frac{1}{\beta}$ and $\alpha\beta$.

(ii) If $\alpha$, $\beta$ be the roots of the equation $x^2 + px + q = 0$, show that $\frac{1}{\alpha + \beta}$ and $\frac{1}{\alpha} + \frac{1}{\beta}$ are the roots of $pqx^2 + (p^2 + q)x + p = 0$.

(iii) Find a quadratic equation whose roots are the **squars** of the sum and difference of the roots of $ax^2 + bx + c = 0$.

*(iv) If $\alpha$ and $\beta$ be the roots of the equation $x^2 + 3x + 4 = 0$, find the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$. **[WBSC – 05]**

(v) If p, q denote the roots of $2x^2 + 5x + 2 = 0$, find the equation whose roots are $p + q$ and $\frac{1}{2}pq$.

*(vi) If $\alpha$, $\beta$ be the roots of the equation $ax^2 + bx + c = 0$, show that $\frac{1}{\alpha + \beta}$ and $\frac{1}{\alpha} + \frac{1}{\beta}$ are the roots of $bcx^2 + (b^2 + ac)x + ab = 0$.

**29.** *(i) If the two equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$, $(a \neq b)$ have a **common root,** show that the other roots are the roots of the equation $x^2 + x + ab = 0$.

(ii) Prove that if the equation $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have only one nonzero **common root** then their other roots satisfy the equation $t^2 + at + bc = 0$

(iii) If the equation $x^2 + abx + c = 0$ and $x^2 + cax + b = 0$ have a **common root** then show that their other roots satisfy the equation $a(b + c)x^2 + (b + c)x - abc = 0$.

*(iv) If the equation $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have only one **common root** different from zero, then show that the reciprocal of their other roots will satisfy the equation $bcx^2 + ax + 1 = 0$.

**30.** (i) If one root of the quadratic equation $x^2 - x - 1 = 0$ is $\alpha$, prove that its other root is $\alpha^3 - 3\alpha$

*(ii) If $\alpha$ be a root of the quadratic equation $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root. **[WBSC – 02]**

(iii) If $\alpha$ be a root of the equation $3x^2 - 4x + 5 = 0$, prove that $2\alpha^2$ is a root of the equation $9x^2 + 28x + 100 = 0$.

(iv) If $\alpha$ be a root of the quadratic equation $ax^2 + bx + c = 0$ then show that $m\alpha^2$ $(m \neq 0)$ is a root of the equation $a^2x^2 + (2ac - b^2)mx + m^2c^2 = 0$

**31.** (i) Find the condition for which the quadratic equation $ax^2 + bx + c = 0$ has exactly one zero root.

(ii) For what value of m the roots of the equation $x^2 - 2(5 + 2m)x + 3(7 + 10m) = 0$, are (a) **equal** (b) **reciprocal** to one another (c) equal in **magnitude** and opposite in **signs** ?

(iii) Find the condition that the roots of the equation $\frac{a}{x - a} + \frac{b}{x - b} = 5$ may be equal in **magnitude** but opposite in **signs.**

*(iv) For what value of m will the equation $\frac{a}{x+a+m} + \frac{b}{x+b+m} = 1$ have roots equal in **magnitude** but opposite in **signs** ?

**[WBSC – 85]**

(v) For what value of m the roots of the equation $\frac{3}{x+3+m} + \frac{5}{x+5+m} = 1$ are equal in **magnitude** and opposite in **signs**.

**32.** Discuss the **nature of the roots** of the following quadratic equations :

(i) $x^2 - 5x + 2 = 0$ *(ii) $x^2 - 2\sqrt{3}x - 22 = 0$ (iii) $x^2 - 10x + 22 = 0$

*(iv) $4x^2 - 6x + 3 = 0$ (v) $\sqrt{5}x^2 - 6x - \sqrt{5} = 0$ (vi) $x^2 - 18x + 81 = 0$

*(vii) $2x^2 + 9x - 11 = 0$ (viii) $x^2 - 7x + 12 = 0$ *(ix) $4x^2 - 4x + 1 = 0$

(x) $2x^2 + \sqrt{17}x - 4 = 0$ *(xi) $x^2 - 2\sqrt{7}x - 2 = 0$ **[WBSC – 95]**

**33.** *(i) Show that the roots of $3x^2 + 4x - 7 = 0$ are **rational**

(ii) Show that roots of $63x^2 - 62x = 221$ are **rational**

**34.** Show that the roots of the equation are **rational** (a, b, c, are rational) :

(i) $(b + c)x^2 - (a + b + c)x + a = 0$

*(ii) $(a - b + c)x^2 + 2cx + (b + c - a) = 0$

**35.** (i) If p, q, r are **rational** and $p + q + r = 0$, prove that the roots of the quadratic equation $px^2 + qx + r = 0$ are **rational.**

(ii) Show that the roots of the equation $(m + n - l)x^2 + (n + l - m)x + (l + m - n) = 0$ are **rational if** l, m, n, are **rational** and $l + m + n = 0$, Also determine these roots.

*(iii) Prove that the roots of $(a + 2b - 3c)x^2 + (b + 2c - 3a)x + (c + 2a - 3b) = 0$ are **rational** if a, b, c are real and **rational.**

**36.** If a, b, c are **real,** show that the roots of each of the following equations are **real**

(i) $(x - a)(x - b) = b^2$ *(ii) $\frac{1}{x-a} + \frac{1}{x-1} + \frac{1}{x-2} = 0$

**37.** (i) If l, m, n are **real,** prove that the roots of the equation $(x - l)(x - m) + (x - m)(x - n) + (x - n)(x - l) = 0$ are always **real** and cannot be equal unless $l = m = n$.

*(ii) If a, b, c are **real,** prove that roots of the equation $\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = 0$ are always **real** and cannot be equal unless $a = b = c$.

(iii) If a, b, c are **real,** prove that the roots of the equation $\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c} = \frac{3}{x}$ are **real.**

*(iv) If p, q, r, s are **real** numbers and $pr = 2(q + s)$. Show that the roots of at least one of the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ are **real.**

**38.** (i) Prove that the values of x obtained from the equations $ax + by = 1$ and $ax^2 + by^2 = 1$ will be **equal** if $a + b = 1$.

*(ii) If the equations $ax + by = 1$ and $cx^2 + dy^2 = 1$ possesses only **one solution,** then prove that $\frac{a^2}{c}+\frac{b^2}{d}=1$ and $x = \frac{a}{c}$, $y = \frac{b}{d}$ **[WBSC – 87]**

(iii) Prove that, if the roots of the equation $(a^2 + b^2)x^2 + 2(bc + ad)x + (c^2 + d^2) = 0$ be real, then they are **equal.**

**39.** *(i) If the roots of the equation $qx^2 + 2px + 2q = 0$ are **real and unequal,** prove that the roots of the equation $(p + q)x^2 + 2qx + (p – q) = 0$ are imaginary

(ii) If the roots of the equation $x^2 + x + a = 0$ be **real and unequal,** then prove that the roots of the equation $2x^2 – 4(1 + a)x + 2a^2+3 = 0$ are **imaginary** (a is real).

(iii) If the roots of the equation $px^2 – 2qx + p = 0$ are **real and unequal** show that the roots of the equation $qx^2– 2px + q = 0$ are **imaginary** (both p and q are real).

**40.** If one root of the equation $qx^2 + px + q = 0$ (p, q are real) be **imaginary**, show that the roots of the equation $x^2– 4qx + p^2 = 0$ are **real and unequal.**

**41.** (i) If the roots of the equation $ax^2 + 4bx + 2a = 0$ are **imaginary,** prove that the roots of $(a + b)x^2 + 2bx + (a – b) = 0$ are **imaginary.**

**42.** (i) If x is real, prove that $5x^2 – 8x + 6$ is always **positive;** find the least value.

*(ii) If x be real then prove that $(x – 1)(x – 2) + 1$ is always **positive.**

(iii) If x is real, find the real value of p, which make $4x^2 + px + 1$ always **positive.**

*(iv) If x is real, find the real value of a which make $x^2 – ax + 1 – 2a^2$ always **positive.**

**43.** (i) For what values of x the expression $x^2 – 2x + 3$ is **negative** ?

*(ii) For what values of x the expression $2x^2 + 5x – 3$ is **negative**? What is its least value.

**44.** (i) If x is real, find the **maximum** value of $3 – 20x – 25x^2$.

*(ii) Find the **maximum** value of $(1 – x)(2 + 3x)$ for real values of x.

**45.** (i) If x is real, find the least value of $2x^2 – 3x + 5$ and the value of x for which the expression is **minimum.**

(ii) If x be real, show that the **least** value of $4x^2 – 4x + 1$ is zero and the corresponding value of x is $\frac{1}{2}$

***46.** Merriman's formula for the weight of roof Principle states $W = \frac{3}{4}ae\left(1+\frac{e}{10}\right)$.

Find the value of e to satisfy the equation if $a = 10$ and $W = 5400$.

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## ANSWERS

**1.** (i) $\frac{b}{ac}$ (ii) $\frac{b^2(b^2-4ac)}{a^2c^2}$ (iii) $\frac{b^2}{ac}$ (iv) $\frac{3abc-b^3}{c^3}$

**2.** (i) $p^2-q$ (ii) $pq^4(p^2-3q)$ (iii) $\frac{p^4-4p^2q+2q^2}{q}$ (iv) $p^4-4p^2q+2q^2$

**3.** (i) $\frac{3pq-p^3}{q^3}$ (ii) $\frac{3pq-p^3}{q^2}$ (iii) $\frac{p^4-4p^2q+2q^2}{q}$ (iv) $(p^2-q)(p^2-3q)$

**4.** (i) $1, -2$ (ii) 0 (iii) $\frac{12}{125}$ 5. (i) 8 (ii) 0 (iii) 0 (iv) $\frac{b^2-2ac}{b^2c^2}$ (v) $\frac{q^3-3pqr}{r^3p^3}$ **6.** (i) $a^3-3ab$ (ii) $\frac{b(5ab^2c-5a^2c^2-b^4)}{a^5}$

**7.** (i) $2(p^2+r^2-2q-2s-pr)$ **8.** (ii) $p=1$ (iv) $k=5$

**10.** (ii) $c=9$ (iii) $k=\pm 24$ (iv) $m=\frac{1}{3}$

**11.** (ii) $a=-\frac{2}{5}, \frac{26}{25}$ **12.** (i) $2p^2=9q$ (v) $b^2=4ac$ (vi) $p^2=4q$. **16.** (i) $m=14$

**20.** (i) $p=-4, q=7$(ii) $x^2-4x+5=0$ (iii) $x^2-6x+7=0$ (iv) $cx^2+bx+a=0$ (v) $4x^2+3x+1=0$
(vi) $y^2-y+16=0$ (vii) $4x^2-24x+9=0$

**21.** (i) $6x^2-12x-19=0$ (ii) $x^2\pm 56x+768=0$

**22.** (i) $ax^2+2bx+4c=0$ (ii) $\pm\frac{3}{4}$, $x^2+x-2=0$ (iii) $x^2-px+(9q-2p^2)=0$

**23.** (i) $acx^2+b(a+c)x+(a+c)^2=0$ (ii) $cx^2+(b-2c)x+(a-b+c)=0$ (iii) $qx^2-p^2x+p^2=0$
(iv) $acx^2-(b^2-2ac)x+ac=0$ **(v)** $(c-b+a)x^2-(2a-b)x+a=0$

**24.** (i) $a^2x^2-(b^2-2ac)x+c^2=0$ (iii) $a^2cx^2+b(b^2-2ac)x+b^2c=0$

**25.** (i) $3x^2-5x-100=0$ (ii) $3x^2-19x+3=0$ (iii) $3x^2+80x+9=0$

**26.** (i) $3x^2+6x+2=0$ (ii) $x^2-x-2=0$ (iii) $4x^2-5x+2=0$ (iv) $4x^2-7x-2=0$

**27.** (i) $acx^2-bx+1=0$ (ii) $a^3c^3x^2-(b^3-3abc)x+1=0$

**28.** (i) $qx^2-(p+q^2)x+pq=0$ (iii) $a^4x^2-2a^2(b^2-2ac)x+b^2(b^2-4ac)=0$ (iv) $x^2-2x-63=0$
(v) $4x^2+8x-5=0$

**31.** (i) $c=0$(ii) (a) $m=2, \frac{1}{2}$ (b) $m=-\frac{2}{3}$ (c) $m=-\frac{5}{2}$ (iii) $a+b=0$ (iv) $m=0$ (v) $m=0$

**32.** (i) Real, Irrational and Unequal (ii) Real, Irrational and Unequal (iii) Real, Irrational and Unequal (iv) Imaginary and Uneuqal (v) Real, Irrational and Unequal (vi) Real, Rational and Equal (vii) Real, Rational and Unequal (viii) Real, Rational and Unequal. (ix) Real, Rational and Equal (x) Real, Irrational and Unequal (xi) Real, Irrational and Unequal.

**35.**(ii) $1, \frac{n}{l}$

**42.** (i) Least value $=\frac{14}{5}$ (iii) $-4\le p\le 4$ (iv) $-\frac{2}{3}\le a\le\frac{2}{3}$

**43.** (i) Cannot be negative for any real value of x (ii) $-3<x<\frac{1}{2}$, Least Value $=-\frac{49}{8}$

**44.** (i) 7 (ii) $\frac{25}{12}$ **45.**(i) $\frac{31}{8}, \frac{3}{4}$ **46.** $e=80, -90$.

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## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1.** (i) **Solution :** Since $\alpha$, $\beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$

$\therefore a\alpha^2 + b\alpha + c = 0$ or, $\alpha(a\alpha + b) = -c$ or, $a\alpha + b = -\frac{c}{\alpha}$ --------- (1)

and $a\beta^2 + b\beta + c = 0$ or, $\beta(a\beta + b) = -c$ or, $a\beta + b = -\frac{c}{\beta}$ -------- (2)

$$\therefore \frac{1}{a\alpha+b} + \frac{1}{a\beta+b} = -\frac{\alpha}{c} - \frac{\beta}{c} = -\frac{\alpha+\beta}{c} = -\frac{-\frac{b}{a}}{c} \quad \left[\because \alpha,\ \beta \text{ are the roots of } ax^2+bx+c=0,\ \therefore \alpha+\beta = -\frac{b}{a}\right]$$

$= \frac{b}{ac}$ **(Ans)**

**Alternative Method :**

Since $\alpha$, $\beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$ $\therefore \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ --------- (1)

$$\therefore \frac{1}{a\alpha+b} + \frac{1}{a\beta+b} = \frac{a\beta+b+a\alpha+b}{(a\alpha+b)(a\beta+b)} = \frac{a(\alpha+\beta)+2b}{a^2\alpha\beta+ab\alpha+ab\beta+b^2} = \frac{a(\alpha+\beta)+2b}{a^2\alpha\beta+ab(\alpha+\beta)+b^2} = \frac{a\left(-\frac{b}{a}\right)+2b}{a^2\times\frac{c}{a}+ab\left(-\frac{b}{a}\right)+b^2} \text{ [from (1)]}$$

$= \frac{-b+2b}{ac-b^2+b^2} = \frac{b}{ac}$ **(Ans)**

**2.** (iii) **Solution :** Since $\alpha$, $\beta$ are the roots of the quadratic equation $x^2 - px + q = 0$ $\therefore \alpha + \beta = p$ and $\alpha\beta = q$------- (1)

$$\therefore \frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha} = \frac{\alpha^4+\beta^4}{\alpha\beta} = \frac{(\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2}{\alpha\beta} = \frac{\{(\alpha+\beta)^2-2\alpha\beta\}^2 - 2(\alpha\beta)^2}{\alpha\beta} = \frac{(p^2-2q)^2-2q^2}{q}$$

$$= \frac{p^4-4p^2q+4q^2-2q^2}{q} = \frac{p^4-4p^2q+2q^2}{q} \textbf{ (Ans)}$$

**3.** (iv) **Solution :** Since $\alpha$, $\beta$ are the roots of the quadratic equation $x^2 + px + q = 0$

$\therefore \alpha + \beta = -p \quad \alpha\beta = q$ ---------(1)

$\therefore \alpha^4 + \alpha^2\beta^2 + \beta^4 = \alpha^4 + 2\alpha^2\beta^2 + \beta^4 - \alpha^2\beta^2 = (\alpha^2 + \beta^2)^2 - (\alpha\beta)^2$

$= \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - (\alpha\beta)^2 = \{(-p)^2 - 2q\}^2 - q^2 = (p^2 - 2q)^2 - q^2 = p^4 - 4p^2q + 4q^2 - q^2 = p^4 - 4p^2q + 3q^2$

$= p^4 - 3p^2q - p^2q + 3q^2 = p^2(p^2 - 3q) - q(p^2 - 3q) = (p^2 - 3q)(p^2 - q)$ **(Ans)**

**4.** (i) **Solution :** Since $\alpha$, $\beta$ are the roots of the quadratic equation $x^2 + \alpha x + \beta = 0$

$\therefore \alpha + \beta = -\alpha$ ----------- (1) and $\alpha\beta = \beta$ ---------- (2)

From (2) we get, $\alpha = 1$ [$\because \beta \neq 0$] and from (1) we get, $\beta = -2\alpha = -2$

$\therefore \alpha = 1, \beta = -2$ **(Ans)**

**5.** (i **Solution :** ) Given $\alpha$ and $\beta$ are the roots of the equation $3x^2 - 6x + 4 = 0$.

Therefore $\alpha + \beta = -\frac{-6}{3} = 2$, $\alpha\beta = \frac{4}{3}$

Now, $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta = \frac{\alpha^2+\beta^2}{\alpha\beta} + 2\frac{\alpha+\beta}{\alpha\beta} + 3\alpha\beta$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} + 2\frac{\alpha+\beta}{\alpha\beta} + 3\alpha\beta = \frac{2^2 - 2\times\frac{4}{3}}{\frac{4}{3}} + 2\times\frac{2}{\frac{4}{3}} + 3\times\frac{4}{3} = \frac{12-8}{4} + 3 + 4 = 8 \textbf{ (Ans)}$$

5. (ii) **Solution :** Since $\alpha$, $\beta$ are the roots of the quadratic equation,

$k^2x^2 + (kx + 1)(x + k) + 1 = 0$ or, $k^2x^2 + kx^2 + x + k^2x + k + 1 = 0$ or, $k(k + 1)x^2 + (k^2 + 1)x + (k + 1) = 0$

$\therefore\ \alpha+\beta = -\frac{k^2+1}{k(k+1)}$ and $\alpha\beta = \frac{k+1}{k(k+1)} = \frac{1}{k}$ ---------- (1)

Now, $\alpha^2\beta^2 + (\alpha\beta + 1)(\alpha + \beta) + 1$

$= \frac{1}{k^2} + \left(\frac{1}{k}+1\right)\left\{-\frac{k^2+1}{k(k+1)}\right\}+1 = \frac{1}{k^2} + \left(\frac{k+1}{k}\right)\left\{-\frac{k^2+1}{k(k+1)}\right\}+1 = \frac{1}{k^2} - \frac{k^2+1}{k^2}+1 = \frac{1}{k^2}-1-\frac{1}{k^2}+1 = 0$ **(Ans)**

5. (v) **Solution :** Since $\alpha$, $\beta$ are the roots of the quadratic equation $px^2 + qx + r = 0$

$\therefore\ \alpha+\beta = -\frac{q}{p}$ and $\alpha\beta = \frac{r}{p}$ ---------- (1)

and $p\alpha^2 + q\alpha + r = 0$ or, $\alpha(p\alpha + q) = -r$ or, $p\alpha + q = -\frac{r}{\alpha}$ ---------- (2)

and, $p\beta^2 + q\beta + r = 0$ or, $\beta(p\beta + q) = -r$ or, $p\beta + q = -\frac{r}{\beta}$ ---------- (3)

Now, $\frac{1}{(p\alpha+q)^3} + \frac{1}{(p\beta+q)^3} = -\frac{\alpha^3}{r^3} - \frac{\beta^3}{r^3}$ [from (2) and (3)]

$= -\frac{\alpha^3+\beta^3}{r^3} = -\frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{r^3} = -\frac{\left(-\frac{q}{p}\right)^3 - 3\frac{r}{p}\left(-\frac{q}{p}\right)}{r^3}$ [from (1)] $= \frac{q^3 - 3pqr}{(pr)^3}$ **(Ans)**

6. (i) **Solution :** Since $\alpha$, $\beta$ are the roots of $x^2 - ax + b = 0$

$\therefore\ \alpha + \beta = a$ and $\alpha\beta = b$

Now, $(\alpha^n + \beta^n)(\alpha + \beta) = \alpha^{n+1} + \alpha^n\beta + \beta^n\alpha + \beta^{n+1} = (\alpha^{n+1} + \beta^{n+1}) + \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$\therefore\ V_n.a = V_{n+1} + bV_{n-1}$ $\therefore\ V_{n+1} = aV_n - bV_{n-1}$ **(Proved)** --------- (1)

Putting $n = 2$ we get, $V_3 = aV_2 - bV_1 = a\{aV_1 - bV_0\} - bV_1$ [from (1)] $= (a^2 - b)V_1 - abV_0$

$= (a^2 - b)(\alpha + \beta) - ab(1 + 1) = (a^2 - b)a - 2ab = a^3 - 3ab$ **(Ans)**

7. (ii) **Solution :** Since $\alpha$, $\beta$ are the roots of the quadratic equation $x^2 + px + q = 0$

$\therefore \alpha + \beta = -p, \quad \alpha\beta = q$ ---------(1)

Again $\gamma$, $\delta$ are the roots of the quadratic equation $x^2 - rx + s = 0$

$\therefore\ \gamma + \delta = r, \ \gamma\delta = s$ ------------(2)

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = \{\alpha^2 - (\gamma + \delta)\alpha + \gamma\delta\}\{\beta^2 - (\gamma + \delta)\beta + \gamma\delta\}$

$= (\alpha^2 - r\alpha + s)(\beta^2 - r\beta + s) = (-p\alpha - q - r\alpha + s)(-p\beta - q - r\beta + s)$

[$\because$ $\alpha$, $\beta$ are the roots of $x^2 + px + q = 0$ $\therefore\ \alpha^2 + p\alpha + q = 0$]

and $\beta^2 + p\beta + q = 0 \Rightarrow \alpha^2 = -p\alpha - q$ and $\beta^2 = -p\beta - q$]

$= \{(s - q) - (p + r)\alpha\}\{(s - q) - (p + r)\beta\} = (s - q)^2 - (p + r)(s - q)(\alpha + \beta) + (p + r)^2\alpha\beta$

$= (s - q)^2 + p(p + r)(s - q) + q(p + r)^2$ [from (1)]

$= (s - q)^2 + (p + r)(ps - pq + pq + qr) = (s - q)^2 + (p + r)(ps + qr)$ ---------- (3) **(Proved)**

**2nd part :** The quadratic equations have a common root, if

either $\alpha = \gamma$ or, $\alpha = \delta$ or, $\beta = \gamma$ or $\beta = \delta$ so, for a common root,

$(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$

$\therefore$ from (3) we get, $(s - q)^2 + (p + r)(ps + qr) = 0$, which is the required condition. **(Ans)**

**7.** (iii) **Solution :** $\because$ $\alpha, \beta$ and $\gamma, \delta$ are the roots of the equation $x^2 - bx + c = 0$ and $x^2 - px + q = 0$,

$\therefore \alpha + \beta = b, \alpha\beta = c \quad \gamma + \delta = p, \ \gamma\delta = q$ --------(1)

Now, $(\alpha - \gamma)(\beta - \delta) + (\beta - \gamma)(\alpha - \delta) = \alpha\beta - \beta\gamma - \delta\alpha + \gamma\delta + \alpha\beta - \alpha\gamma - \beta\delta + \gamma\delta$

$= 2(\alpha\beta + \gamma\delta) - \beta\gamma - \alpha\gamma - \alpha\delta - \beta\delta = 2(\alpha\beta + \gamma\delta) - (\beta + \alpha)\gamma - (\alpha + \beta)\delta$

$= 2(\alpha\beta + \gamma\delta) - (\alpha + \beta)(\gamma + \delta) = 2(c + q) - bp$ **(Proved)**

**7.** (v) **Solution :** Since a, b are the roots of the quadratic equation $x^2 + px + 1 = 0$

$\therefore a + b = -p, \ ab = 1$ ----------- (1)

Again since c, d are the roots of the quadratic equation $x^2 + qx + 1 = 0$

$\therefore c + d = -q, \ cd = 1$ ----------- (2)

Now, $(a - c)(b - c)(a + d)(b + d) = \{ab - (a + b)c + c^2\}\{ab + d(a + b) + d^2\}$

$= (1 + pc + c^2)(1 - pd + d^2) = (pc - qc)(-pd - dq)$

[$\because$ c, d are the roots of $x^2 + qx + 1 = 0$ $\therefore c^2 + qc + 1 = 0, d^2 + qd + 1 = 0 \Rightarrow 1 + c^2 = -qc$ and $1 + d^2 = -qd$]

$= (q - p)(q + p)cd = (q^2 - p^2).1$ [from (2)] $= q^2 - p^2$

$\therefore q^2 - p^2 = (a - c)(b - c)(a + d)(b + d)$ **(Proved)**

**8.** (iii) **Solution :** Replacing x by $\frac{1}{x}$ in the equation $px^2 + qx + r = 0$ we get,

$p \cdot \frac{1}{x^2} + q \cdot \frac{1}{x} + r = 0$ or, $p + qx + rx^2 = 0$ or, $rx^2 + qx + p = 0$

By the problem, roots of $ax^2 + bx + c = 0$ and $rx^2 + qx + p = 0$ are same

Hence, $\frac{a}{r} = \frac{b}{q} = \frac{c}{p}$ or, $a : b : c = r : q : p$ **(Proved)**

**8.** (iv) **Solution :** Given equation is $5x^2 + 13x + k = 0$

Let, $\alpha, \frac{1}{\alpha}$ be the roots. Then $\alpha \times \frac{1}{\alpha} = \frac{k}{5}$ or, $k = 5$ **(Ans)**

**9.** (i) **Solution :** $(a - b)x^2 + (b - c)x + (c - a) = 0$ or, $(a - b)x^2 - \{(a - b) + (c - a)\}x + (c - a) = 0$

or, $(a - b)x^2 - (a - b)x - (c - a)x + (c - a) = 0$

or, $(a - b)x\{x - 1\} - (c - a)(x - 1) = 0$ or, $(x - 1)\{(a - b)x - (c - a)\} = 0$

$\therefore$ either, $x - 1 = 0$ or, $(a - b)x - (c - a) = 0$ or, $x = 1$, or $x = \frac{c - a}{a - b}$ ------- (1)

Again, $(b - c)x^2 + (c - a)x + (a - b) = 0$ or, $(b - c)x^2 - \{(b - c) + (a - b)\}x + (a - b) = 0$

or, $(b - c)x^2 - (b - c)x - (a - b)x + (a - b) = 0$

or, $(b-c)x(x-1)-(a-b)(x-1)=0$ or, $(x-1)\{(b-c)x-(a-b)\}=0$

$\therefore$ either, $x-1=0$ or, $(b-c)x-(a-b)=0$ or, $x=1$ or, $x=\frac{a-b}{b-c}$ ------- (2)

From (1) and (2) we see that $x=1$ is the common root. Hence the result. **(Proved)**

**9.** (i) **Alternative Method :**

Given equations are $(a-b)x^2+(b-c)x+(c-a)=0$ ------------(1)

and $(b-c)x^2+(c-a)x+(a-b)=0$ ------------(2)

Putting $x=1$ in equation (1) we get,

L.H.S. $=(a-b).1+(b-c).1+(c-a)=a-b+b-c+c-a=0=$ R.H.S.

Putting $x=1$ in equation (2) we get,

L.H.S. $=(b-c).1+(c-a).1+(a-b).1=b-c+c-a+a-b=0=$ R.H.S.

Hence $x=1$ satisfies both equations (1) and (2). So, $x=1$ is the common root. **(Proved)**

**Alternative Method.** Given equations can be written as

$px^2+qx+r=0$ -------------- (1), $qx^2+rx+p=0$------------ (2) where, $p+q+r=0$ --------- (3)

Let $\alpha$ be the common root of equations (1) and (2).

$p\alpha^2+q\alpha+r=0$ and $q\alpha^2+r\alpha+p=0$

$\therefore$ by the rule of cross multiplication we get,

$$\frac{\alpha^2}{pq-r^2}=\frac{\alpha}{qr-p^2}=\frac{1}{pr-q^2}\quad\therefore\ \alpha^2=\frac{pq-r^2}{pr-q^2},\ \alpha=\frac{qr-p^2}{pr-q^2}\quad\therefore\ \frac{(qr-p^2)^2}{(pr-q^2)^2}=\frac{pq-r^2}{pr-q^2}$$

or, $(pq-r^2)(pr-q^2)=(qr-p^2)^2$ or, $(pq-r^2)(pr-q^2)-(qr-p^2)^2=0$

$\therefore$ for common root, $(pq-r^2)(pr-q^2)-(qr-p^2)^2=0$ where $p+q+r=0$

Now, L.H.S. $=(pq-r^2)(pr-q^2)-(qr-p^2)^2=p^2qr-pr^3-pq^3+q^2r^2-q^2r^2+2p^2qr-p^4$

$=3p^2qr-pr^3-pq^3-p^4=-p(p^3+q^3+r^3-3pqr)=-p\{(p+q+r)(p^2+q^2+r^2-pq-qr-rp)\}$

$=0$ [$\because$ $p+q+r=0$] $=$ R. H. S.

Hence the given two equations have an common root. **(Proved)**

**9.** (iv) **Solution :** Let $\alpha$ be the common root, $\therefore$ $\alpha^2+p\alpha+q=0$ and $\alpha^2+q\alpha+p=0$

$\therefore$ by the rule of cross-multiplication,

$$\frac{\alpha^2}{p^2-q^2}=\frac{\alpha}{q-p}=\frac{1}{q-p}\quad\therefore\ \alpha^2=\frac{p^2-q^2}{q-p},\ \alpha=1\quad\therefore\ \frac{p^2-q^2}{q-p}=1$$

or, $p^2-q^2=-(p-q)$

or, $(p+q)(p-q)+(p-q)=0$ or, $(p-q)(p+q+1)=0$

$\therefore$ either $p-q=0$ or $p+q+1=0$

$\therefore$ either $p=q$ or $p+q+1=0$ **(Proved)**

**10.** (iii) **Solution :** For equal roots of $9x^2 - kx + 16 = 0$, discriminant $(B^2 - 4AC) = 0$

or, $(-k)^2 - 4 \times 9 \times 16 = 0$ or, $k^2 = 24^2 \Rightarrow k = \pm 24$ **(Ans)**

**10.** (vi) **Solution :** he roots of the quadratic equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ will be equal,

if Discriminant $(B^2 - 4AC) = 0$

or, $\{b(c-a)\}^2 - 4a(b-c).\,c(a-b) = 0$ or, $b^2(c-a)^2 - 4ac(b-c)(a-b) = 0$

or, $b^2\{(c+a)^2 - 4ac\} - 4ac(b-c)(a-b) = 0$ or, $b^2(c+a)^2 - 4acb^2 - 4ac(ab - ac - b^2 + bc) = 0$

or, $b^2(c+a)^2 - 4ac(b^2 + ab - ac - b^2 + bc) = 0$ or, $\{b(c+a)\}^2 - 4ac(ab+bc) + 4a^2c^2 = 0$

or, $\{b(a+c)\}^2 - 2b(a+c).2ac + (2ac)^2 = 0$ or, $\{b(a+c) - 2ac\}^2 = 0$ or, $b(a+c) - 2ac = 0$

or, $b(a+c) = 2ac$ or, $\frac{a+c}{ac} = \frac{2}{b}$ or, $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ **(Proved)**

**10.** (ix) **Solution :** Given equation is $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$

or, $3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$

The root will be equal if, Discriminant $(B^2 - 4AC) = 0$

or, $\{-2(a+b+c)\}^2 - 4.3.(ab+bc+ca) = 0$ or, $4(a+b+c)^2 - 12(ab+bc+ca) = 0$

or, $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3ab - 3bc - 3ca = 0$

or, $a^2 + b^2 + c^2 - ab - bc - ca = 0$ or, $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

or, $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0 \Rightarrow a - b = 0,\ b - c = 0,\ c - a = 0$

$[\because a^2 + b^2 + c^2 = 0 \Rightarrow a = 0, b = 0, c = 0]$

or, $a = b, b = c, c = a \therefore a = b = c$ **(Proved)**

**11.** (ii) **Solution :** Let $\alpha$, $5\alpha$ be the two roots of the equation, $x^2 + (5a+2)x + (5a+2) = 0$

$\therefore \alpha + 5\alpha = -(5a+2)$ or, $6\alpha = -(5a+2)$ $\therefore \alpha = -\frac{5a+2}{6}$ ------- (1)

and, $\alpha.5\alpha = 5a + 2$ or, $5\alpha^2 = 5a + 2$ or, $5\left(-\frac{5a+2}{6}\right)^2 = 5a + 2$ [by (1)]

or, $5(5a+2)^2 - 36(5a+2) = 0$ or, $(5a+2)\{5(5a+2) - 36\} = 0$

or, $(5a+2)(25a - 26) = 0$

$\therefore$ either, $5a + 2 = 0 \Rightarrow a = -\frac{2}{5}$ or, $25a - 26 = 0 \Rightarrow a = \frac{26}{25}$

$\therefore a = -\frac{2}{5}, \frac{26}{25}$ **(Ans)**

**12.**(i) **Solution :** Let $\alpha$, $2\alpha$ be the roots of the equation $x^2 - px + q = 0$

Therefore, $\alpha + 2\alpha = p$ or, $3\alpha = p \Rightarrow \alpha = \frac{p}{3}$ .......................... (1)

and $\alpha \,.\, 2\alpha = q$ or, $2\alpha^2 = q$ or, $2\left(\frac{p}{3}\right)^2 = q$ or, $2p^2 = 9q$ **(Ans)**

**12.** (iv) **Solution :** Let $a\alpha$, $b\alpha$ be the roots of the equation $px^2 + rx + r = 0$

$\therefore a\alpha + b\alpha = -\frac{r}{p}$ or, $\alpha(a+b) = -\frac{r}{p}$ ------- (1) and $a\alpha \cdot b\alpha = \frac{r}{p}$ or, $ab\alpha^2 = \frac{r}{p}$ ------- (2)

From (1) we get, $\alpha^2(a+b)^2 = \left(-\frac{r}{p}\right)^2$ or, $\alpha^2(a+b)^2 = \frac{r^2}{p^2}$ ------- (3)

Dividing (3) by (2) we get, $\frac{\alpha^2(a+b)^2}{ab\alpha^2} = \frac{\frac{r^2}{p^2}}{\frac{r}{p}}$ or, $\frac{(a+b)^2}{ab} = \frac{r}{p}$ or, $p(a+b)^2 = rab$ **(Proved)**

**12.** **(ix) Solution :** Let $\alpha$ and $\beta$ be the roots of the equation $ax^2 + bx + c = 0$

$\therefore \alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$ ------- (1)

And by the given condition $\frac{m}{n} = \frac{\alpha}{\beta}$ $\therefore$ $\frac{n}{m} = \frac{\beta}{\alpha}$

$\therefore \sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = \frac{\sqrt{\alpha}}{\sqrt{\beta}} + \frac{\sqrt{\beta}}{\sqrt{\alpha}} = \frac{\alpha+\beta}{\sqrt{\alpha\beta}} = \sqrt{\frac{(\alpha+\beta)^2}{\alpha\beta}} = \sqrt{\frac{\left(-\frac{b}{a}\right)^2}{\frac{c}{a}}} = \sqrt{\frac{\frac{b^2}{a^2}}{\frac{c}{a}}} = \sqrt{\frac{b^2}{a^2} \times \frac{a}{c}} = \sqrt{\frac{b^2}{ac}}$ **(Proved)**

**13.** (i) **Solution :** Let $\alpha$, $\beta$ be the roots of $x^2 - 2px + q^2 = 0$ $\therefore$ $\alpha + \beta = 2p$, $\alpha\beta = q^2$ --------- (1)

Again, let $\gamma$, $\delta$ be the roots of $x^2 - 2rx + s^2 = 0$ $\therefore$ $\gamma + \delta = 2r$, $\gamma\delta = s^2$ ---- (2)

Now, by the problem $\frac{\alpha}{\beta} = \frac{\gamma}{\delta}$ -------- (3)

or, $\frac{\alpha}{\beta} + 1 = \frac{\gamma}{\delta} + 1$ or, $\frac{\alpha+\beta}{\beta} = \frac{\gamma+\delta}{\delta}$ -------- (4)

From (3) we get, $\frac{\beta}{\alpha} = \frac{\delta}{\gamma}$ or, $\frac{\beta}{\alpha} + 1 = \frac{\delta}{\gamma} + 1$ or, $\frac{\beta+\alpha}{\alpha} = \frac{\delta+\gamma}{\gamma}$ -------- (5)

Multiplying (4) and (5) we get, $\frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{(\gamma+\delta)^2}{\gamma\delta}$ or, $\frac{4p^2}{q^2} = \frac{4r^2}{s^2}$ [from (1) and (2)]

or, $p^2s^2 = q^2r^2$ **(Proved)**

**14. (i) Solution :** Let $\alpha$, $\alpha^2$ be the roots of the equation $ax^2 + bx + c = 0$

$\therefore \alpha + \alpha^2 = -\frac{b}{a}$ -------- (1), and $\alpha \cdot \alpha^2 = \frac{c}{a}$ or, $\alpha^3 = \frac{c}{a}$ -------- (2)

Cubing both sides of (1) we get,

$(\alpha + \alpha^2)^3 = \left(-\frac{b}{a}\right)^3$

or, $\alpha^3 + (\alpha^2)^3 + 3\alpha \cdot \alpha^2(\alpha + \alpha^2) = -\frac{b^3}{a^3}$

or, $\alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2) = -\frac{b^3}{a^3}$

or, $\frac{c}{a} + \frac{c^2}{a^2} + 3 \cdot \frac{c}{a} \cdot \left(-\frac{b}{a}\right) = -\frac{b^3}{a^3}$ or, $\frac{c}{a} + \frac{c^2}{a^2} - 3 \cdot \frac{bc}{a^2} + \frac{b^3}{a^3} = 0$

or, $a^2c + ac^2 - 3abc + b^3 = 0$ [Multiplying both sides by $a^3$]

or, $b^3 + a^2c + ac^2 = 3abc$ **(Proved)**

**15.** (i) **Solution :** Let $\alpha$, $\beta$ be the roots of the equation $ax^2 + bx + c = 0$

$\therefore\ \alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}$ -------- (1)

Now, $b^3 + a^2c + ac^2 = 3abc$ or, $\frac{b^3}{a^3}+\frac{a^2c}{a^3}+\frac{ac^2}{a^3}=\frac{3abc}{a^3}$ or, $\left(\frac{b}{a}\right)^3+\frac{c}{a}+\left(\frac{c}{a}\right)^2=3\cdot\frac{b}{a}\cdot\frac{c}{a}$

or, $(-\alpha-\beta)^3 + \alpha\beta + \alpha^2\beta^2 = 3(-\alpha-\beta).\alpha\beta$ or, $-(\alpha+\beta)^3 + \alpha\beta + \alpha^2\beta^2 = -3(\alpha+\beta).\alpha\beta$

or, $-\alpha^3 - \beta^3 - 3\alpha\beta(\alpha+\beta) + \alpha\beta + \alpha^2\beta^2 + 3\alpha\beta(\alpha+\beta) = 0$

or, $\alpha\beta - \alpha^3 - \beta^3 + \alpha^2\beta^2 = 0 \Rightarrow \alpha(\beta-\alpha^2) - \beta^2(\beta-\alpha^2) = 0$

or, $(\alpha-\beta^2)(\beta-\alpha^2) = 0$

$\therefore$ either, $\alpha-\beta^2 = 0\ \therefore\ \alpha = \beta^2$ or, $\beta-\alpha^2 = 0\ \therefore\ \beta = \alpha^2$

$\therefore$ one root of the given equation is square of the other. **(Proved)**

**15.** (iii) **Solution :** Given, $\alpha$, $\beta$ be the roots of the equation $ax^2 + bx + c = 0$

$\therefore\ \alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}$ -------- (1)

Now, $3b^3 + 9a^2c + ac^2 = 9abc$ or, $3\times\frac{b^3}{a^3}+9\times\frac{a^2c}{a^3}+\frac{ac^2}{a^3}=9\times\frac{abc}{a^3}$ or, $3\times\left(\frac{b}{a}\right)^3+9\times\frac{c}{a}+\left(\frac{c}{a}\right)^2=9\times\frac{b}{a}\times\frac{c}{a}$

or, $3(-\alpha-\beta)^3 + 9\alpha\beta + \alpha^2\beta^2 = 9(-\alpha-\beta).\alpha\beta$ or, $-3(\alpha+\beta)^3 + 9\alpha\beta + \alpha^2\beta^2 = -9(\alpha+\beta).\alpha\beta$

or, $-3\alpha^3 - 3\beta^3 - 9\alpha\beta(\alpha+\beta) + 9\alpha\beta + \alpha^2\beta^2 + 9\alpha\beta(\alpha+\beta) = 0$

or, $9\alpha\beta - 3\alpha^3 - 3\beta^3 + \alpha^2\beta^2 = 0 \Rightarrow 3\alpha(3\beta-\alpha^2) - \beta^2(3\beta-\alpha^2) = 0$

or, $(3\alpha-\beta^2)(3\beta-\alpha^2) = 0$

$\therefore$ either, $3\alpha-\beta^2 = 0\ \therefore\ 3\alpha = \beta^2$ or, $3\beta-\alpha^2 = 0\ \therefore\ 3\beta = \alpha^2$ or, $\alpha^2 = 3\beta$ **(Proved)**

**17.** (iii) **Solution :** Let $\alpha$, $\beta$ be the roots of the equation $x^2 - px + q = 0\ \therefore\ \alpha+\beta = p$ and $\alpha\beta = q$

Again let $\gamma$, $\delta$ are the roots of the equation $x^2 - qx + p = 0\ \therefore\ \gamma+\delta = q,\ \gamma\delta = p$

By the problem, $\alpha-\beta = \gamma-\delta$ or, $(\alpha-\beta)^2 = (\gamma-\delta)^2$ [squaring both sides]

or, $(\alpha+\beta)^2 - 4\alpha\beta = (\gamma+\delta)^2 - 4\gamma\delta$ or, $p^2 - 4q = q^2 - 4p$

or, $p^2 - q^2 + 4(p-q) = 0$ or, $(p-q)(p+q+4) = 0$

$\therefore\ p+q+4 = 0$ if $p \neq q$. Hence the result.

**18. Solution :** $\alpha$, $\beta$ are the roots of $ax^2 + 2bx + c = 0\ \therefore\ \alpha+\beta=-\frac{2b}{a},\ \alpha\beta=\frac{c}{a}$ ------ (1)

Again $\alpha+\delta$, $\beta+\delta$ are the roots of $Ax^2 + 2Bx + C = 0$

$\therefore\ (\alpha+\delta)+(\beta+\delta)=-\frac{2B}{A}$ and $(\alpha+\delta)(\beta+\delta)=\frac{C}{A}$ ------ (2)

From (1), $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = \left(-\frac{2b}{a}\right)^2-4\cdot\frac{c}{a}=\frac{4b^2}{a^2}-\frac{4c}{a}=\frac{4}{a^2}\left(b^2-ac\right)$ ------ (3)

From (2), $\{(\alpha+\delta)-(\beta+\delta)\}^2 = \{(\alpha+\delta)+(\beta+\delta)\}^2 - 4(\alpha+\delta)(\beta+\delta)$

$\therefore\ (\alpha-\beta)^2 = \left(-\frac{2B}{A}\right)^2-4\cdot\frac{C}{A}=\frac{4B^2}{A^2}-\frac{4C}{A}=\frac{4}{A^2}\left(B^2-AC\right)$ ----- (4)

From (3) and (4) we get, $\frac{4}{a^2}\left(b^2 - ac\right) = \frac{4}{A^2}\left(B^2 - AC\right)$

or, $\frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2}$ **(Proved)**

**19. Solution :** Since, the roots of the equation $ax^2 + bx + c = 0$ be of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$

$\therefore$ $\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a}$ ------ (1) and $\left(\frac{k+1}{k}\right)\left(\frac{k+2}{k+1}\right) = \frac{c}{a}$ or, $\frac{k+2}{k} = \frac{c}{a}$ or, $1 + \frac{2}{k} = \frac{c}{a}$

or, $\frac{2}{k} = \frac{c}{a} - 1 = \frac{c-a}{a}$ or, $k = \frac{2a}{c-a}$ ------- (2)

From (1) we get, $\frac{k+1}{k} + 1 + \frac{1}{k+1} = -\frac{b}{a}$ or, $\frac{k+1}{k} + \frac{1}{k+1} = \frac{-b-a}{a}$ or, $\frac{\frac{2a}{c-a}+1}{\frac{2a}{c-a}} + \frac{1}{\frac{2a}{c-a}+1} = -\frac{a+b}{a}$ or, $\frac{a+c}{2a} + \frac{c-a}{a+c} = -\frac{a+b}{a}$

or, $(a+c)^2 + 2a(c-a) = -\frac{a+b}{a}\cdot 2a(a+c)$ or, $(a + c)^2 + 2ac - 2a^2 = -2a(a + c) - 2b(a + c)$

or, $(a + c)^2 + 2(a + c).b + b^2 = b^2 - 2ac + 2a^2 - 2a^2 - 2ac$

or, $(a + c + b)^2 = b^2 - 4ac$ or, $(a + b + c)^2 = b^2 - 4ac$ **(Proved)**

**20.** (iv) **Solution :** Replacing x by $\frac{1}{x}$ in the equation $ax^2 + bx + c = 0$ we get,

$a\cdot\frac{1}{x^2} + b\cdot\frac{1}{x} + c = 0$ or, $cx^2 + bx + a = 0$ is the required equation. **(Ans)**

**20.** (vi) **Solution :** Given equation is $x^2 + 3x + 4 = 0$ or, $x^2 + 4 = -3x$ or, $(x^2 + 4)^2 = 9x^2$

Replacing $x^2$ by y we get, $(y + 4)^2 = 9y$

or, $y^2 + 8y + 16 = 9y$ or, $y^2 - y + 16 = 0$ is the required equation. **(Ans)**

**21.** (i) **Solution :** Let $\alpha$, $\beta$ be the roots of the equation, then $\alpha + \beta = 2$ --------(1)

and $\alpha^3 + \beta^3 = 27$ or, $(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = 27$ or, $2^3 - 3\alpha\beta \times 2 = 27$

or, $6\alpha\beta = 8 - 27 = -19$ or, $\alpha\beta = -\frac{19}{6}$

Therefore, the required equations is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ or, $x^2 - 2x - \frac{19}{6} = 0$ or, $6x^2 - 12x - 19 = 0$ **(Ans)**

**22.** (iii) **Solution :** $\because$ $\alpha$, $\beta$ are the roots of the equation $x^2 - px + q = 0$ $\therefore$ $\alpha + \beta = p$, $\alpha\beta = q$

Now, $(2\alpha - \beta) + (2\beta - \alpha) = 2(\alpha + \beta) - (\alpha + \beta) = \alpha + \beta = p$

and $(2\alpha - \beta)(2\beta - \alpha) = 4\alpha\beta - 2\beta^2 - 2\alpha^2 + \alpha\beta = 5\alpha\beta - 2(\alpha^2 + \beta^2) = 5\alpha\beta - 2\{(\alpha + \beta)^2 - 2\alpha\beta\}$

$= 5\alpha\beta - 2(\alpha + \beta)^2 + 4\alpha\beta = 9\alpha\beta - 2(\alpha + \beta)^2 = 9q - 2p^2$

$\therefore$ the required equation is, $x^2$ – (sum of the roots)x + product of the roots = 0

or, $x^2 - px + (9q - 2p^2) = 0$ **(Ans)**

**23.** (iii) **Solution :** $\because$ $\alpha$, $\beta$ are the roots of $x^2 - px + q = 0$ $\therefore$ $\alpha + \beta = p$, $\alpha\beta = q$

$\therefore \left(\frac{\alpha}{\beta} + 1\right) + \left(\frac{\beta}{\alpha} + 1\right) = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2 = \frac{\alpha^2 + \beta^2 + 2\alpha\beta}{\alpha\beta} = \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{p^2}{q}$

and $\left(\frac{\alpha}{\beta}+1\right)\left(\frac{\beta}{\alpha}+1\right) = 1+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+1 = \frac{\alpha^2+\beta^2+2\alpha\beta}{\alpha\beta} = \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{p^2}{q}$

$\therefore$ the required equation is $x^2$ – (sum of the roots)x + product of the roots = 0

or, $x^2-\frac{p^2}{q}x+\frac{p^2}{q}=0$ or, $qx^2-p^2x+p^2=0$ **(Ans)**

**23.**(v) **Solution :** $\because$ $\alpha$, $\beta$ are the roots of $ax^2+bx+c=0$ $\therefore$ $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$

$$\therefore \frac{1}{\alpha+1}+\frac{1}{\beta+1}=\frac{\beta+1+\alpha+1}{(\alpha+1)(\beta+1)}=\frac{\alpha+\beta+2}{\alpha\beta+\alpha+\beta+1}=\frac{-\frac{b}{a}+2}{\frac{c}{a}-\frac{b}{a}+1}=\frac{2a-b}{c-b+a}$$

and $\left(\frac{1}{\alpha+1}\right)\left(\frac{1}{\beta+1}\right)=\frac{1}{(\alpha+1)(\beta+1)}=\frac{1}{\alpha\beta+\alpha+\beta+1}=\frac{1}{\frac{c}{a}-\frac{b}{a}+1}=\frac{a}{c-b+a}$

$\therefore$ the required equation is, $x^2$ – (sum of the roots)x + product of the roots = 0

or, $x^2-\frac{2a-b}{c-b+a}x+\frac{a}{c-b+a}=0$ or, $(c-b+a)x^2-(2a-b)x+a=0$ **(Ans)**

**24.** (iii) **Solution :** $\because$ $\alpha$ , $\beta$ are the roots of the equation $ax^2+bx+c=0$ $\therefore$ $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$

$$\therefore \left(\alpha+\frac{\alpha^2}{\beta}\right)+\left(\beta+\frac{\beta^2}{\alpha}\right) = \frac{\alpha}{\beta}(\alpha+\beta)+\frac{\beta}{\alpha}(\alpha+\beta)=(\alpha+\beta)\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)=(\alpha+\beta)\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)$$

$$=(\alpha+\beta)\left\{\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\right\} = -\frac{b}{a}\left\{\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c}{a}}\right\} = -b\left\{\frac{b^2-2ac}{a^2c}\right\}$$

and $\left(\alpha+\frac{\alpha^2}{\beta}\right)\left(\beta+\frac{\beta^2}{\alpha}\right) = \frac{\alpha}{\beta}(\alpha+\beta)\times\frac{\beta}{\alpha}(\alpha+\beta)=(\alpha+\beta)^2=\frac{b^2}{a^2}$

Therefore, the required equation is, $x^2$ – (sum of the roots)x + product of the roots = 0

or, $x^2+\frac{b(b^2-2ac)}{a^2c}x+\frac{b^2}{a^2}=0$ or, $a^2cx^2+b(b^2-2ac)x+b^2c=0$ **(Ans)**

**25.** **(ii) Solution :** $a^2=5a-3$ ----------- (1), $b^2=5b-3$ ---------- (2)

$\therefore$ $a^2-b^2=5(a-b)$ [ subtracting (2) from (1)] or, $a+b=5$ --------- (3) ($\because$ $a\neq b$)

Again adding (1) and (2) we get,

$a^2+b^2=5(a+b)-6$ or, $(a+b)^2-2ab=5(a+b)-6$

or, $5^2-2ab=5.5-6$ or, $25-2ab=25-6$ or, $2ab=6$ or, $ab=3$

$\therefore$ equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$ is

$x^2-\left(\frac{a}{b}+\frac{b}{a}\right)x+\frac{a}{b}\times\frac{b}{a}=0$ or, $x^2-\left(\frac{a^2+b^2}{ab}\right)x+1=0$

or, $x^2-\frac{(a+b)^2-2ab}{ab}x+1=0$ or, $x^2-\frac{5^2-2\times3}{3}x+1=0$

or, $3x^2-19x+3=0$ **(Ans)**

**26. (iv) Solution :** Since $\alpha$, $\beta$ are the roots of the equation $2x^2 + x - 1 = 0$

$\therefore\ \alpha+\beta=-\frac{1}{2}$ and $\alpha\beta=-\frac{1}{2}$

$$\therefore\ \frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{\left(-\frac{1}{2}\right)^3-3\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{-\frac{1}{2}}=\frac{-\frac{1}{8}-\frac{3}{4}}{-\frac{1}{2}}=\frac{\frac{7}{8}}{\frac{1}{2}}=\frac{7}{4}$$

and $\frac{\alpha^2}{\beta}\times\frac{\beta^2}{\alpha}=\alpha\beta=-\frac{1}{2}$

$\therefore$ the required equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$ is

$$x^2-\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)x+\frac{\alpha^2}{\beta}\times\frac{\beta^2}{\alpha}=0 \text{ or, } x^2-\frac{7}{4}x-\frac{1}{2}=0 \text{ or, } 4x^2-7x-2=0 \textbf{ (Ans)}$$

**28.(iv) Solution :** Given $\alpha$ and $\beta$ are the roots of the equation $x^2 + 3x + 4 = 0$.

Therefore $\alpha+\beta=-3,\ \alpha\beta=4$

Sum of the roots $(\alpha+\beta)^2+(\alpha-\beta)^2=2(\alpha^2+\beta^2)=2\{(\alpha+\beta)^2-2\alpha\beta\}=2\{(-3)^2-2\times4\}=2$

Product of the roots $(\alpha+\beta)^2(\alpha-\beta)^2$

$=(\alpha+\beta)^2\{(\alpha+\beta)^2-4\alpha\beta\}=(-3)^2\times\{(-3)^2-4\times4\}=9\times\{9-16\}=-63$

Hence the required equation is, $x^2$ – (Sum of the roots )x + Product of the roots. = 0

or, $x^2-2x-63=0$ **(Ans)**

**28. (vi) Solution :** $\because$ $\alpha$, $\beta$ are the roots of the equation, $ax^2 + bx + c = 0$

$\therefore\ \alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}$

Now, $\frac{1}{\alpha+\beta}+\frac{1}{\alpha}+\frac{1}{\beta}=\frac{1}{\alpha+\beta}+\frac{\alpha+\beta}{\alpha\beta}=-\frac{a}{b}-\frac{\frac{b}{a}}{\frac{c}{a}}=-\frac{a}{b}-\frac{b}{c}=-\left(\frac{ac+b^2}{bc}\right)$

and $\frac{1}{\alpha+\beta}\times\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=\frac{1}{\alpha+\beta}\times\frac{\alpha+\beta}{\alpha\beta}=\frac{1}{\alpha\beta}=\frac{a}{c}$

Therefore, the required equation is. $x^2$ – (sum of the roots) x + product of the roots = 0

or, $x^2+\frac{ac+b^2}{bc}x+\frac{a}{c}=0$ or, $bcx^2+(ac+b^2)x+ab=0$ **(Ans)**

**29. (i) Solution :** Let $\alpha$, $\beta$ and $\alpha$, $\gamma$ be respectively the roots of the equations

$x^2 + ax + b = 0$ and $x^2 + bx + a = 0$.

Then $\alpha+\beta=-a,\ \alpha+\gamma=-b$ and $\alpha\beta=b,\ \alpha\gamma=a$ ------------- (1)

Here $\alpha$ is the common root.

$\alpha^2+a\alpha+b=0$ ------------ (2), $\alpha^2+b\alpha+a=0$ ------------ (3)

Subtracting $(a-b)\alpha=a-b$ or, $\alpha=1$ [$\because$ $a\neq b$]

Putting $\alpha=1$ in (2) we get, $1+a+b=0$ ---------(4)

From (1) we get, $\alpha+\beta+\alpha+\gamma=-a-b$

or, $\beta + \gamma = -(a+b) - 2\alpha = 1 - 2 = -1$ [from (4) $a + b = -1$ and $\alpha = 1$]

and $\alpha\beta.\alpha\gamma = ab$ or, $\alpha^2\beta\gamma = ab$ or, $\beta\gamma = ab$ [$\because \alpha = 1$]

Therefore, the required equation whose roots are, $\beta, \gamma$ is

$x^2 - (\beta+\gamma)x + \beta\gamma = 0$ or, $x^2 - (-1)x + ab = 0$ or, $x^2 + x + ab = 0$ **(Proved)**

**29.** (iv) **Solution :** Let $\alpha, \beta$ and $\alpha, \gamma$ be the roots of the equaiton

$x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ respectively.

$\therefore \alpha + \beta = -b, \alpha\beta = ca, \alpha + \gamma = -c, \alpha\gamma = ab$ ------ (1)

Here, $\alpha$ is the common root. $\therefore \alpha^2 + b\alpha + ca = 0$ ----------- (2) and $\alpha^2 + c\alpha + ab = 0$ ----------- (3)

Subtracting we get, $\alpha(b-c) = a(b-c) \Rightarrow \alpha = a$ [$\because b \neq c$]

Putting $\alpha = a$ in (2) we get, $a^2 + ab + ac = 0 \Rightarrow a + b + c = 0$ ------ (4) [$\because a \neq 0$]

Now, $\alpha + \beta + \alpha + \gamma = -b - c$ [from (1)] or, $\beta + \gamma = -2\alpha - (b+c) = -2a - (-a)$ [from (4)] $= -a$

and $\alpha\beta.\alpha\gamma = ca.ab$ or, $\alpha^2\beta\gamma = a^2bc$ or, $a^2\beta\gamma = a^2bc$ $\therefore \beta\gamma = bc$

$\therefore$ the required equation is, $x^2 - \left(\frac{1}{\beta} + \frac{1}{\gamma}\right)x + \frac{1}{\beta}\cdot\frac{1}{\gamma} = 0$ or, $\beta\gamma x^2 - (\beta+\gamma)x + 1 = 0$ or, $bcx^2 + ax + 1 = 0$ **(Proved)**

**30.** (ii) **Solution :** Let $\beta$ be the other root of the given equation $4x^2 + 2x - 1 = 0$.

$\therefore \alpha + \beta = -\frac{2}{4} = -\frac{1}{2}$ or, $\beta = -\alpha - \frac{1}{2}$ ................ (1)

Since $\alpha$ be a root of $4x^2 + 2x - 1 = 0$ $\therefore 4\alpha^2 + 2\alpha - 1 = 0$ or, $4\alpha^2 = 1 - 2\alpha$ -------- (2)

Now, $4\alpha^3 - 3\alpha = \alpha(4\alpha^2) - 3\alpha = \alpha(1-2\alpha) - 3\alpha = \alpha - 2\alpha^2 - 3\alpha = -2\alpha^2 - 2\alpha$

$= -\frac{1}{2}\left(4\alpha^2\right) - 2\alpha = -\frac{1}{2}(1-2\alpha) - 2\alpha = -\frac{1}{2} + \alpha - 2\alpha = -\frac{1}{2} - \alpha = \beta$ [from (1)],

which shows that $4\alpha^3 - 3\alpha$ is the other root. **(Proved)**

**31.** (iv) **Solution :** Given equation is, $\frac{a}{x+a+m} + \frac{b}{x+b+m} = 1$ or, $\frac{a(x+b+m)+b(x+a+m)}{(x+a+m)(x+b+m)} = 1$

or, $(x+a+m)(x+b+m) = a(x+b+m) + b(x+a+m)$

or, $x^2 + bx + mx + ax + ab + am + mx + mb + m^2 = ax + ab + am + bx + ab + bm$

or, $x^2 + 2mx + m^2 = ab$ or, $x^2 + 2mx + (m^2 - ab) = 0$

By the problem, let $\alpha, -\alpha$ be the roots of this equation.

$\therefore \alpha - \alpha = -\frac{2m}{1}$ or, $0 = -2m$ $\therefore m = 0$ **(Ans)**

**32.** (ii) **Solution :** Given equation is $x^2 - 2\sqrt{3}x - 22 = 0$

Discriminant $(B^2 - 4AC) = \left(-2\sqrt{3}\right)^2 - 4\cdot1\cdot(-22) = 12 + 88 = 100 = (10)^2$

$\therefore$ the discriminant is a perfect square, but $b = -2\sqrt{3}$ is an irrational number.

Hence the roots are real, irrational and unequal. **(Ans)**

**32.** (iv) **Solution :** Given equation is $4x^2 - 6x + 3 = 0$

Discriminant $(B^2 - 4AC) = (-6)^2 - 4 \times 4 \times 3 = 36 - 48 = -12 < 0$

Therefore, the roots are imaginary and unequal. **(Ans)**

**32.** (vii) **Solution :** Given equation is $2x^2 + 9x - 11 = 0$

Discriminant $(B^2 - 4AC) = (9)^2 - 4 \times 2\,(-11) = 81 + 88 = 169 = (13)^2$

$\therefore$ the discriminant is a perfect square and the coefficients $a = 2$, $b = 9$ and $c = -11$ are rational rumbers.

Hence, the roots are, real, rational and unequal. **(Ans)**

**32.** (ix) **Solution :** Given equation is $4x^2 - 4x + 1 = 0$

Discriminant $(B^2 - 4AC) = (-4)^2 - 4 \times 4 \times 1 = 16 - 16 = 0$

Hence the roots are real, equal and rational [$\because$ $a = 4$, $b = -4$, $c = 1$ are rational] **(Ans)**

**32.** (xi) **Solution :** Given equation is $x^2 - 2\sqrt{7}\,x - 2 = 0$

Discriminant $(B^2 - 4AC) = \left(-2\sqrt{7}\right)^2 - 4 \cdot 1 \cdot (-2) = 28 + 8 = 36 = 6^2$

$\therefore$ the discriminant is a perfect square. But $b = -2\sqrt{7}$ = an irrational number,

Hence, the roots are real, irrational and unequal. **(Ans)**

**33.** (i) **Solution :** Given equation is $3x^2 + 4x - 7 = 0$

Here discriminant $(b^2 - 4ac) = 4^2 - 4 \times 3\,(-7) = 16 + 84 = 100 = (10)^2$, a perfect square.

Also, each of the coefficients of $x^2$ and x are rational,

Therefore, two roots are rational. **(Ans)**

**34.** (ii) **Solution :** Given equation is, $(a - b + c)x^2 + 2cx + (b + c - a) = 0$

Here, discriminant $(B^2 - 4AC) = (2c)^2 - 4.(a - b + c)(b + c - a) = 4\{c^2 - (c + a - b)(c - a + b)\}$

$= 4\{c^2 - c^2 + (a - b)^2\} = 4(a - b)^2 = (2a - 2b)^2$, a perfect square,

Again, a, b, c are rational, so each of the coefficients of $x^2$ and x are rational. So the roots are rational. **(Ans)**

**35.** (iii) **Solution :** Given equation is, $(a + 2b - 3c)x^2 + (b + 2c - 3a)x + (c + 2a - 3b) = 0$

or, $px^2 + qx + r = 0$ ---------- (1)

where, $p + q + r = a + 2b - 3c + b + 2c - 3a + c + 2a - 3b = 0$ ---------- (2)

Here, discriminant $(B^2 - 4AC)$ of equation (1)

$= q^2 - 4pr = (-p - r)^2 - 4pr$ [from (2)] $= (p + r)^2 - 4pr = (p - r)^2$, a perfect square,

Again, a, b, c are rational, so each of the coefficients of $x^2$ and x are rational.

Hence the roots of the given equation are rational. **(Ans)**

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**36.** (ii) **Solution :** Given equation, $\frac{1}{x-a}+\frac{1}{x-1}+\frac{1}{x-2}=0$ --------- (1)

Let $\alpha + i\beta$ be a root of equation (1), where $\alpha$, $\beta$ are both real.

Then $\alpha - i\beta$ is also a root.

$$\therefore \frac{1}{\alpha+i\beta-a}+\frac{1}{\alpha+i\beta-1}+\frac{1}{\alpha+i\beta-2}=0 \text{ or, } \frac{1}{\alpha-a+i\beta}+\frac{1}{\alpha-1+i\beta}+\frac{1}{\alpha-2+i\beta}=0 \quad \text{--------- (2)}$$

Similarly, for the root $\alpha - i\beta$, $\frac{1}{\alpha-a-i\beta}+\frac{1}{\alpha-1-i\beta}+\frac{1}{\alpha-2-i\beta}=0$ --------- (3)

From (2) and (3) we get [(3) – (2)]

$$\left(\frac{1}{\alpha-a-i\beta}-\frac{1}{\alpha-a+i\beta}\right)+\left(\frac{1}{\alpha-1-i\beta}-\frac{1}{\alpha-1+i\beta}\right)+\left(\frac{1}{\alpha-2-i\beta}-\frac{1}{\alpha-2+i\beta}\right)=0$$

$$\text{or, } \frac{2i\beta}{(\alpha-a)^2+\beta^2}+\frac{2i\beta}{(\alpha-1)^2+\beta^2}+\frac{2i\beta}{(\alpha-2)^2+\beta^2}=0 \text{ or, } \beta\left[\frac{1}{(\alpha-a)^2+\beta^2}+\frac{1}{(\alpha-1)^2+\beta^2}+\frac{1}{(\alpha-2)^2+\beta^2}\right]=0$$

$$\therefore \beta = 0 \quad \left[\because \frac{1}{(\alpha-a)^2+\beta^2}+\frac{1}{(\alpha-1)^2+\beta^2}+\frac{1}{(\alpha-2)^2+\beta^2}\neq 0\right]$$

which shows that imaginary part of $\alpha + i\beta$ is zero.

Hence, the roots are real. (**Proved**)

**37.** (ii) **Hints :** $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=0$

or, $(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0$ or, $3x^2-2(a+b+c)x+(bc+ca+ab)=0$

Discriminant $= \{-2(a+b+c)\}^2 - 4.3\,(bc+ca+ab)$

$= 4\{(a+b+c)^2-3(bc+ca+ab)\} = 4\{a^2+b^2+c^2-ab-bc-ca\}$

$= 2\{(a-b)^2+(b-c)^2+(c-a)^2\} \geq 0$

$\Rightarrow$ Roots are real. Discriminant = 0 if a = b = c. Hence the result.

**37.** (iv) **Solution :** Let the roots of both the equations be not real.

Therefore, discriminant of both the equations are negative and so their sum will be negative.

Discriminant of $x^2+px+q=0$ is $p^2-4q$

Discriminant of $x^2+rx+s=0$ is $r^2-4s$

Sum of the discriminants $= p^2-4q+r^2-4s<0$

or, $p^2+r^2-4(q+s)<0$ or, $p^2+r^2-2pr<0$ [$\because$ $pr=2(q+s)$]

or, $(p-r)^2<0$, which is not possible since p, q are real.

Hence, the roots of one of them must be real. (**Proved**)

**38.** (ii) **Hints :** $ax+by=1$ or, $y=\frac{1-ax}{b}$ --------(1)

and $cx^2+dy^2=1$ or, $cx^2+d\left(\frac{1-ax}{b}\right)^2=1$ or, $b^2cx^2+d(1-ax)^2=b^2$

or, $(b^2c+a^2d)x^2-2adx-(b^2-d)=0$ --------(2)

$\therefore$ given set of equations possesses only one solution, if the roots of equation (2) are equal.

$\therefore$ discriminant of (2) = 0

or, $(-2ad)^2 + 4(b^2c + a^2d)(b^2 - d) = 0$ or, $a^2d^2 + b^4c + a^2b^2d - b^2cd - a^2d^2 = 0$

or, $b^4c + a^2b^2d = b^2cd$ or, $\frac{a^2}{c} + \frac{b^2}{d} = 1$ **(Proved)**

or, $b^2c + a^2d = cd$ -------- (3)

From (2), $2x = \frac{2ad}{b^2c + a^2d}$ [$\because$ roots are equal, discriminant = 0]

$= \frac{2ad}{cd}$ [from (3)] or, $x = \frac{a}{c}$ and sum of the roots $= -\frac{b}{a}$

From (1), $y = \frac{1 - a.\frac{a}{c}}{b} = \frac{1 - \frac{a^2}{c}}{b} = \frac{\frac{b^2}{d}}{b}$ [from (3)] or, $y = \frac{b}{d}$

$\therefore$ $x = \frac{a}{c}$, $y = \frac{b}{d}$ **(Proved)**

**39.** (i) **Solution :** Since the roots of the quadratic equation $qx^2 + 2px + 2q = 0$ are real and unequal, therefore, discriminant $(B^2 - 4AC) > 0$ or, $(2p)^2 - 4.q.2q > 0$ or, $4p^2 - 8q^2 > 0$ or, $p^2 - 2q^2 > 0$ or, $2q^2 - p^2 < 0$----- (1)

Now, the discriminant $(B^2 - 4AC)$ of the quadratic equation,

$(p + q)x^2 + 2qx + (p - q) = 0$ --------- (2)

is $(2q)^2 - 4(p + q)(p - q) = 4\{q^2 - p^2 + q^2\} = 4(2q^2 - p^2) < 0$ [ by (1)]

Hence the roots of the equations (2) are imaginary. **(Proved)**

**42.** (ii) **Solution :** Given expression, $(x - 1)(x - 2) + 1 = x^2 - 3x + 2 + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 3 = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4}$

Since, x is real, $\therefore$ $\left(x - \frac{3}{2}\right)^2$ is always positive and so, $\left(x - \frac{3}{2}\right)^2 \geq 0$

$\therefore$ $\left(x - \frac{3}{2}\right)^2 + \frac{3}{4}$ also always positive.

Hence, given expression is always positive, for real x. **(Proved)**

**42.** (iv) **Solution :** Given, $x^2 - ax + 1 - 2a^2 = \left(x - \frac{a}{2}\right)^2 + 1 - 2a^2 - \frac{a^2}{4} = \left(x - \frac{a}{2}\right)^2 + \frac{4 - 8a^2 - a^2}{4} = \left(x - \frac{a}{2}\right)^2 + \frac{4 - 9a^2}{4}$

Since, $\left(x - \frac{a}{2}\right)^2$ is always positive for real x, therefore, given expression will be positive

if $\frac{4 - 9a^2}{4} \geq 0$ or, $9a^2 \leq 4$ or, $a^2 \leq \frac{4}{9}$ or, $-\frac{2}{3} \leq a \leq \frac{2}{3}$ **(Ans)**

**43.** (ii) **Solution :** Given expression, $2x^2 + 5x - 3 = 2\left\{x^2 + \frac{5}{2}x\right\} - 3 = 2\left\{\left(x + \frac{5}{4}\right)^2 - \frac{25}{16}\right\} - 3$

$= 2\left(x + \frac{5}{4}\right)^2 - 3 - \frac{25}{8} = 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$ ------- (1)

The given expression will be negative

if $2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8} < 0$ or, $\left(x + \frac{5}{4}\right)^2 < \frac{49}{16}$ or, $-\frac{7}{4} < x + \frac{5}{4} < \frac{7}{4}$

or, $-\frac{7}{4} - \frac{5}{4} < x < \frac{7}{4} - \frac{5}{4}$ or, $-3 < x < \frac{1}{2}$ **(Ans)**

Now, since $\left(x + \frac{5}{4}\right)^2 \geq 0$ $\therefore$ least value of $\left(x + \frac{5}{4}\right)^2 = 0$

Hence from (1), least value of the given expression $= -\frac{49}{8}$ **(Ans)**

**44.** (ii) **Solution :** Given expression, $(1-x)(2+3x) = 2-2x+3x-3x^2$

$$= 2+x-3x^2 = -3\left\{x^2-\frac{1}{3}x\right\}+2 = -3\left\{\left(x-\frac{1}{6}\right)^2-\frac{1}{36}\right\}+2 = -3\left(x-\frac{1}{6}\right)^2+\frac{1}{12}+2 = \frac{25}{12}-3\left(x-\frac{1}{6}\right)^2$$

For real x, $\left(x-\frac{1}{6}\right)^2$ is always positive.

$\therefore$ minimum value of $3\left(x-\frac{1}{6}\right)^2 = 0$.

Hence maximum value of the given expression $= \frac{25}{12}$ **(Ans)**

**46. Solution :** Given, $a = 10$, $W = 5400$

$\therefore$ $W = \frac{3}{4}ae\left(1+\frac{e}{10}\right)$ or, $5400 = \frac{3}{4}\times 10e\left(1+\frac{e}{10}\right)$ or, $\frac{3}{4}e(10+e) = 5400$

or, $3e(e+10) = 21600$ or, $3e^2+30e-21600 = 0$

or, $e^2+10e-7200 = 0$ or, $e^2+90e-80e-7200 = 0$

or, $e(e+90)-80(e+90) = 0$ or, $(e+90)(e-80) = 0$ or, $(e+90)(e-80) = 0$

$\therefore$ either, $e+90 = 0$ $\therefore$ $e = -90$ or, $e-80 = 0$ $\therefore$ $e = 80$ $\therefore$ $e = 80, -90$ **(Ans)**

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N. Tute

# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. The value of k, for which the roots of the quadratic equation $9x^2 - kx + 16 = 0$ are equal, are
   (a) $\pm 24$, (b) $\pm 16$, (c) $\pm 12$, (d) none of these. [WBSC – 03, 10]
2. If one root of the equation $2x^2 - 3x + c = 0$ is 1, then other root is –
   (a) $-1$, (b) 2, (c) $\frac{1}{2}$ (d) none of these. [WBSC – 10]
3. If one root of $5x^2 + 13x + k = 0$ is reciprocals of the other, then k is –
   (i) 0, (b) 5, (c) $\frac{1}{6}$, (d) 6. [WBSC – 05, 06, 11]
4. Write down the quadratic equation whose roots are $3+\sqrt{2}$ and $3-\sqrt{2}$. [WBSC – 11]
5. If the roots of $ax^2 + bx + c = 0$ are reciprocals of those of $px^2 + qx + r = 0$, then :
   (a) $a : b : c = r : p : q$ (b) $a : b : c = p : q : r$ (c) $a : b : c = r : q : p$ (d) $b : a : c = r : q : p$
   [WBSC – 04]
6. If $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$, then the equation whose roots are $\alpha^{-1}$ and $\beta^{-1}$ is –
   (a) $ax^2 + bx + c = 0$ (b) $bx^2 + cx + a = 0$ (c) $cx^2 + bx + a = 0$ (d) none of these. [WBSC – 07]
7. If x be real, then the least value of $x^2 - x + 1$ is – (a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) 0 (d) none of these. [WBSC – 07]
8. If x be real, then the least value of $4x^2 - 4x + 1$ is –
   (a) 0 (b) $-1$ (c) 1 (d) none of these. [WBSC – 06]
9. If x is real, then the least value of $f(x) = 4x^2 - 4x + 2$ is – (a) 0, (b) $-1$, (c) 1, (d) None [WBSC – 10]
10. The maximum value of $5 + 4x - 4x^2$ is – (a) 5 (b) 8 (c) 6 (d) none of these. [WBSC – 09, 12]
11. If x is real the least value of $f(x) = 4x^2 - 4x + 1$ is – (a) 0 (b) $\frac{1}{2}$ (c) $-1$ (d) 1 [WBSC – 11]
12. If the sum of the roots of $ax^2 + 2x + 3a = 0$ is equal to their product, then a is –
    (a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $-\frac{3}{2}$ (d) $-\frac{2}{3}$ [WBSC – 09, 12]
13. The product of roots of the equation $3x^2 - 5mx + (m^2 + 5) = 0$ is 2 then 'm' is –
    (a) $\pm\sqrt{2}$ (b) $\pm i$ (c) $\pm 1$ (d) none of these. [WBSC – 10]
14. The roots of the equation $x+\frac{1}{x}=2$ are – (a) $-1$ (b) 1, $-1$ (c) $-1$, $-1$ (d) $-1$, $-5$ [WBSC – 08]
15. If the roots of the equation $x^2 - px + q = 0$ be 1 : 2 then the relation between p and q is –
    (a) $2p^2 = 9q$ (b) $p^2 = q^2$ (c) $2p^2 = 3q^2$ (d) none of these. [WBSC – 07, 08]
16. If roots of the equation $x^2 - px + q = 0$ are equal then find a relation between p and q. [WBSC – 12]

## SUBJECTIVE TYPE

1. If $\alpha$ and $\beta$ be the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of $\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)+2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+3\alpha\beta$. **[WBSC – 04, 11]**

2. If $\alpha$ and $\beta$ be the roots of the eqation $3x^2 - 6x + 3 = 0$, then find the value of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+2\alpha\beta+3(\frac{1}{\alpha}+\frac{1}{\beta})+1$ **[WBSC – 17]**

3. If $\alpha$ and $\beta$ be the roots of the equation $3x^2 - 6x + 4 = 0$ find the value of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+3\alpha\beta$ **[WBSC – 08]**

4. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + (c^2 + d^2) = 0$ are equal then prove that $ad = bc$. **[WBSC – 08, 12]**

5. If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then show that $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$ **[WBSC – 10, 16]**

6. If the roots of the equation $ax^2 + 2bx + c = 0$ be $\alpha$ and $\beta$ and the roots of the equation $Ax^2 + 2Bx + C = 0$ be $\alpha + \delta$ and $\beta + \delta$, prove that $\frac{b^2 - ac}{B^2 - AC} = \left(\frac{a}{A}\right)^2$ **[WBSC – 03, 09]**

7. If one root of the equation $ax^2 + bx + c = 0$ is the square of the other, show that $b^3 + ac^2 + a^2c = 3abc$. **[WBSC – 03]**

8. If one root of the equation $x^2 + px + q = 0$ is the square of the other, prove that $p^3 - q(3p - 1) + q^2 = 0$ **[WBSC – 14]**

9. If $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$, then prove that $\alpha^2 = 3\beta$ where $3b^3 + 9a^2c + ac^2 = 9abc$. **[WBSC – 06]**

10. If $\alpha$ and $\beta$ be the roots of the quadratic equation $2x^2 + 6x + 3 = 0$, find the equation whose roots are $\alpha^2$ and $\beta^2$. **[WBSC – 07]**

11. If $\alpha$ and $\beta$ be the roots of the quadratic equation $4x^2 + 6x + 3 = 0$. find the equation whose roots are $\alpha^2$ and $\beta^2$ **[WBSC – 09]**

12. If $\alpha$ and $\beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$. find the quadratic equation whose roots are $\frac{1}{\alpha+1}$ and $\frac{1}{\beta+1}$ **[WBSC – 07, 16]**

13. If $\alpha$ and $\beta$ be the roots of the equation $x^2 + 3x + 4 = 0$, find the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$. **[WBSC – 05]**

14. If $\alpha$ is a root of $4x^2 + 2x - 1=0$, prove that $4\alpha^3 - 3\alpha$ is the other root. **[WBSC – 18, 19]**

Chapter 4

# BINOMIAL THEOREM

## 4.1 Factorial :

The product of the natural numbers from 1, 2, 3,........etc. upto n is denoted by the symbol n! (read as **factorial** n).

Thus $2! = 2.1 = 2$, $3! = 3.2.1 = 6$, $4! = 4.3.2.1 = 24$ etc.

**Note :** $0! = 1$

## 4.2 Permutation :

Each of the different **arguments,** which can be made by taking **some** or **all** of a given number of things is called a permutations. Taking two letters a and b together, **two** arguments ab and ba can be made. So the number of permutations in this case is 2. Similarly, for three letters a, b, c clearly, six different arguments can be made by taking two at a time, namely ab, ba, bc, cb, ca, ac; each of these arguments is a permutation. So, the number of permutations in this case is 6. Again taking all three letters a, b, c at a time we have 6 permutations abc, acb, bca, bac, cab, cba.

The number of permutations of n things taken r at a time is denoted by the symbol ${}^nP_r$ $(r \le n)$

It can be proved that,

$${}^nP_r = n(n-1)(n-2)\text{..............................} \text{ to r factors.} = n(n-1)(n-2)\text{..........................}(n-r+1)$$

$$= \frac{n(n-1)(n-1).......(n-r+1)(n-r)(n-r-1).......3.2.1}{(n-r)(n-r-1).......3.2.1} = \frac{n!}{(n-r)!}$$

## 4.3 Combination :

Each of the different **selections** (or **group**) which can be formed by taking **some** or **all** of a given number of things (without regard to the **order** of the things in each group) is called a combination.

Taking two letters a and b together, **only one** selection ab can be made. So the number of combinations in this case is **1.**

Similarly, for three letters a, b, c clearly, three different selections can be made by taking two at a time, namely ab, bc, ca and each of these selections is a combination. So, the number of combinations in this case is **3.**

Again taking all three letters a, b, c at a time we have **1** combination abc.

The number of combinations of n things taken r at a time is denoted by the symbol ${}^nC_r$ $(r \le n)$

It can be proved that, $${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)(n-3).......(n-r+1)}{r(r-1)(r-2)(r-3).......3.2.1}$$

## 4.4 Important Results :

(i) ${}^{n}P_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$. (ii) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

**4.5 Binomial expression :** It consist of two terms. a + b, 2a + 3b, $4x^3 - \frac{1}{2}y^4$ etc. are examples of Binomial expression. **Similarly,** a + b + c, $3x^2 - 4y^2 + 9z$ are examples of **Trinomials** etc.

You know that, $(a + b)^2 = a^2 + 2ab + b^2$, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Here, $a^2 + 2ab + b^2$ and $a^3 + 3a^2b + 3ab^2 + b^3$ are the expansions of $(a + b)^2$ and $(a + b)^3$ respectively which we can easily be obtained by direct multiplication.

Now, what will be the expansion of $(a + b)^8$, or $(2a + 3b)^{99}$ or $\left(4x^3 - \frac{1}{2}y^4\right)^{\frac{1}{2}}$ or $(p + 2q)^{-4}$ etc. Expansions of any power of a binomial expression are obtained in the form of a series (finite or infinite) by the use of a general formula is known as the **Binomial Theorem.**

## 4.6 Binomial Theorem for a Positive Integral Index :

**Statement : –**

If n be a positive integer, then

$$(a + x)^n = a^n + {}^{n}C_1a^{n-1}x + {}^{n}C_2a^{n-2}x^2 + {}^{n}C_3a^{n-3}x^3 + \ldots\ldots + {}^{n}C_ra^{n-r}x^r + \ldots\ldots + x^n \quad \text{--------- (1)}$$

$$= a^n + n.a^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots\ldots$$

$$\ldots\ldots + \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots\ldots + x^n \text{ for all values of a and x} \quad \text{----------- (2)}$$

## Expansion of $(1 + x)^n$ :

Put a = 1 in the formuale (1) and (2) we obtain, for a positive integer n,

$$(1 + x)^n = 1 + {}^{n}C_1x + {}^{n}C_2x^2 + {}^{n}C_3x^3 + \ldots\ldots + {}^{n}C_rx^r + \ldots\ldots + x^n \quad \text{---------- (3)}$$

$$= 1 + n.x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots + \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)}{r!}x^r + \ldots\ldots + x^n \quad \text{------- (4)}$$

for all values of x.

## Expansion of $(a - x)^n$ and $(1 - x)^n$ :

Replacing x by –x in the expansion of $(a + x)^n$ we obtain, for a postitive integer n,

$$(a - x)^n = a^n - {}^{n}C_1a^{n-1}x + {}^{n}C_2a^{n-2}x^2 - {}^{n}C_3a^{n-3}x^3 + \ldots\ldots + (-1)^r\,{}^{n}C_ra^{n-r}x^r + \ldots\ldots + (-1)^nx^n \quad \text{----------- (5)}$$

for all values of n and x.

Put a = 1 in the formula (5) we obtain, for a positive integer n,

$$(1 - x)^n = 1 - {}^{n}C_1x + {}^{n}C_2x^2 - {}^{n}C_3x^3 + \ldots\ldots + (-1)^r\,{}^{n}C_rx^r + \ldots\ldots + (-1)^nx^n \quad \text{----------- (6)}$$

## 4.7 General Term in the expansion of $(a + x)^n$ :

From the expansion of $(a + x)^n$ we see that,

1st term = (0 + 1)th term = $t_1 = a^n = {}^nC_0 a^{n-0} x^0 = t_{0+1}$

2nd term = (1 + 1)th term = $t_2 = {}^nC_1 a^{n-1} x^1 = t_{1+1}$

3rd term = (2 + 1)th term = $t_3 = {}^nC_2 a^{n-2} x^2 = t_{2+1}$

4th term = (3 + 1)th term = $t_4 = {}^nC_3 a^{n-3} x^3 = t_{3+1}$

Similarly = (r + 1)th term = $t_{r+1} = {}^nC_r a^{n-r} x^r$

Therefore, (r + 1)th term, $t_{r+1} = {}^nC_r a^{n-r} x^r$ is known as the general term.

## 4.8 Middle Term(s) in the expansion of $(a + x)^n$ :

(i) When n is **even** the number of terms in the expansion of $(a + x)^n$ is $(n + 1)$ which is an **odd** number.

In this case the middle term is $\left(\frac{n}{2}+1\right)$th term. Therefore, the middle term $t_{\frac{n}{2}+1} = {}^nC_{\frac{n}{2}} \cdot a^{n-\frac{n}{2}} \cdot x^{\frac{n}{2}}$

(ii) When n is **odd** number the number of terms in the expansion of $(a + x)^n$ is $(n + 1)$ which is an **even** number and therefore there are **two** middle terms.

1st middle term = $\left(\frac{n-1}{2}+1\right)$th term = $\left(\frac{n+1}{2}\right)$th term

$$= t_{\frac{n-1}{2}+1} = {}^nC_{\frac{n-1}{2}} \cdot a^{n-\frac{n-1}{2}} \cdot x^{\frac{n-1}{2}} = {}^nC_{\frac{n-1}{2}} \cdot a^{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}}$$

2nd middle term $\left(\frac{n-1}{2}+2\right)$th term = $\frac{n+3}{2}$th term

$$= t_{\frac{n+1}{2}+1} = {}^nC_{\frac{n+1}{2}} \cdot a^{n-\frac{n+1}{2}} \cdot x^{\frac{n+1}{2}} = {}^nC_{\frac{n+1}{2}} \cdot a^{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}}$$

## PROBLEM SET – I

[Problems with '*' marks are solved at the end of the problem set]

**1.** Find the value of :

*(i) $(\sqrt{3}+1)^6-(\sqrt{3}-1)^6$ (ii) $(2+\sqrt{3})^7+(2-\sqrt{3})^7$ *(iii) $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$ **[WBSC – 99]**

**2.** Simplify :

(i) $\left(a+\sqrt{a^2-1}\right)^8-\left(a-\sqrt{a^2-1}\right)^8$ *(ii) $\left(a+\sqrt{a^2-1}\right)^7+\left(a-\sqrt{a^2-1}\right)^7$

**3.** Find the coefficients of

*(i) $\frac{1}{x^2}$ in the expansion of $\left(1-\frac{1}{x}\right)^{10}$ **[WBSC – 99]** *(ii) $\frac{1}{x^3}$ in the expansion of $\left(1+\frac{1}{x}\right)^{10}$

(iii) $x^4$ in the expansion of $\left(x^4+\frac{1}{x^3}\right)^{15}$ (iv) $a^{-11}$ in the expansion of $\left(5a^3-\frac{2}{a^2}\right)^{13}$

*(v) x in the expansion of $\left(x^2+\frac{a^2}{x}\right)^5$ **[WBSC – 08]** *(vi) $x^{-17}$ in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$

(vii) $x^{-2}$ in the expansion of $\left(2x^3-\frac{1}{x^2}\right)^6$ *(viii) $x^{16}$ in the expansion of $(x^2-2x)^{10}$ **[WBSC – 82]**

*(ix) $x^{16}$ in the expansion of $x^{10}(x-2)^{10}$ *(x) $x^{-2}$ in the expansion of $\left(3x-\frac{7}{x}\right)^8$ **[WBSC – 04]**

*(xi) $x^{2r+1}$ in the expansion of $\left(x-\frac{1}{x}\right)^{2n+1}$

**4.** (i) Examine whether the expansion $\left(2x^2-\frac{1}{x}\right)^{20}$ will contain $x^9$ term?

*(ii) Examine whether the expansion $\left(x-\frac{3}{x}\right)^7$ will contain the term independent of x ?

**5.** (a) Find the term independent of x in the expansion of

(i) $\left(x+\frac{1}{x}\right)^{2n}$ **[WBSC – 85]** *(ii) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

(iii) $\left(9x^2-\frac{1}{3x}\right)^{12}$ (iv) $\left(x^2+\frac{1}{x}\right)^{12}$

*(v) $\left(2x+\frac{1}{3x^2}\right)^{12}$

**(b)** *(i) $(1+4x)^p\cdot\left(1+\frac{1}{4x}\right)^q$ p, q, are positive integers. (ii) $(1+x)^3\cdot\left(x-\frac{1}{x}\right)^6$

*(c) If the x-independent term in the expansion of $\left(\sqrt{x}-\frac{\sqrt{m}}{x^2}\right)^{10}$ is 450, find m.

**6.** Find the middle term(s) in the expansion of

*(i) $\left(ax-\frac{1}{ax}\right)^6$ **[WBSC – 00]** *(ii) $\left(x^2-\frac{1}{x}\right)^9$ (iii) $\left(x+\frac{1}{x}\right)^8$

(iv) $\left(\frac{x^2}{3}+\frac{3}{x^2}\right)^8$ (v) $\left(2x-\frac{1}{3x}\right)^{2n}$

7. (i) Show that the middle term in the expansion of $\left(x+\frac{1}{2x}\right)^{2n}$ is $\frac{1.3.5......................(2n-1)}{n!}$

*(ii) Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1.3.5......................(2n-1)}{n!}.2^n \cdot x^n$

8. (i) If in the expansion of $(1 + x)^{2n+1}$ the coefficient of $x^r$ and $x^{r+1}$ be equal, find r.

*(ii) Prove that the coefficient of $x^n$ in the expansion of $(1 + x)^{2n}$ is double the coefficient of $x^n$ in the expansion of $(1 + x)^{2n-1}$ **[WBSC – 83]**

*(iii) If the coefficient of $x^7$ in the expansion of $\left(px^2+\frac{1}{qx}\right)^{11}$ be equal to the coefficient of $x^{-7}$ in the expansion of $\left(px-\frac{1}{qx^2}\right)^{11}$, show that, $pq = 1$. **[WBSC – 92, 94, 01, 03]**

9. *(i) Find the coefficient of x in the expansion of $\left(1-2x^3+3x^5\right)\left(1+\frac{1}{x}\right)^{10}$ **[WBSC – 07]**

*(ii) Find the coefficient of $x^{10}$ in the expansion of $(1 + x + x^2)(1 - x)^{15}$

10. *(i) The coefficients of pth, (p + 1)th and (p + 2)th terms in the expansion of $(1 + x)^n$ are in A.P., show that, $n^2 - n(4p + 1) + 4p^2 - 2 = 0$.

(ii) The three successive coefficients in the expansion of $(1 + x)^n$ are in the ratio 1 : 2 : 3. Find n.

*(iii) If the 2nd, 3rd and 4th terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, find x, a and n.

(iv) If the 3rd, 4th and 5th terms in the expansion of $(x + a)^n$ are 84, 280 and 560 respectively, find x, a and n.

(v) If the coefficients of three successive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462 respectively, find n.

*(vi) If the coefficients of three successive terms in the expansion of $(1 + x)^n$ are 120, 210 and 252 respectively, find n.

(vii) If the coefficients of three successive terms in the expansion of $(1 + x)^n$ are 252, 210 and 120 respectively, find n.

*(viii) If a, b, c are any three consecutive coefficients in the expansion of $(1 + x)^n$, show that, $n=\frac{2ac+b(a+c)}{b^2-ac}$ and that the serial number of the term of which a is the coefficient, is $\frac{a(b+c)}{b^2-ac}$ **[WBSC – 87]**

11. (i) If $a_1$, $a_2$, $a_3$, $a_4$, are four consecutive coefficients in the expansion of $(1 + x)^n$, show that, $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$ **[WBSC – 88]**

*(ii) If the coefficients of the 2nd, 3rd, 4th and 5th terms in the expansion of $(1 + x)^n$ be $a_1$, $a_2$, $a_3$, $a_4$, respectively, show that, $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$

*(iii) If n be a positive integer and if the 3rd, 4th, 5th and 6th terms in the expansion of $(x + A)^n$, when expanding in ascending powers of x be a, b, c and d respectively, show that, $\frac{b^2-ac}{c^2-bd}=\frac{5a}{3c}$

## ANSWERS

**1.** (i) $240\sqrt{3}$ (ii) 10084 (iii) 82 **2.** (i) $16a\sqrt{a^2-1}\,[16a^6-24a^4+10a^2-1]$ (ii) $2a(64a^6-112a^4+56a^2-7)$
**3.** (i) 45 (ii) 120 (iii) 6435 (iv) 3,66,08,000 (v) $10a^6$ (vi) $-1365$ (vii) 60 (viii) 3360 (ix) 3360 (x) 2,54,12,184
(xi) $(-1)^{n-r}\dfrac{(2n+1)!}{(n-r)!.(n+r+1)!}$

**4.**(i) No.(ii) No. **5. (a)** (i) $\dfrac{(2n)!}{n!.n!}$ (ii) $\frac{7}{18}$ (iii) 495 (iv) 495 (v) $\frac{14080}{9}$ **5. (b)** (i) $\dfrac{(p+q)!}{p!.q)!}$ (ii) 25 **5. (c)** m = 10

**6.** (i) $-20$ (ii) $126x^6, -126x^3$ (iii) 70 (iv) 70 (v) $(-1)^n.\left(\frac{2}{3}\right)^n.\left[\frac{(2n)!}{(n!)^2}\right]$ **8.** (i) r = n **9.** (i) 540 (ii) 4433

**10.** (ii) n = 14 (iii) a = 3, x = 2 n = 5 (iv) n = 7, x = 1, a = 2 (v) n = 11 (vi) n = 10 (vii) 10

## SOLUTION OF THE PROBLEMS WITH '*' MARKS

**1. (i) Solution :** Given expression, $(\sqrt{3}+1)^6-(\sqrt{3}-1)^6$

$$=\left\{(\sqrt{3})^6+{}^6C_1\cdot(\sqrt{3})^5\cdot 1+{}^6C_2\cdot(\sqrt{3})^4\cdot 1^2+{}^6C_3\cdot(\sqrt{3})^3\cdot 1^3+{}^6C_4\cdot(\sqrt{3})^2\cdot 1^4+{}^6C_5\cdot(\sqrt{3})\cdot 1^5+1^6\right\}$$
$$-\left\{(\sqrt{3})^6-{}^6C_1\cdot(\sqrt{3})^5\cdot 1+{}^6C_2\cdot(\sqrt{3})^4\cdot 1^2-{}^6C_3\cdot(\sqrt{3})^3\cdot 1^3+{}^6C_4\cdot(\sqrt{3})^2\cdot 1^4-{}^6C_5\cdot(\sqrt{3})\cdot 1^5+1^6\right\}$$
$$=2\left\{{}^6C_1\cdot(\sqrt{3})^5+{}^6C_3\cdot(\sqrt{3})^3+{}^6C_5\cdot(\sqrt{3})\right\}=2\sqrt{3}\{6\times 9+20\times 3+6\}$$
$$\left[\because {}^6C_1=6,\ {}^6C_5={}^6C_{6-1}=6\text{ and }{}^6C_3=\frac{6!}{3!3!}=\frac{6.5.4.3!}{3.2.1.3!}=20\right]$$
$$=2\sqrt{3}(54+60+6)=240\sqrt{3}\ \textbf{(Ans)}$$

**1.** **(iii) Solution:** Given expression : $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$

$$=\left\{(\sqrt{2})^5+{}^5C_1\cdot(\sqrt{2})^4\cdot 1+{}^5C_2\cdot(\sqrt{2})^3\cdot 1^2+{}^5C_3\cdot(\sqrt{2})^2\cdot 1^3+{}^5C_4\cdot(\sqrt{2})\cdot 1^4+1^5\right\}$$
$$-\left\{(\sqrt{2})^5-{}^5C_1\cdot(\sqrt{2})^4\cdot 1+{}^5C_2\cdot(\sqrt{2})^3\cdot 1^2-{}^5C_3\cdot(\sqrt{2})^2\cdot 1^3+{}^5C_4\cdot(\sqrt{2})\cdot 1^4-1^5\right\}$$
$$=2\left\{{}^5C_1\cdot(\sqrt{2})^4+{}^5C_3\cdot(\sqrt{2})^2+1\right\}=2(5\times 4+10\times 2+1)=82\ \textbf{(Ans)}$$

**2.** **(ii) Solution:** Given expression : $\left(a+\sqrt{a^2-1}\right)^7+\left(a-\sqrt{a^2-1}\right)^7$

$$=a^7+{}^7C_1\cdot a^6\cdot\sqrt{a^2-1}+{}^7C_2\cdot a^5\cdot\left(\sqrt{a^2-1}\right)^2+{}^7C_3\cdot a^4\cdot\left(\sqrt{a^2-1}\right)^3$$
$$+{}^7C_4\cdot a^3\cdot\left(\sqrt{a^2-1}\right)^4+{}^7C_5\cdot a^2\cdot\left(\sqrt{a^2-1}\right)^5+{}^7C_6\cdot a\cdot\left(\sqrt{a^2-1}\right)^6+\left(\sqrt{a^2-1}\right)^7$$
$$+a^7-{}^7C_1\cdot a^6\cdot\sqrt{a^2-1}+{}^7C_2\cdot a^5\cdot\left(\sqrt{a^2-1}\right)^2-{}^7C_3\cdot a^4\cdot\left(\sqrt{a^2-1}\right)^3$$
$$+{}^7C_4\cdot a^3\cdot\left(\sqrt{a^2-1}\right)^4-{}^7C_5\cdot a^2\cdot\left(\sqrt{a^2-1}\right)^5+{}^7C_6\cdot a\cdot\left(\sqrt{a^2-1}\right)^6-\left(\sqrt{a^2-1}\right)^7$$

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$$= 2\left\{a^7 + {}^7C_2 \cdot a^5 \cdot \left(\sqrt{a^2-1}\right)^2 + {}^7C_4 \cdot a^3 \cdot \left(\sqrt{a^2-1}\right)^4 + {}^7C_6 \cdot a \cdot \left(\sqrt{a^2-1}\right)^6\right\}$$

$$= 2\{a^7 + 21.a^5.(a^2-1) + 35.a^3(a^2-1)^2 + 7a\,(a^2-1)^3\}$$

$$= 2\{a^7 + 21a^7 - 21a^5 + 35a^7 - 70a^5 + 35a^3 + 7a^7 - 21a^5 + 21a^3 - 7a\}$$

$$= 2\{64a^7 - 112a^5 + 56a^3 - 7a\} = 2a(64a^6 - 112a^4 + 56a^2 - 7)$$ **(Ans)**

**3.** (i) **Solution:** Let (r + 1)th term contains $\frac{1}{x^2}$ i. e., $x^{-2}$ the expansion of $\left(1-\frac{1}{x}\right)^{10}$

Here, (r + 1)th term, $t_{r-1} = {}^{10}C_r \cdot (1)^{10-r} . \left(-\frac{1}{x}\right)^r = {}^{10}C_r \cdot (-1)^r \cdot x^{-r}$

$\therefore -r = -2 \Rightarrow r = 2 \quad \therefore$ (r + 1)th term = (2 + 1)th = 3rd term contains $\frac{1}{x^2}$.

The coefficient of $x^{-2} = {}^{10}C_2 \cdot (-1)^2 = 45$ **(Ans)**

**3.(ii) Solution:** Let (r + 1)th term contains $\frac{1}{x^3}$ i. e., $x^{-3}$ in the expansion of $\left(1+\frac{1}{x}\right)^{10}$

Here, (r + 1)th term, $t_{r-1} = {}^{10}C_r \cdot (1)^{10-r} . \left(\frac{1}{x}\right)^r = {}^{10}C_r \cdot x^{-r}$

$\therefore -r = -3 \Rightarrow r = 3 \quad \therefore$ (r + 1)th term = (3 + 1)th = 4th term contains $\frac{1}{x^3}$.

The coefficient of $x^{-3} = {}^{10}C_3 = \frac{10!}{3!.7!} = \frac{10.9.8.7!}{3.2.1.7!} = 120$ **(Ans)**

**3.** (v) **Solution:** Let (r+1)th term contains x in the expansion of $\left(x^2 + \frac{a^2}{x}\right)^5$

Here, (r + 1)th term, $t_{r+1} = {}^5C_r \cdot \left(x^2\right)^{5-r} \cdot \left(\frac{a^2}{x}\right)^r = {}^5C_r \cdot x^{10-2r} \cdot a^{2r} \cdot x^{-r} = {}^5C_r.x^{10-3r}.a^{2r}$

$\therefore$ 10 – 3r = 1 or, 3r = 9, or, r = 3 $\quad\therefore$ (r + 1)th term = 4th term contains x.

The coefficient of x = ${}^5C_3 \cdot a^6 = 10a^6$ **(Ans)**

**3.** (vi) **Solution:** Suppose the (r + 1)th term in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ contains $x^{-17}$.

Here $t_{r-1} = {}^{15}C_r \cdot \left(x^4\right)^{15-r} \cdot \left(-\frac{1}{x^3}\right)^r = {}^{15}C_r(-1)^r.x^{60-4r}.x^{-3r} = {}^{15}C_r(-1)^r.x^{60-7r}$

$\therefore$ 60 – 7r = – 17 or, 7r = 77 $\therefore$ r = 11 $\quad\therefore$ 11 + 1 = 12th term contains $x^{-17}$.

The required coefficient = ${}^{15}C_{11} \cdot (-1)^{11} = -\frac{15!}{11!4!} = -\frac{15.14.13.12.11!}{11!4.3.2.1} = -1365$ **(Ans)**

**3** (viii) **Solution:** Let (r + 1)th term contains $x^{16}$ in the expansion of $(x^2 - 2x)^{10}$

Here, (r + 1)th term, $t_{r-1} = {}^{10}C_r\,(x^2)^{10-r}.(-2x)^r = {}^{10}C_r\,(-2)^r.x^{20-2r}.x^r = {}^{10}C_r\,(-2)^r.x^{20-r}$

$\therefore$ 20 – r = 16 or, r = 4 $\quad\therefore$ (r + 1)th term = 5th term contains $x^{16}$.

The coefficient of $x^{16} = {}^{10}C_4\,(-2)^4 = {}^{10}C_4 \times 16 = 210 \times 16 = 3360$ **(Ans)**

**3.** (ix) **Solution:** Since $x^{10} \times x^6 = x^{16}$

$\therefore$ the coefficient of $x^{16}$ in the expansion of $x^{10}\,(x-2)^{10}$ will be the coefficient of $x^6$ in the expansion of $(x-2)^{10}$.

Let (r + 1)th term contains $x^6$ in the expansion of $(x-2)^{10}$.

Here, $t_{r-1} = {}^{10}C_r\,x^{10-r}\,(-2)^r = (-2)^r.\,{}^{10}C_r.\,x^{10-r}$

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$\therefore\ 10 - r = 6 \ \therefore\ r = 4$

$\therefore$ (r + 1)th term = 5th term contains $x^6$.

The coefficient of $x^6 = (-2)^4 . {}^{10}C_4 = 16 \times \frac{10!}{4!\,6!} = 16 \times \frac{10.9.8.7.6!}{4.3.2.1.6!} = 16 \times 210 = 3360$ **(Ans)**

(x) **Solution:** Suppose the (r + 1)th term in the expansion of $\left(3x - \frac{7}{x}\right)^8$ contains $x^{-2}$.

Here $t_{r-1} = {}^8C_r \cdot (3x)^{8-r} \cdot \left(-\frac{7}{x}\right)^r = {}^8C_r \cdot 3^{8-r} \cdot x^{8-r} \cdot (-7)^r \cdot x^{-r}$

$= {}^8C_r \cdot 3^{8-r} \cdot (-7)^r \cdot x^{8-2r}$

$\therefore\ 8 - 2r = -2$ or, $2r = 10 \ \therefore\ r = 5$

$\therefore\ 5 + 1 = 6$ th term contains $x^{-2}$.

The required coefficient

$= {}^8C_5 \cdot 3^{8-5} \cdot (-7)^5 = -{}^8C_5 \times 3^3 \times 7^5 = 2,54,12,184$ **(Ans)**

(xi) **Solution:** Let (k + 1)th term in the expansion of $\left(x - \frac{1}{x}\right)^{2n+1}$ contains $x^{2r+1}$

Here, $t_{k+1} = {}^{2n+1}C_k \cdot x^{2n+1-k} \cdot \left(-\frac{1}{x}\right)^k$

$= (-1)^k . {}^{2n+1}C_k . x^{2n+1-2k}$

$\therefore\ 2n + 1 - 2k = 2r + 1$ or, $n - k = r$ or, $k = n - r$

$\therefore$ (n − r + 1)th term conatins $x^{2r+1}$

The coefficient of $x^{2r+1}$ in the expansion $= (-1)^{n-r} . {}^{2n+1}C_{n-r}$

$= (-1)^{n-r} \cdot \frac{(2n+1)!}{(n-r)!(2n+1-n+r)!} = (-1)^{n-r} \cdot \frac{(2n+1)!}{(n-r)!(n+r+1)!}$ **(Ans)**

**4.** **(ii) Solution:** Let (r +1)th term in the expansion of $\left(x - \frac{3}{x}\right)^7$ is independent of x.

Here, $t_{r+1} = {}^7C_r \cdot x^{7-r} \cdot \left(-\frac{3}{x}\right)^r = (-3)^r . {}^7C_r \cdot x^{7-r} . x^{-r} = (-3)^r . {}^7C_r \cdot x^{7-2r}$

$\therefore\ 7 - 2r = 0$ or, $2r = 7$ or, $r = \frac{7}{2}$ which is a fraction.

But the fractional term cannot occur in the expansion of $\left(x - \frac{3}{x}\right)^7$.

Therefore, the given expansion cannot contain a term independent of x.

**5. a.** (ii) **Solution:** Let (r +1)th term contains the term independent of x in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$

Now, (r + 1)th term, $t_{r+1} = {}^9C_r \cdot \left(\frac{3x^2}{2}\right)^{9-r} \cdot \left(-\frac{1}{3x}\right)^r = {}^9C_r \cdot \left(\frac{3}{2}\right)^{9-r} \cdot \left(-\frac{1}{3}\right)^r \cdot x^{18-2r-r} = {}^9C_r \cdot \left(\frac{3}{2}\right)^{9-r} \cdot \left(-\frac{1}{3}\right)^r \cdot x^{18-3r}$

$\therefore\ 18 - 3r = 0$ or, $3r = 18$ or, $r = 6$ $\quad\therefore$ (6 + 1)th = 7th term is independent of x.

The term is $= {}^9C_6 \cdot \left(\frac{3}{2}\right)^{9-6} \cdot \left(-\frac{1}{3}\right)^6 = {}^9C_6 \cdot \left(\frac{3}{2}\right)^3 \cdot \frac{1}{3^6} = \frac{9!}{6!\,3!} \cdot \frac{3^3}{2^3} \cdot \frac{1}{3^6} = \frac{9.8.7.6!}{6!\,3.2.1} \cdot \frac{1}{2^3.3^3} = 3.4.7.\frac{1}{8.27} = \frac{7}{18}$ **(Ans)**

**5. a.** (v) **Solution:** Let (r + 1)th term contains the term independent of x, in the expansion of $\left(2x+\frac{1}{3x^2}\right)^{12}$

Now, (r + 1)th term, $t_{r+1}={}^{12}C_r\cdot(2x)^{12-r}\cdot\left(\frac{1}{3x^2}\right)^r={}^{12}C_r\cdot 2^{12-r}\cdot x^{12-r}\cdot\left(\frac{1}{3}\right)^r\cdot\frac{1}{x^{2r}}={}^{12}C_r\cdot 2^{12-r}\cdot\left(\frac{1}{3}\right)^r\cdot x^{12-3r}$

$\therefore$ 12 − 3r = 0 or, 3r = 12 or, r = 4 $\therefore$ (4 + 1)th = 5th term is independent of x.

The term is, ${}^{12}C_4\cdot 2^{12-4}\cdot\frac{1}{3^4}=\frac{12!}{4!8!}\cdot 2^8\cdot\frac{1}{3^4}=\frac{12.11.10.9}{4.3.2.1}\cdot 2^8\cdot\frac{1}{3^4}=\frac{11.5.2^8}{3^2}=1564\frac{4}{9}$ **(Ans)**

**5. b.** (i) **Solution:** $(1+4x)^p\left(1+\frac{1}{4x}\right)^q=(1+4x)^p\frac{(1+4x)^q}{(4x)^q}=\frac{(1+4x)^{p+q}}{(4x)^q}$

$$=\frac{1+{}^{p+q}C_1.4x+{}^{p+q}C_2(4x)^2+\ldots\ldots+{}^{p+q}C_q(4x)^q+\ldots\ldots+(4x)^{p+q}}{(4x)^q}$$

which shows that the term independent of x in the expansion of

$(1+4x)^p.\left(1+\frac{1}{4x}\right)^q$ is ${}^{p+q}C_q=\frac{(p+q)!}{q!\ p!}$ **(Ans)**

**5. c. Solution:** Let (r + 1)th term is independent of x in the expansion of $\left(\sqrt{x}-\frac{\sqrt{m}}{x^2}\right)^{10}$

Here, $t_{r-1}={}^{10}C_r.(\sqrt{x})^{10-r}.\left(-\frac{\sqrt{m}}{x^2}\right)^r=(-\sqrt{m})^r.\,{}^{10}C_r.(\sqrt{x})^{10-r}.x^{-2r}$

$=(-\sqrt{m})^r.\,{}^{10}C_r.x^{5-\frac{r}{2}-2r}=(-\sqrt{m})^r.\,{}^{10}C_r.x^{5-\frac{5r}{2}}$

$\therefore 5-\frac{5r}{2}=0$ or, $1-\frac{r}{2}=0$ or, r = 2

$\therefore$ the x-independent term is (2 + 1)rd term = 3rd term and the term is

$=(-\sqrt{m})^2.\,{}^{10}C_2=m.\,{}^{10}C_2=m.\frac{10!}{2!\ 8!}=m\frac{10.9.8!}{2.1.8!}=45m$

By the problem, 45m = 450, or, m = 10 **(Ans)**

6. (i) **Solution:** Number of terms in the expansion of $\left(ax-\frac{1}{ax}\right)^6$ is (6 + 1) = 7, which is an odd number.

$\therefore$ only middle term is $\left(\frac{6}{2}+1\right)$ = 4th term.

Now, $t_4=t_{3+1}={}^6C_3(ax)^{6-3}.\left(-\frac{1}{ax}\right)^3=-{}^6C_3.a^3x^3.\frac{1}{a^3x^3}=-\frac{6!}{3!3!}=-\frac{6.5.4\,3!}{3!\ 3!}=-\frac{6.5.4}{3.2.1}=-20$ **(Ans)**

6. (ii) **Solution:** Here number of terms are (9 + 1) = 10, which is an even number.

$\therefore$ number of middle terms is 2.

These two terms are, $\frac{10}{2}$ and $\left(\frac{10}{2}+1\right)$ i.e. 5th and 6th terms.

Now, 5th term, $t_5=t_{4+1}={}^9C_4(x^2)^{9-4}.\left(-\frac{1}{x}\right)^4={}^9C_4.x^{5\times 2}\frac{1}{x^4}={}^9C_4.x^6$

$=\frac{9!}{4!5!}.x^6=\frac{9.8.7.6\,5!}{4.3.2.1\,5!}.x^6=126x^6$ **(Ans)**

6th term $t_6=t_{5+1}={}^9C_5.(x^2)^{9-5}\left(-\frac{1}{x}\right)^5=-126.x^8.\frac{1}{x^5}=-126x^3$ **(Ans)**

**7. (ii) Solution:** Number of terms in the expansion of $(1+x)^{2n}$ is $2n+1$, which is an odd number.

$\therefore$ only middle term is, $\left(\frac{2n}{2}+1\right) = (n+1)$th term.

Here, $t_{n+1} = {}^{2n}C_n \cdot 1^{2n-n} \cdot x^n = \frac{2n!}{n!\,n!}x^n = \frac{2n(2n-1)(2n-2)........6.5.4.3.2.1}{n!\,n!} \cdot x^n$

$= \frac{\{2n(2n-2)......6.4.2\}\{(2n-1)......5.3.1\}}{n!\,n!} \cdot x^n = \frac{2^n \cdot \{n(n-1).......3.2.1\}\{(2n-1).........5.3.1\}}{n!\;n!} \cdot x^n$

$= \frac{n! \cdot \{1.3.5........(2n-1)\} \cdot 2^n \cdot x^n}{n! \cdot n!} = \frac{1.3.5.......(2n-1)}{n!} \cdot 2^n \cdot x^n$ **(Proved)**

**8.(ii) Solution:** The expansion of $(1+x)^{2n} = 1 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + ..... + {}^{2n}C_n \cdot x^n + ..... + x^{2n}$

$\therefore$ coefficient of $x^n = {}^{2n}C_n$ ----------(1)

Again, the expansion of $(1+x)^{2n-1} = 1 + {}^{2n-1}C_1 x + {}^{2n-1}C_2 x^2 + .......... + {}^{2n-1}C_n x^n + .... + x^{2n-1}$

coefficient of $x^n = {}^{2n-1}C_n = \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n)!}{2n \cdot n!(n-1)!} = \frac{(2n)!}{2 \cdot n!\,n\,(n-1)!} = \frac{(2n)!}{2 \cdot n! \cdot n!} = \frac{1}{2} \cdot {}^{2n}C_n$ ---- (2)

From (1) and (2) we see that coefficient of $x^n$ in the expansion of $(1+x)^{2n}$ is double the coefficient of $x^n$ in the expansion of $(1+x)^{2n-1}$ **(Proved)**

**8.(iii) Solution:** Let $(r+1)$th term contains $x^7$ in the expansion of $\left(px^2 + \frac{1}{qx}\right)^{11}$

Here, $(r+1)$th term, $t_{r+1} = {}^{11}C_r \cdot \left(px^2\right)^{11-r} \cdot \left(\frac{1}{qx}\right)^r = {}^{11}C_r \cdot p^{11-r} \cdot \frac{1}{q^r} x^{22-2r-r} = {}^{11}C_r \cdot p^{11-r} \cdot \frac{1}{q^r} x^{22-3r}$

By the problem, $22 - 3r = 7$ or, $3r = 15$ $\therefore$ $r = 5$

$\therefore$ coefficient of $x^7$ in the expansion of $\left(px^2 + \frac{1}{qx}\right)^{11}$ is ${}^{11}C_5 \cdot p^{11-5} \cdot \frac{1}{q^5} = {}^{11}C_5 \cdot p^6 \cdot \frac{1}{q^5}$

Again let $(u+1)$th term contains $x^{-7}$ in the expansion of $\left(px - \frac{1}{qx^2}\right)^{11}$

Here, $t_{u+1} = {}^{11}C_u \cdot (px)^{11-u} \cdot \left(-\frac{1}{qx^2}\right)^u = {}^{11}C_u \cdot p^{11-u} \cdot \left(-\frac{1}{q}\right)^u x^{11-u-2u} = {}^{11}C_u \cdot p^{11-u} \cdot \left(-\frac{1}{q}\right)^u x^{11-3u}$

$\therefore$ $11 - 3u = -7$ or, $3u = 18$ $\therefore$ $u = 6$

$\therefore$ coefficient of $x^{-7}$ in the expansion of $\left(px - \frac{1}{qx^2}\right)^{11} = {}^{11}C_6 \cdot p^{11-6} \cdot \left(-\frac{1}{q}\right)^6 = {}^{11}C_6 \cdot p^5 \cdot \frac{1}{q^6}$

By the problem, ${}^{11}C_5 \cdot \frac{p^6}{q^5} = {}^{11}C_6 \cdot \frac{p^5}{q^6}$ or, $\frac{11!}{5! \cdot 6!} p = \frac{11!}{6! \cdot 5!} \frac{1}{q}$ or, $pq = 1$ **(Proved)**

**9.(i) Solution:** The given expression $\left(1 - 2x^3 + 3x^5\right)\left(1 + \frac{1}{x}\right)^{10}$

$= \left(1 - 2x^3 + 3x^5\right) \times \left(1 + {}^{10}C_1 \frac{1}{x} + {}^{10}C_2 \frac{1}{x^2} + {}^{10}C_3 \frac{1}{x^3} + {}^{10}C_4 \frac{1}{x^4} + {}^{10}C_5 \frac{1}{x^5} + {}^{10}C_6 \frac{1}{x^6} + ......\right)$

Now, in the multiplication $-2x^3 \times {}^{10}C_2 \frac{1}{x^2} = -2 \times {}^{10}C_2\, x$, $3x^5 \times {}^{10}C_4 \frac{1}{x^4} = 3 \times {}^{10}C_4\, x$

and there is no other term containing x in the product.

$\therefore$ the required coefficient of $x = -2\times {}^{10}C_2 + 3\times {}^{10}C_4 = -2\times\frac{10!}{2!\,8!}+3\times\frac{10!}{4!\,6!}$

$$= -2\times\frac{10\times 9}{2}+3\times\frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = -90+630 = 540 \text{ (Ans)}$$

**9.** (ii) **Solution:** The given expansion $= (1+x+x^2)(1-x)^{15}$

$$= (1+x+x^2)\left(1-{}^{15}C_1x+{}^{15}C_2x^2-{}^{15}C_3x^3+{}^{15}C_4x^4-{}^{15}C_5x^5+{}^{15}C_6x^6-{}^{15}C_7x^7+{}^{15}C_8x^8-{}^{15}C_9x^9 + {}^{15}C_{10}x^{10}-\ldots\ldots\right)$$

Now, in the multiplication $1\times {}^{15}C_{10}x^{10} = {}^{15}C_{10}x^{10}$, $x\times\left(-{}^{15}C_9x^9\right) = -{}^{15}C_9x^{10}$,

$x^2\times\left({}^{15}C_8x^8\right) = {}^{15}C_8x^{10}$, and there is no other term containing $x^{10}$ in the product.

$\therefore$ the required coefficient of $x^{10} = {}^{15}C_{10}-{}^{15}C_9+{}^{15}C_8 = \frac{15!}{10!\,5!}-\frac{15!}{9!\,6!}+\frac{15!}{8!\,7!}$

$$= \frac{15.14.13.12.11.10!}{10!5.4.3.2.1}-\frac{15.14.13.12.11.10.9!}{9!.6.5.4.3.2.1}+\frac{15.14.13.12.11.10.9.8!}{8!.\,7.6.5.4.3.2.1} = 3003-5005+715\times 9 = 4433 \text{ (Ans)}$$

**10.(i) Solution:** In the expansion of $(1+x)^n$, pth term $t_p = t_{(p-1)+1} = {}^nC_{p-1}\,.\,x^{p-1}$

Similarly $t_{p+1} = {}^nC_p.x^p$ and $t_{p+2} = t_{(p+1)+1} = {}^nC_{p+1}.x^{p+1}$

By the problem, ${}^nC_{p-1}, {}^nC_p, {}^nC_{p+1}$ are in A.P. $\therefore 2.\,{}^nC_p = {}^nC_{p-1}+{}^nC_{p+1}$

or, $\frac{2\cdot n!}{p!\cdot(n-p)!} = \frac{n!}{(p-1)!\cdot(n-p+1)!}+\frac{n!}{(p+1)!\cdot(n-p-1)!}$ or, $\frac{2}{p\cdot(n-p)} = \frac{1}{(n-p)\cdot(n-p+1)}+\frac{1}{p\cdot(p+1)}$

or, $2(n-p+1)(p+1) = p(p+1)+(n-p)(n-p+1)$

or, $2(np-p^2+p+n-p+1) = p^2+p+n^2-2np+p^2+n-p$

or, $2np-2p^2+2n+2 = 2p^2+n^2-2np+n$ or, $n^2-4np-n+4p^2-2 = 0$

or, $n^2-n(4p+1)+4p^2-2 = 0$ **(Proved)**

**10.(iii) Solution:** 2nd term, $t_2 = {}^nC_1.x^{n-1}.a$, 3rd term, $t_3 = {}^nC_2.x^{n-2}.a^2$, 4th term. $t_4 = {}^nC_3x^{n-3}.a^3$

By the problem, ${}^nC_1.x^{n-1}a = 240$ or, $an.x^{n-1} = 240$ ---------- (1)

${}^nC_2x^{n-2}.a^2 = 720$ or, $\frac{n(n-1)}{2}a^2x^{n-2} = 720$ or, $n(n-1)a^2.x^{n-2} = 1440$ --------(2)

and ${}^nC_3.x^{n-3}.a^3 = 1080$ or, $\frac{n(n-1)(n-2)}{6}a^3x^{n-3} = 1080$ or, $n(n-1)(n-2).x^{n-3}.a^3 = 6480$ -----(3)

From (1) and (2) we get, $\frac{n(n-1).a^2x^{n-2}}{na\,x^{n-1}} = \frac{1440}{240}$ or, $\frac{(n-1)a}{x} = 6$ ---------(4)

From, (2) and (3) we get, $\frac{n(n-1)(n-2)x^{n-3}a^3}{n(n-1).x^{n-2}a^2} = \frac{6480}{1440}$ or, $\frac{(n-2)a}{x} = \frac{9}{2}$ -------(5)

From (4) and (5) we get, [ (5) ÷ (4)] $\frac{n-2}{n-1} = \frac{3}{4}$ or, $4n-8 = 3n-3$ or, $n = 5$

From (4) we get, $\frac{(5-1)a}{x} = 6$ or, $4a = 6x$ or, $a = \frac{3x}{2}$

From (1) we get, $\frac{3x}{2}.5.x^{5-1} = 240$ or, $x^5 = 32 = 2^5 \Rightarrow x = 2$

$\therefore a = \frac{3}{2}\times 2 = 3$ $\therefore a = 3$ $x = 2$ ' $n = 5$ **(Ans)**

**10.(vi) Solution:** Let in the expansion of $(1 + x)^n$, the coefficients of rth, (r + 1)th and (r + 2)th terms respectively be 120, 210 and 252.

Now, $t_r = t_{(r-1)+1} = {}^nC_{r-1}.x^{r-1}$

$\therefore$ by the problem, ${}^nC_{r-1} = 120$ --------- (1)

Similarly, ${}^nC_r = 210$ ----------- (2) and ${}^nC_{r+1} = 252$ --------- (3)

From (1) and (2) we get, $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{210}{120}$ or, $\frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{7}{4}$

or, $\frac{(r-1)!.(n-r+1)(n-r)!}{r(r-1)!(n-r)!} = \frac{7}{4}$ or, $\frac{n-r+1}{r} = \frac{7}{4}$ --------- (4)

Similarly, from (2) and (3) we get, $\frac{n-r}{r+1} = \frac{252}{210}$ or, $\frac{n-r}{r+1} = \frac{6}{5}$ --------- (5)

From (4) we get, $7r = 4n - 4r + 4$ or, $11r = 4n + 4$ --------- (6)

From (5) we get, $6r + 6 = 5n - 5r$ or, $11r = 5n - 6$ --------- (7)

From (6) and (7) we get, $5n - 6 = 4n + 4$ or, $n = 10$ **(Ans)**

**10.(viii) Solution:** Let a, b, c be the coefficients in the expansion of $(1 + x)^n$ of (r + 1)th, (r + 2)th and (r + 3)th terms respectively.

Now, (r + 1)th term, $t_{r+1} = {}^nC_r.x^r$ Therefore, by the problem, ${}^nC_r = a$ --------- (1)

Similarly, ${}^nC_{r+1} = b$ ------- (2) and ${}^nC_{r+2} = c$ --------- (3)

From (1) and (2) we get,

$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{b}{a}$ or, $\frac{n!}{(r+1)!\cdot(n-r-1)!} \times \frac{r!\cdot(n-r)!}{n!} = \frac{b}{a}$ or, $\frac{r!\cdot(n-r)\cdot(n-r-1)!}{(r+1)\cdot r!\cdot(n-r-1)!} = \frac{b}{a}$ or, $\frac{n-r}{r+1} = \frac{b}{a}$ --------(4)

From (2) and (3) we get, (similarly) $\frac{n-r-1}{r+2} = \frac{c}{b}$ --------(5)

From (4) we get, $br + b = an - ar$ or, $(a + b)r = an - b$ -------- (6)

From (5) we get, $cr + 2c = bn - br - b$ or, $(b + c)r = bn - 2c - b$ --------- (7)

From (6) and (7) eliminating r we get,

$(an - b)(b + c) = (bn - 2c - b)(a + b)$ or, $n(ab + ac - ab - b^2) = b(b + c) - (a + b)(2c + b)$

or, $n(ac - b^2) = b^2 + bc - 2ac - 2bc - ab - b^2 = -2ac - ab - bc$

or, $n(b^2 - ac) = 2ac + b(a+c)$ or, $n = \frac{2ac + b(a+c)}{b^2 - ac}$ **(Proved)**

From (6) we get, $an = (a + b)r + b$ and from (7) we get, $bn = (b + c)r + 2c + b$

Eliminating n we get, $\frac{a}{b} = \frac{(a+b)r + b}{(b+c)r + 2c + b}$ or, $a(b + c)r + 2ac + ab = b(a + b)r + b^2$

or, $r(ab + ac - ab - b^2) = b^2 - 2ac - ab$ or, $r(ac - b^2) = b^2 - 2ac - ab$ or, $r = \frac{2ac + ab - b^2}{b^2 - ac}$

$\therefore$ $r + 1 = \frac{2ac + ab - b^2}{b^2 - ac} + 1 = \frac{ac + ab}{b^2 - ac} = \frac{a(b+c)}{b^2 - ac}$ Hence **proved**.

**11.(ii) Solution:** In the expansion of $(1 + x)^n$, 2nd, 3rd, 4th and 5th terms are respectively, ${}^nC_1x$, ${}^nC_2x^2$, ${}^nC_3x^3$ and ${}^nC_4x^4$

$\therefore$ by the problem, ${}^nC_1 = a_1$, ${}^nC_2 = a_2$, ${}^nC_3 = a_3$, ${}^nC_4 = a_4$

$$\therefore \frac{a_1}{a_1+a_2} = \frac{{}^nC_1}{{}^nC_1+{}^nC_2} = \frac{{}^nC_1}{{}^{n+1}C_2} = \frac{n}{\dfrac{(n+1)!}{2!\,(n-1)!}} = \frac{2.n\,(n-1)!}{(n+1).n\,(n-1)!} = \frac{2}{n+1}$$

$$\frac{a_3}{a_3+a_4} = \frac{{}^nC_3}{{}^nC_3+{}^nC_4} = \frac{{}^nC_3}{{}^{n+1}C_4} = \frac{n!}{3!\,(n-3)!} \times \frac{4!\,(n-3)!}{(n+1)!} = \frac{n! \times 4.3!}{3!\,(n+1)\,n!} = \frac{4}{n+1}$$

$$\therefore \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2}{n+1} + \frac{4}{n+1} = \frac{6}{n+1}$$

Again, $$2.\frac{a_2}{a_2+a_3} = 2.\frac{{}^nC_2}{{}^nC_2+{}^nC_3} = 2.\frac{{}^nC_2}{{}^{n+1}C_3} = 2.\frac{n!}{2!\,(n-2)!} \times \frac{3!\,(n-2)!}{(n+1)!} = 2.\frac{n!}{2!} \times \frac{3.2!}{(n+1)\,n!} = \frac{6}{n+1}$$

$$\therefore \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = 2.\frac{a_2}{a_2+a_3} \quad \textbf{(proved)}$$

**11.(iii) Solution:** By the problem,

3rd term $t_3 = {}^nC_2.x^{n-2}.A^2 = a$ --------(1), 4th term $t_4 = {}^nC_3.x^{n-3}.A^3 = b$ --------(2)

5th term $t_5 = {}^nC_4.x^{n-4}.A^4 = c$ --------(3), 6th term $t_6 = {}^nC_5.x^{n-5}.A^5 = d$ --------(4)

From (1) and (2), $$\frac{b}{a} = \frac{{}^nC_3.x^{n-3}.A^3}{{}^nC_2.x^{n-2}.A^2} = \frac{n!}{3!\,(n-3)!} \times \frac{2!(n-2)!}{n!} \times \frac{A}{x} = \frac{2!.(n-2)\,(n-3)!}{3.2!\,(n-3)!}.\frac{A}{x} = \frac{n-2}{3}.\frac{A}{x} \quad \text{--------(5)}$$

From (2) and (3), $$\frac{c}{b} = \frac{{}^nC_4x^{n-4}.A^4}{{}^nC_3.x^{n-3}A^3} = \frac{n!}{4!\,(n-4)!} \times \frac{3!\,(n-3)!}{n!}.\frac{A}{x} = \frac{3!\,(n-3)\,(n-4)!}{4.3!\,(n-4)!}.\frac{A}{x} = \frac{n-3}{4}.\frac{A}{x} \quad \text{--------(6)}$$

Similarly from (3) and (4), $$\frac{d}{c} = \frac{{}^nC_5.x^{n-5}.A^5}{{}^nC_4.x^{n-4}.A^4} = \frac{n-4}{5}.\frac{A}{x} \quad \text{-------(7)}$$

From (5) and (6),

$$\frac{b}{a} - \frac{c}{b} = \left(\frac{n-2}{3} - \frac{n-3}{4}\right)\frac{A}{x} = \frac{4n-8-3n+9}{12}.\frac{A}{x} \quad \text{or, } \frac{b^2-ac}{ab} = \frac{n+1}{12}.\frac{A}{x} \quad \text{-----------(8)}$$

From (6) and (7), $$\frac{c}{b} - \frac{d}{c} = \left(\frac{n-3}{4} - \frac{n-4}{5}\right).\frac{A}{x} = \left(\frac{5n-15-4n+16}{20}\right).\frac{A}{x} \quad \text{or, } \frac{c^2-bd}{bc} = \frac{n+1}{20}.\frac{A}{x} \quad \text{-----------(9)}$$

From (8) and (9) we get,

$$\frac{b^2-ac}{ab} \times \frac{bc}{c^2-bd} = \frac{n+1}{12} \times \frac{20}{n+1} \quad \text{or, } \frac{b^2-ac}{c^2-bd} = \frac{5a}{3c} \quad \textbf{(proved)}$$

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## 4.9 Infinite series:

If a series contains no last term, then it is known as infinite series.

Hence, an infinite series is an expression of the form

$u_1 + u_2 + u_3 + \dots\dots\dots + u_n + \dots\dots\dots\dots\dots\dots\dots$

### • Definition of Infinite series:

Any expression of the form $u_1 + u_2 + u_3 + \dots\dots + u_n + \dots\dots$ ...... (i) or in short $\sum_{n=1}^{\infty} u_n$ or simply, $\sum u_n$ in which every term is followed by another term is called an infinite series. Here $u_1, u_2, u_3, \dots\dots, u_n, \dots\dots\dots$ are called its terms. $u_n$ is the n-th term of the infinite series.

$\therefore \sum u_n = u_1 + u_2 + u_3 + \dots\dots\dots + u_n + \dots\dots\dots$

The sum to n terms or the n-th partial sum of the series $\Sigma u_n$ is generally denoted by $S_n$

### • Convergent:

The infinite series (i) is said to be convergent if $\Sigma u_n$ exist and = a finite real number = S (say).

$\therefore S = \sum u_n = u_1 + u_2 + u_3 + \dots\dots\dots + u_n + \dots\dots\dots$

### • Divergent:

If $\sum u_n = +\infty$ or $-\infty$, then the series $\Sigma u_n$ is said to be diverge properly to $+\infty$ or $-\infty$ as the case may be.

## 4.10 Binomial Theorem for negative and fractional indices:

For any value of n the infinite series $1 + nx + \frac{n(n-1)}{2!}.x^2 + \dots\dots + \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}.x^r + \dots\dots$ ---------- (1)

will be convergent if and only if $-1 < x < 1$ i. e., $|x| < 1$ and the sum of the series will be $(1 + x)^n$.

$$\therefore (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}.x^2 + \dots\dots + \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}.x^r + \dots\dots, \; -1 < x < 1$$

**• General Term:** The general term is the (r + 1)-th term and is given by $t_{r+1} = \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}.x^r$

### Note :

**1. (i)** The limiting value of the sum of the infinite series (1) is $(1 + x)^n$.

**(ii)** The infinite series (1) converges to the sum $(1 + x)^n$, when $-1 < x < 1$.

**2. (i)** The above theorem is sometimes called the General Binomial Theorem.

**(ii)** The proof of this theorem is beyond the scope of the present treatise.

**(iii)** The above infinite series is divergent if $|x| > 1$, i.e., $x > 1$ and $x < -1$, and no conclusion can be made if, $|x| = 1$.

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(iv) The coefficient of each term after $(r + 1)$-th term from the beginning contains the factor $(n - r + 1)$, which becomes zero when $r = n + 1$. Hence, If n is a positive integer, the series (1) terminates after $(n + 1)$ terms. In this case the theorem becomes Binomial Theorem for a positive integral index.

(v) If n is negative integer or fraction (positive or negative), the series (1) does not terminate, since, $r = n + 1$ is not possible in this case as r assumes positive integral values only. Therefore it contains infinite number of terms.

(vi) Since ${}^nC_r$ is defined only for positive integral values of n and r, the coefficients here can not be written as ${}^nC_0, {}^nC_1, {}^nC_2, \dots\dots\dots, {}^nC_r$.

- **Expansion of $(a + x)^n$, $a > 0$ when n is a negative integer or fraction:**

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left[1+n.\frac{x}{a}+\frac{n(n-1)}{2!}.\left(\frac{x}{a}\right)^2+\dots\dots+\frac{n(n-1)(n-2)\dots.(n-r+1)}{r!}.\left(\frac{x}{a}\right)^r+\dots\dots\right],\quad -1<\frac{x}{a}<1$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \dots. + \frac{n(n-1)(n-2)\dots.(n-r+1)}{r!}a^{n-r}x^r + \dots.,\quad -a < x < a$$

- **GENERAL TERM :** The general term is the $(r + 1)$-th term and is given by $t_{r+1} = \frac{n(n-1)(n-2)\dots.(n-r+1)}{r!}a^{n-r}x^r$

- **SOME USEFUL EXPANSIONS:**

(i) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots\dots\dots \infty$

(ii) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots\dots \infty$

(iii) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots\dots\dots \infty$

(iv) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots\dots\dots \infty$

(v) $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + \dots\dots\dots \infty$

(vi) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + \dots\dots\dots \infty$

(vii) $(1 - x)^{-4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + \dots\dots\dots \infty$

(viii) $(1 + x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + \dots\dots\dots \infty$

# PROBLEM SET – II

[Problems with '*' marks are solved at the end of the problem set]

1. Expand upto four terms:

**(i)** $(1-2a)^{-2}$ **(ii)** $(1-x)^{\frac{3}{4}}$ **(iii)** $(4-3a)^{\frac{3}{2}}$ **(iv)** $(1-2a)^{-\frac{3}{2}}$ ***(v)** $\frac{1}{\sqrt[3]{(1-3x)^2}}$

2. **(i)** Write down the first four terms in the infinite series expansion of $(1+x)^{\frac{3}{4}}$ in ascending powers of x and state the condition of validity of the expansion..

**(ii)** If the value of n is other than a positive integer write down the expansion of $(1 + x)^n$. State the condition of validity (if any ) of the expansion. **[HS – 93]**

**(iii)** Find the first three terms in the expansion of $(2 + 3x)^{-3}$ and state the condition of validity of the expansion.

**(iv)** Write down the first four terms in the expansion of $(2 + x)^{-2}$ in ascending power of $\frac{1}{x}$, stating the condition of validity of the expansion.

3. ***(i)** Under what condition $(1-2x)^{-\frac{3}{4}}$ can be expanded into an infinite series? Find the (r + 1)th term in that expansion. **[HS – 94]**

***(ii)** What condition must be satisfied in order that $(1-2x)^{-\frac{1}{2}}$ may be expanded in ascending powers of x ? If the condition is satisfied, show that the (r + 1)th term of the expansion is $\frac{1.3.5.........(2r-1)}{r!}.x^r$

**(iii)** The condition of validity of expansion being assumed to be satisfied, find the 5th term in the expansion of $(1-2x)^{-\frac{1}{2}}$ in ascending powers of x.

4. ***(i)** Prove that the **coefficient** of $x^r$ in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\frac{(2r)!}{(r!)^2}$. **[HS – 97]**

**(ii)** If $|x|<\frac{1}{2}$, find the **coefficient** of $x^n$ in the expansion of $(1-2x)^{-\frac{1}{2}}$.

**(iii)** If $|x|<1$, find the **coefficient** of $x^3$ in the expansion of $\sqrt{1+x+x^2+x^3+.........\infty}$

**(iv)** If n is a positive integer and $|x|<1$, show that the **coefficient** of the nth term in the expansion of $(1 - x)^{-n}$ is twice the **coefficient** of (n – 1)th term.

**(v)** Find the **coefficient** of $x^r$ in the expansion of $(1-nx)^{-\frac{1}{n}}$

***(vi)** Find the **coefficient** of $x^5$ in the expansion of $\frac{1+x}{1-x}$

***(vii)** Find the **coefficient** of $x^r$ in the expansion of $\frac{1+x}{(1-x)^2}$.

**(viii)** Find the **coefficient** of $x^{50}$ in the expansion of $\frac{1-x}{(1+x)^2}$.

(ix) Find the **coefficient** of $x^{10}$ in the expansion of $\frac{1+x}{(1-x)^3}$.

*(x) Find the **coefficient** of $x^r$ in the expansion of $\frac{(1+x)^2}{(1-x)^3}$.

5. (i) Find the **coefficient** of $x^{10}$ in the expansion of $\frac{1+x+2x^2}{(1+x)^4}$.

(ii) Find the **coefficient** of $x^7$ in the expansion of $\frac{1-2x}{3+2x-x^2}$

*(iii) If $|x| < 1$, find the **coefficient** of $x^{10}$ in the expansion of $\frac{x}{(1-2x)(1-3x)}$

6. (i) Find the sum of the coefficients of first $(r + 1)$ terms in the expansion of $(1 - x)^{-3}$.

*(ii) Find the sum of the coefficients of first $(r + 1)$ terms in the expansion of $(1 - x)^{-5}$.

7. (i) If the numerical value of x is less than 1, find the expansion of $(1 + x + x^2 + x^3 + \ldots\ldots \text{ to } \infty)^{-3}$ **[HS – 91]**

(ii) Find the coefficient of x in the expansion of $(1 + x + x^2 + x^3 + \ldots\ldots \text{to } \infty)^{-n}$.

(iii) Find the coefficient of $x^r$ and $x^{15}$ in the expansion of $\left(1-3x+6x^2-10x^3+\ldots\ldots\right)^{\frac{1}{3}}$

(iv) If the numerical value of x is less than 1 and $x \neq 0$, show that the coefficient of any term in the expansion of $\left(1+2x+3x^2+4x^3+\ldots\ldots\right)^{\frac{1}{2}}$ is 1. **[HS – 90]**

(v) Find the coefficient of $x^r$ in the expansion of $(1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \text{to } \infty)^2$

*(vi) If $x = 1 + a + a^2 + \ldots\ldots \text{ to } \infty\ (|a| < 1)$ & $y = 1 + b + b^2 + \ldots\ldots \text{ to } \infty\ (|b| < 1)$,

Prove that, $1+ab+a^2b^2+\ldots\ldots\infty = \frac{xy}{x+y-1}$

(vii) Show that, $(1 + x + x^2 + \ldots\ldots \infty)^2 = 1 + 2x + 3x^2 + \ldots\ldots + nx^{n-1} + \ldots\ldots$, if x is numerically less than 1.

*(viii) If $-1 < x < 1$, prove that $(1 + x + x^2 + \ldots\ldots \infty)(1 + 2x + 3x^2 + \ldots\ldots \infty)$

$$= \frac{1}{2}(1.2 + 2.3\,x + 3.4\,x^2 + \ldots\ldots \infty)$$ **[HS – 02]**

(ix) Prove that $(1 + x + x^2 + \ldots\ldots \infty)(1 - x + x^2 - \ldots\ldots \infty) = 1 + x^2 + x^4 + \ldots\ldots \infty$ **[WBSC – 19]**

(x) If $0 < a, x < 1$, prove that, $\frac{1}{(1-x)(1-ax)} = 1 + (1 + a)x + (1 + a + a^2)x^2 + (1 + a + a^2 + a^3)x^3 + \ldots\ldots$

8. (i) If If $y = \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \ldots\ldots$, show that $y^2 + 2y - 2 = 0$

(ii) If $y = x - x^2 + x^3 - x^4 + \ldots\ldots$ to $\infty$, show that, $x = y + y^2 + y^3 + \ldots\ldots$ to $\infty$

(iii) If $x = \frac{1}{3}y - \frac{1.4}{3^2.2!}y^2 + \frac{1.4.7}{3^3.3!}y^3 - \ldots\ldots\infty$ show that, $y = 3x + 6x^2 + 10x^3 + \ldots\ldots$

*(iv) If $y = 2x + 3x^2 + 4x^3 + \ldots\ldots$, $-1 < x < 1$, show that $x = \frac{1}{2}y - \frac{3}{8}y^2 + \frac{5}{16}y^3 - \ldots\ldots$

(v) If $y = x + x^2 + 2x^3 + \ldots\ldots + \frac{(2n)!.x^{n+1}}{n!.(n+1)!} + \ldots\ldots$, show that $y^2 - y + x = 0$.

# ANSWERS

**1. (i)** $1 + 4a + 12a^2 + 32a^3 + \ldots\ldots,\ -\frac{1}{2} < a < \frac{1}{2}$ **(ii)** $1 - \frac{3}{4}x - \frac{3}{32}x^2 - \frac{5}{128}x^3 - \ldots\ldots\ldots,\ -1 < x < 1$

**(iii)** $8 - 9a + \frac{27}{16}a^2 + \frac{27}{128}a^3 + \ldots\ldots\ldots\ -\frac{4}{3} < a < \frac{4}{3}$ **(iv)** $1 + 3a + \frac{15}{2}a^2 + \frac{35}{2}a^3 + \ldots\ldots\ldots\ -\frac{1}{2} < a < \frac{1}{2}$

**(v)** $1 + 2x + 5x^2 + \frac{40}{3}x^3 + \ldots\ldots\ldots\ -\frac{1}{3} < x < \frac{1}{3}$

**2. (i)** $1 + \frac{3}{4}x - \frac{3}{32}x^2 + \frac{5}{128}x^3 - \ldots\ldots\ldots,\ -1 < x < 1$

**(ii)** $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots,\ -1 < x < 1$

**(iii)** $\frac{1}{8} - \frac{9}{16}x + \frac{27}{16}x^2 - \ldots\ldots\ldots\ |x| < \frac{2}{3}$ **(iv)** $\frac{1}{x^2} - \frac{4}{x^3} + \frac{12}{x^4} - \frac{32}{x^5} + \ldots\ldots,\ -1 < \frac{2}{x} < 1$

**3. (i)** $-\frac{1}{2} < x < \frac{1}{2};\ \frac{3.7.11\ldots\ldots\ldots(4r-1).x^r}{2^r.r!}$ **(ii)** $-\frac{1}{2} < a < \frac{1}{2}$ . **(iii)** $\frac{35}{8}x^4$

**4. (ii)** $\frac{1.3.5\ldots\ldots\ldots(2n-1)}{n!}$ **(iii)** $\frac{5}{16}$ **(v)** $\frac{(n+1)(2n+1)\ldots\ldots\ldots\{(r-1)n+1\}}{r!}$ **(vii)** $2r + 1$ **(viii)** $101$ **(ix)** $(r+1)^2$ **(x)** $2r^2 + 2r + 1$

**5. (i)** $396$ **(ii)** $-\frac{1}{4}\left(3 + \frac{5}{3^8}\right)$ **(iii)** $3^{10} - 2^{10}$ **6. (i)** $\frac{1}{6}(r+1)(r+2)(r+3)$ **(ii)** $\frac{1}{120}(r+1)(r+2)(r+3)(r+4)(r+5)$

**7. (i)** $1 - 3x + 3x^2 - x^3$ **(ii)** $-n$ **(iii)** $(-1)^r, -1$ **(v)** $\frac{1}{6}(r+1)(r+2)(r+3)$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1(v) Solution :** $\frac{1}{\sqrt[3]{(1-3x)^2}} = \frac{1}{(1-3x)^{\frac{2}{3}}} = (1 - 3x)^{-\frac{2}{3}}$

$$= 1 + \left(-\frac{2}{3}\right)(-3x) + \frac{-\frac{2}{3}\left(-\frac{2}{3}-1\right)}{2!}(-3x)^2 + \frac{-\frac{2}{3}\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{3!}(-3x)^3 + \ldots\ldots\ldots\text{to } \infty$$

$$= 1 + 2x + \frac{\frac{2}{3}\times\frac{5}{3}}{2}3^2 . x^2 + \frac{\frac{2}{3}\times\frac{5}{3}\times\frac{8}{3}}{6}3^3 . x^3 + \ldots\ldots\text{to } \infty \quad \left[-\frac{1}{3} < x < \frac{1}{3}\right]$$

$$= 1 + 2x + 5x^2 + \frac{40}{3}x^3 + \ldots\ldots\ldots\text{to } \infty \quad \left[-\frac{1}{3} < x < \frac{1}{3}\right] \quad \textbf{(Ans)}$$

**3.(i) Solution:** The required condition so that $(1-2x)^{-\frac{3}{4}}$ can be expanded into an infinite series is $|2x| < 1$ or,

$|x| < \frac{1}{2}$ or, $-\frac{1}{2} < x < \frac{1}{2}$ and the $(r + 1)$th term,

$$t_{r+1} = \frac{-\frac{3}{4}\left(-\frac{3}{4}-1\right)\left(-\frac{3}{4}-2\right)\left(-\frac{3}{4}-3\right)\ldots\ldots\ldots\ldots\left(-\frac{3}{4}-r+1\right)}{r!}(-2x)^r$$

$$= (-1)^{2r} . \frac{3.7.11\ldots\ldots\ldots\ldots(4r-1)}{4^r.r!}2^r.x^r = \frac{3.7.11\ldots\ldots\ldots\ldots(4r-1)}{2^r.r!}.x^r \quad \textbf{(Ans)}$$

**3(ii) Solution :** The condition for the expansion of $(1-2x)^{-\frac{1}{2}}$ in ascending powers of x is $|2x|<1$

or, $-1<|2x|<1 \Rightarrow -\frac{1}{2}<x<\frac{1}{2}$

If the above condition is satisfied, then the (r + 1) th term in the expansion of $(1-2x)^{-\frac{1}{2}}$ is

$$t_{r+1}=\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\ldots\left(-\frac{1}{2}-r+1\right)}{r!}\times(-2x)^r=\frac{(-1)^r\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\ldots\ldots\frac{2r-1}{2}}{r!}\times 2^r\times x^r\times(-1)^r$$

$$=(-1)^{2r}\frac{1\cdot3\cdot5\ldots\ldots(2r-1)}{2^r\cdot r!}\cdot2^r\cdot x^r=\frac{1\cdot3\cdot5\ldots\ldots(2r-1)}{r!}\cdot x^r \text{ (Proved)}$$

**4. (i) Solution:** The (r + 1)th term in the expansion of $(1-4x)^{-\frac{1}{2}}$

is, $t_{r+1}=\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}.(-4x)^r$, here $n=-\frac{1}{2}$

$$=\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\left(-\frac{1}{2}-r+1\right)}{r!}.(-4)^r(x)^r=(-1)^r\frac{1.3.5\ldots\ldots(2r-1)}{r!\,2^r}.(-1)^r.4^r.x^r$$

$$=(-1)^{2r}\frac{1.2.3.4.5.6\ldots\ldots(2r-1).2r}{r!2^r(2.4.6\ldots\ldots2r)}.4^r.x^r=\frac{(2r)!.2^{2r}.x^r}{r!.2^r.2^r.(1.2.3\ldots\ldots r)}=\frac{(2r)!.x^r}{r!.r!}=\frac{(2r)!}{(r!)^2}x^r$$

Hence the required coefficient of $x^r$ in the expansion of $(1-4x)^{-\frac{1}{2}}=\frac{(2r)!}{(r!)^2}$ **(Proved)**

**4.(vi) Solution :** $\frac{1+x}{1-x}=(1+x)(1-x)^{-1}=(1+x)(1+x+x^2+x^3+x^4+x^5+\ldots\ldots)$

The required **coefficient** of $x^5$ in the expansion = $1\times1+1\times1=2$ **(Ans)**

**4(vii) Solution :** Given expression : $\frac{1+x}{(1-x)^2}=(1+x)(1-x)^{-2}$

$$=(1+x)\left\{1+(-2)(-x)+\frac{-2(-2-1)}{2!}(-x)^2+\ldots\ldots\ldots\right.$$

$$\left.+\frac{-2(-2-1)\ldots\ldots(-2-r+2)}{(r-1)!}(-x)^{r-1}+\frac{-2(-2-1)\ldots\ldots(-2-r+1)}{r!}(-x)^r+\ldots\ldots\right\}$$

$$=(1+x)\left\{1+2x+3x^2+\ldots\ldots\ldots+\frac{(-1)^{r-1}\cdot r!}{(r-1)!}\cdot(-1)^{r-1}\cdot x^{r-1}+\frac{(-1)^r\cdot(r+1)!}{r!}\cdot(-1)^r\cdot x^r+\ldots\ldots\ldots\right\}$$

$$=(1+x)\left\{1+2x+3x^2+\ldots\ldots+r\cdot x^{r-1}+(r+1)\cdot x^r+\ldots\ldots\ldots\right\}$$

$$\left[\because (-1)^{r-1}(-1)^{r-1}=(-1)^{2(r-1)}=1 \text{ and } (-1)^r(-1)^r=(-1)^{2r}=1\right]$$

Hence, the required coefficient of $x^r$ in the above expansion $=1\times(r+1)+1\times r=r+1+r=2r+1$ **(Ans)**

**4(x) Solution :** Given expression : $\frac{(1+x)^2}{(1-x)^3}=(1+x)^2(1-x)^{-3}=(1+x)^2\left[1+(-3)(-x)+\frac{-3(-3-1)}{2!}(-x)^2+\ldots\ldots\ldots\right.$

$$+\frac{-3(-3-1)\ldots\ldots(-3-r+3)}{(r-2)!}(-x)^{r-2}+\frac{-3(-3-1)\ldots\ldots(-3-r+2)}{(r-1)!}(-x)^{r-1}$$

$$\left.+\frac{-3(-3-1)\ldots\ldots(-3-r+1)}{r!}(-x)^r+\ldots\ldots\right]$$

$$= \left(1+2x+x^2\right)\left[1+3x+\frac{3\cdot 4}{2!}x^2+\ldots\ldots+\frac{(-1)^{r-2}\cdot 3\cdot 4\ldots\ldots r}{(r-2)!}(-1)^{r-2}x^{r-2}\right.$$

$$\left.+\frac{(-1)^{r-1}\cdot 3\cdot 4\ldots\ldots r(r+1)}{(r-1)!}(-1)^{r-1}x^{r-1}+\frac{(-1)^r\cdot 3\cdot 4\ldots\ldots\ldots r(r+1)(r+2)}{r!}(-1)^r x^r+\ldots\ldots\ldots\ldots\right]$$

$$= \left(1+2x+x^2\right)\left[1+3x+\frac{3\cdot 4}{2}x^2+\ldots\ldots\ldots+\frac{(r-1)\cdot r}{2}x^{r-2}+\frac{r\cdot(r+1)}{2}x^{r-1}+\frac{(r+1)\cdot(r+2)}{2}x^r+\ldots\ldots\ldots\ldots\right]$$

By actual multiplication we get, coefficient of $x^r$

$$= 1\times\frac{(r+1)(r+2)}{2}+2\times\frac{r(r+1)}{2}+1\times\frac{(r-1)r}{2} = \frac{r^2+3r+2+2r^2+2r+r^2-r}{2} = \frac{4r^2+4r+2}{2} = 2r^2+2r+1 \text{ (Ans)}$$

**5(iii) Solution :** $\frac{x}{(1-2x)(1-3x)} = \frac{1}{1-3x}-\frac{1}{1-2x} = (1-3x)^{-1}-(1-2x)^{-1}$

$=\{1+(3x)+(3x)^2+(3x)^3+\ldots\ldots+(3x)^{10}+\ldots\ldots\} - \{1+(2x)+(2x)^2+(2x)^3+\ldots\ldots+(2x)^{10}+\ldots\ldots\}$

Therefore the coefficient of $x^{10} = 3^{10}-2^{10}$ **(Ans)**

**6(ii) Solution :** Let $(1-x)^{-5} = a_0+a_1x+a_2x^2+\ldots\ldots+a_rx^r+\ldots\ldots\ldots\infty$

Then sum of the coefficients of first $(r+1)$ terms in the given expansion $= a_0+a_1+a_2+\ldots\ldots\ldots+a_r$

Now, $(1-x)^{-1}(1-x)^{-5} = (1-x)^{-6}$

or, $(1-x)^{-1}(a_0+a_1x+a_2x^2+\ldots\ldots\ldots\ldots+a_rx^r+\ldots\ldots\ldots\infty) = (1-x)^{-6}$

or, $(1+x+x^2+\ldots\ldots+x^{r-1}+x^r+\ldots\ldots\infty)\times(a_0+a_1x+a_2x^2+\ldots\ldots\ldots+a_rx^r+\ldots\ldots\infty) = (1-x)^{-6}$ ...................(1)

Now, the coefficient of $x^r$ on L.H.S. of (1) is $a_0+a_1+a_2+\ldots\ldots\ldots+a_r$

Again, the coefficient of $x^r$ in the expansion of $(1-x)^{-6}$

$$= \frac{-6(-6-1)(-6-2)\ldots\ldots\ldots\ldots(-6-r+1)}{r!}\cdot(-1)^r = \frac{(-1)^r 6\cdot 7\cdot 8\ldots\ldots\ldots\ldots(r+5)\cdot(-1)^r}{r!}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\ldots\ldots\ldots\ldots r\cdot(r+1)\ldots\ldots\ldots(r+5)}{5!\cdot r!} = \frac{r!(r+1)(r+2)(r+3)(r+4)(r+5)}{5!\cdot r!}$$

$$= \frac{(r+1)(r+2)(r+3)(r+4)(r+5)}{5!}$$

$$\therefore\ a_0+a_1+a_2+\ldots\ldots\ldots+a_r = \frac{(r+1)(r+2)(r+3)(r+4)(r+5)}{5!} \text{ (Ans)}$$

**7. (vi) Solution:** $x = 1+a+a^2+\ldots.. \text{ to } \propto = (1-a)^{-1}$

or, $x = \frac{1}{1-a}$ Similarly, $y = \frac{1}{1-b}$ $\therefore\ xy = \frac{1}{(1-a)(1-b)}$

and $x+y-1 = \frac{1}{1-a}+\frac{1}{1-b}-1 = \frac{1-a+1-b-1+a+b-ab}{(1-a)(1-b)} = \frac{1-ab}{(1-a)(1-b)}$

$$\therefore\ \frac{xy}{x+y-1} = \frac{\frac{1}{(1-a)(1-b)}}{\frac{1-ab}{(1-a)(1-b)}} = \frac{1}{1-ab} = (1-ab)^{-1} = 1+ab+a^2b^2+\ldots\ldots \text{ to } \propto$$

$$\therefore\ 1+ab+a^2b^2+\ldots\ldots \text{ to } \propto = \frac{xy}{x+y-1} \text{ (Proved)}$$

**7(viii) Solution :** Given expression :

$$(1 + x + x^2 + \dots\dots\dots \infty)(1 + 2x + 3x^2 + \dots\dots\dots\dots \infty)$$

$$= (1-x)^{-1}(1-x)^{-2} = (1-x)^{-3}$$

$$= 1 + (-3)(-x) + \frac{-3(-3-1)}{2!}(-x)^2 + \dots\dots = 1 + 3x + \frac{3\cdot 4}{2}x^2 + \dots\dots\dots$$

$$= \frac{1}{2}\left(1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \dots\dots\dots\dots\dots\right) \textbf{(Proved)}$$

**8.(iv) Solution:** Given, $y = 2x + 3x^2 + 4x^3 + \dots\dots\dots$

or, $1 + y = 1 + 2x + 3x^2 + 4x^3 + \dots\dots\dots = (1-x)^{-2}$ or, $1 - x = (1+y)^{-\frac{1}{2}}$

or, $1 - x = 1 + \left(-\frac{1}{2}\right)y + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}y^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}y^3 + \dots\dots\dots$

or, $-x = -\frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3 + \dots\dots$ or, $x = \frac{1}{2}y - \frac{3}{8}y^2 + \frac{5}{16}y^3 - \dots\dots$ **(Proved)**

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# MISCELLANEOUS :: OBJECTIVE TYPE MULTIPLE CHOICE

1. The coefficient of $\frac{1}{x^2}$ in the expansion of $\left(1-\frac{1}{x}\right)^{10}$ is : (a) –45 (b) 54 (c) 45 (d) None. [WBSC – 04, 07]

2. The coefficient of $\frac{1}{x^3}$ in the expansion of $\left(1+\frac{1}{x}\right)^{10}$ is : (a) 120 (b) 45 (c) 60 (d) none of these. [WBSC – 05]

3. Coefficient of the 4th term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$ is – (a) 20 (b) 120 (c) 45 (d) None. [WBSC – 07]

4. The coefficient of $\frac{1}{x^6}$ in the expansion of $\left(1-\frac{1}{x^2}\right)^{10}$ is – (a) 45, (b) –120, (c) 120, (d) 90. [WBSC – 08]

5. Sixth term in the expansion of $\left(x-\frac{1}{x}\right)^9$ is (a) $-{}^9C_5\cdot\frac{1}{x}$ (b) ${}^9C_5\cdot\frac{1}{x}$ (c) ${}^9C_6$ (d) None [WBSC – 09]

6. Coefficient of 6th term in the expansion of $\left(a-\frac{1}{2a}\right)^{10}$ is – (i) $\frac{245}{8}$ (b) $-\frac{63}{8}$ (c) –63 (d) None. [WBSC – 10, 12]

7. Coefficient of 6th term in the expansion of $\left(x-\frac{1}{3x}\right)^{10}$ is – (i) –252 (b) $-\frac{28}{27}$ (c) $-\frac{63}{8}$ (d) None. [WBSC – 11]

8. In the expansion of $\left(x+\frac{1}{x}\right)^6$, the 5th term will be $6x^4$ – (a) True (b) False. [WBSC – 08]

9. The second term in the expansion of $(2x + 3y)^5$ is – (a) $46x^2y^3$ (b) $30x^3y^2$ (c) $240x^4y$ (d) $810xy^4$ [WBSC – 08]

## SUBJECTIVE TYPE

1. Find the coefficient of x in the expansion of $\left(x^2+\frac{a^2}{x}\right)^5$ [WBSC – 08, 11]

2. Find the coefficient of $x^{10}$ in the expansion of $\left(x^2-\frac{1}{x^3}\right)^{10}$ [WBSC – 09, 12]

3. Find the co-efficient of x in the expansion of $\left(1-2x^3+3x^5\right)\left(1+\frac{1}{x}\right)^{10}$ [WBSC – 07, 15]

4. Find the coefficient of $x^{-2}$ in the expansion of $\left(3x-\frac{7}{x}\right)^8$. [WBSC – 04]

5. Find the middle term in the expansion of $\left(3x-\frac{1}{2x}\right)^8$ [WBSC – 10, 17]

6. Prove that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5\ldots\ldots(2n-1)}{n}2^n x^n$ [WBSC – 18]

7. If the coefficient of $x^7$ in the expansion of $\left(px^2+\frac{1}{qx}\right)^{11}$ be equal to the coefficient of $x^{-7}$ in the expansion of $\left(px-\frac{1}{qx^2}\right)^{11}$, prove that pq = 1. [WBSC – 03, 19]

8. If the third and fourth terms in the expansion of $\left(2x+\frac{1}{8}\right)^{10}$ are equal find the value of x. [WBSC – 14, 16]

9. Prove that $(1 + x + x^2 + \ldots\ldots \infty)(1 - x + x^2 - \ldots\ldots \infty) = 1 + x^2 + x^4 + \ldots\ldots \infty$ [WBSC – 19]

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## 4.11 The series e :

The infinite series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\dots+\frac{1}{r!}+\dots\dots\infty$ is denoted by the letter e and is called e series.

Thus, $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\dots+\frac{1}{r!}+\dots\dots\infty$.

### • Properties of the e series :

**(i) The value of e is finite and lies between 2 and 3 i.e., $2<e<3$**

**(ii) e is an incommensurable quantity.**

## 4.12 The Exponential Series :

For all real values of x, $e^x$ and $a^x$ $[a>0,\ a\neq 1]$ are called exponential functions and the expansion of $e^x$ and $a^x$ are called exponential series,

For all real x, the infinite series,

$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots+\frac{x^r}{r!}+\dots\dots$to $\infty$ is convergent and converges to $e^x$.

$\therefore\ e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots+\frac{x^r}{r!}+\dots\dots$to $\infty$ [ Its proof is beyond the scope of the present treatise]

Expansion of $e^x$: $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots+\frac{x^r}{r!}+\dots\dots$to $\infty$ ............ (1)

Replacing x by $-x$ in (1) we get,

Expansion of $e^{-x}$: $e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots\dots+(-1)^r.\frac{x^r}{r!}+\dots\dots$to $\infty$ ............ (2)

Putting $x=-1$ in (1) we get,

Expansion of $e^{-1}$: $e^{-1}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots\dots+\frac{(-1)^r}{r!}+\dots\dots$to $\infty$ ............ (3)

Replacing x by $x.\log_e a$, $[a>0,\ a\neq 1]$ in (1) we get,

$$e^{x.\log a}=1+\frac{x.\log_e a}{1!}+\frac{x^2.(\log_e a)^2}{2!}-\frac{x^3.(\log_e a)^3}{3!}+\dots\dots\text{to }\infty$$

Expansion of $a^x$: $a^x=1+\frac{x}{1!}(\log_e a)+\frac{x^2}{2!}(\log_e a)^2-\frac{x^3}{3!}(\log_e a)^3+\dots\dots$to $\infty$ ............ (4)

$[\because$ for all $a>0$ and $a\neq 1$, $e^{x\log a}=e^{\log a^x}=a^x]$

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## 4.13 LOGARITHMIC SERIES :

If $-1 < x \le 1$, then the infinite series

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots + (-1)^{n-1}.\frac{x^n}{n} + \ldots\ldots\ldots$ to $\infty$ is convergent and converges to $\log_e(1+x)$.

[ Its proof is beyond the scope of the present practise.]

$$\therefore \quad \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{ to } \infty \quad -1 < x \le 1 \quad \text{------------ (1)}$$

Replacing x by $-x$ on both sides of (1) we get,

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots \text{ to } \infty \quad -1 \le x < 1 \quad \text{------------ (2)}$$

Putting $x = 1$ in (1) we get,

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots \text{ to } \infty \quad \text{------------ (3)}$$

Subtracting (2) from (1) we get,

$$\log_e(1+x) - \log_e(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots\right)$$

$$\text{or, } \quad \frac{1}{2}.\log\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots \quad -1 < x \le 1 \quad \text{------------ (4)}$$

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# UNIT – 2

## VECTOR ALGEBRA　　PAGE NO



N. Tute

**DEFINITION AND VECTOR REPRESENTATION**

# 5.1. Introduction.

## 1. Scalar and vector:

• A **scalar quantity** or briefly a scalar, has magnitude but is not related to any direction in space.

It is completly specified by a number and is subject to ordinary arithmetical laws.

• If water of mass 200 grammes be mixed with milk of mass 400 grammes, then the total quantity becomes 600 grammes, which is obtained by the simple arithmetical summation. Thus **mass** is a scalar quantity

• The meaning of scalar quantity is fully conveyed by expressing its magnitude alone in terms of unit quantity of the same type. Thus, when we say that the volume of a body is 50 cubic centimeters it is understood that the volume of the body is 50 times as much as the volume of a cube on a side of one centimeter and no information as regards direction is required since volume is not related to any direction in space. Therefore **volume** is a scalar quantity.

• Similarly, quantities expressing **length**, **density**, **temperature**, **potential**, **electric charge** etc. are all scalars.

• A **vector quantity** or briefly a **vector** is a physical quantity having both **magnitude** and definite **direction** in space. To give an idea of a displacement completely, both its magnitude and direction must be mentioned. So **displacement** is a vector quantity. Similarly, **acceleration**, **velocity**, **force**, **momentum** etc. are examples of vector quantities.

## 2. Directed line segment:

Let AB be the **line segment**[Fig. (i)] of a given indefinite straight line L. We may consider the line segments AB[Fig. (ii)] and BA[Fig. (iii)] as different segments by giving each a **direction** or **sence**. Thus AB shall be directed from A to B and BA shall be directed from B to A i. e., oppositely to AB. These line segments are then called **directed line segments**. Such directed line segments are represented vectors and are usually denoted by $\overrightarrow{AB}$ and $\overrightarrow{BA}$, to distinguish them from the undirected line segments AB and BA.

• In $\overrightarrow{AB}$ we call A the **initial point** and B the **terminal point** and in $\overrightarrow{BA}$ we say B is the **initial point** and A is the **terminal point**.

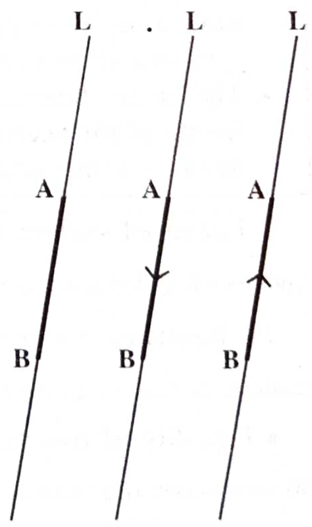



Fig-(i) Fig-(ii) Fig-(iii)

• In the physical sense $\overrightarrow{AB}$ means the displacement of the point A to the point B and $\overrightarrow{BA}$ means the displacement of the point B to the point A.

• With every **directed line segment** we attribute three characteristics: **(i) Length**; **(ii) Support**; **(iii) Sence**

**(i) Length:** The length of the directed line segment $\overrightarrow{AB}$ is the length of the line segment AB, a scalar. The notation for this length is $|\overrightarrow{AB}|$; read as **absolute value** of $\overrightarrow{AB}$, or **length** of $\overrightarrow{AB}$.

• Note that $|\overrightarrow{AB}| = |\overrightarrow{BA}|$.

**(ii) Support:** The support of $\overrightarrow{AB}$ is the line L of indefinite length of which the AB is a portion.

**(iii) Sence:** The sence of $\overrightarrow{AB}$ is indicated by the order in which the letters are stated i.e., from A towards B.

## Note:

(i) $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have same **length**, same **support** but are of opposite **sences**; hence they are different (directed) line segments. We call them **equal and opposite** directed line segments.

(ii) $\overrightarrow{AB}$ and $\overrightarrow{CD}$ [Fig. (iv)] have same **support** and same **sence**. Now if $|\overrightarrow{AB}| = |\overrightarrow{CD}|$, then they will be treated as **equal directed segments**.

(iii) $\overrightarrow{AB}$ and $\overrightarrow{EF}$ [Fig.(iv) & (v)] have parallel supports L and M. If $|\overrightarrow{AB}| = |\overrightarrow{EF}|$ we may treat them as **equal segments** in the **wider sence** of the term.

• If two segments are not on the **same** or **parallel** suports they can never be equal.

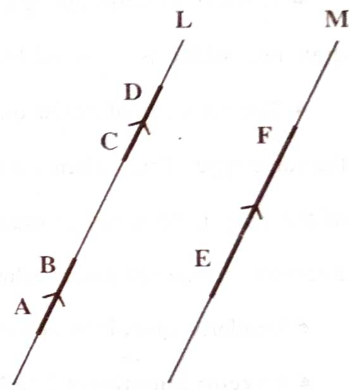


Fig-(iv) Fig-(v)

## 3. Vector notation:

• A vector from A to B [Fig. (ii)] is denoted by $\overrightarrow{AB}$, where it is necessary to specify its end points **otherwise**, by a single letter with an **arrow overhead**, i..e., $\vec{a}$, $\vec{b}$, $\vec{c}$, .........., or by a single symbol such as **a** printed in **bold–faced** type (Clarendon type).

• The length of the vector $\overrightarrow{AB}$ or $\vec{a}$ or **a** is denoted by $|\overrightarrow{AB}|$ or $|\vec{a}|$ or $|\mathbf{a}|$ or simply by a; read: **length of the vector** $\overrightarrow{AB}$ or $\vec{a}$ or **a** or **absolute value** of $\overrightarrow{AB}$ or $\vec{a}$ or **a**; $|\vec{a}|$ is also known as modulus or magnitude of the vector $\vec{a}$.

**4. Localised vector:** When a line segment is restricted to **pass through a given point** in space, the vector it represents is called a localized vector.[ Otherwise the vector is called a free or non–localized vector.]

The importance of a localized vector lies in the fact that effect depends also on the point through which it is restricted to pass, e. g., the effect of a force acting on a body.

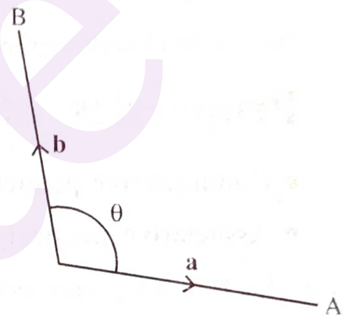
• **Equality of two vectors:** Two vectors **a** and **b** are equal, if they have **(i)** same length i. e., $|\mathbf{a}| = |\mathbf{b}|$; **(ii)** same sence; **(iii)** same or parallel supports and written as **a** = **b**.

From definition it is clear that it does not demand that the vectors must have the same support. The vectors conforming to such a definition of equality are said to be **free**.

- **Free vector:** A free vector is a directed line segment occupying any position in space; its initial point may be choosen arbitrarily on parallel supports.

- **Line vector:** In some applications, however, vectors which are restricted to lie on a given line (e. g., forces acting on a rigid body are restricted to lie along their lines of action). Such a vector which is confind to a definite line of support is called a **Line vector**. Two such line vectors are equal if their **lengths**, **support** and **sences** are same.

## Angle between two vectors:




Let the two vectors **a** and **b** be represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Then the angle AOB is said to be the angle between the vectors $\overrightarrow{OA}$ (= **a** ) and $\overrightarrow{OB}$ (= **b** ). Evidently, this angle does not exceed $\pi$. If we denote this angle by $\theta$, then $0 < \theta < \pi$.

- When this angle is $\frac{\pi}{2}$, the vectors are said to be **perpendicular** and when it is 0 or $\pi$, they are said to be **parallel** or **coincident**.

- **Unit vector:** A vector whose **length is unity** is a unit vector. Unit vector thus indicates the direction. The unit vector corresponding to any proper vector **a** is $\frac{\mathbf{a}}{|\mathbf{a}|}$.

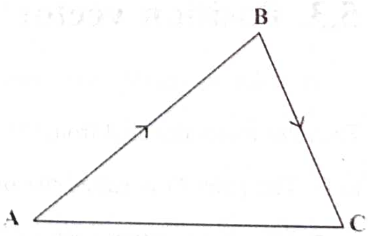
- **Zero vector:** A line vector whose initial and terminal points coincide is called zero or a null vector. The length or modulus of a null vector is zero but its direction is **indeterminate**; i. e., it may be regarded to possess any direction. All zero vectors are regarded as equal and so we have used **zero** in **bold–faced** type to denote a null vector or zero vector i. e., by **0**.

- **Proper vector:** Every vector which is not a null vector is called a **proper vector**.

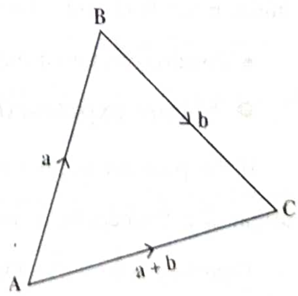
- **Co–initial vectors:** Vectors having the same initial point are called co–initial vectors.

**Unless otherwise stated, we mean a free vector when we use the term vector.**

## 5.2. Addition of vectors:

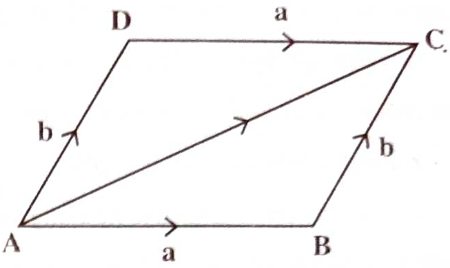



A rectilinear displacement or a translation from A to B may be represented by the vector $\overrightarrow{AB}$. If a particle be given two displacements successively, one from A to B and a second from B to C, the result is the same as if the particle were given a single displacement from A to C. This suggests that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.




- **Triangular law of addition:** Given two vectors **a** and **b**; draw **b** from the terminal point of **a**, then the vector directed from the initial point of **a** to the terminal point of **b** is called the sum of **a** and **b**, written as **a** + **b**. In Fig. $\overrightarrow{AB}$ = **a**, $\overrightarrow{BC}$ = **b**; then complete the triangle ABC; $\overrightarrow{AC}$ will now represent the sum **a** + **b**.

• **Parallelogram law of addition:** Completing the parallelogram ABCD with AB and BC as sides, we note $\overrightarrow{AD} = \overrightarrow{BC} = \mathbf{b}$, since their supports are parallel, lengths and sences are same.




Similarly, $\overrightarrow{AB} = \overrightarrow{DC} = \mathbf{a}$

Hence, $\overrightarrow{AC} = \mathbf{a} + \mathbf{b} = \overrightarrow{AB} + \overrightarrow{BC}$ [from triangular law] or $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$.

Thus the sum of two co–initial vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is given by $\overrightarrow{AC}$, where AC is the diagonal of the parallelogram ABCD having AB and AD as adjacent sides.

This is called **Parallelogram law of addition** of two vectors.

## Properties of addition:

• **Commutative property:** Addition of two vectors **a** and **b** obey the commutative property, i. e., $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

• **Associative property:** Vector addition is independent of the way its elements are associated in groups, i. e., if **a**, **b**, **c** be any three vectors, then $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.

• **Negative of a vector:** The vector **a** is negative of the vector **b** or vice versa, if $\mathbf{a} + \mathbf{b} = \mathbf{0}$.

From triangle law, $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \mathbf{0}$. Hence, $\overrightarrow{AB}$ is the negative of $\overrightarrow{BA}$ or vice versa. If $\overrightarrow{AB} = \mathbf{a}$, we can write $\overrightarrow{BA} = -\mathbf{a}$. Thus the negative of $\overrightarrow{AB}$ is a vector $\overrightarrow{BA}$ of same length, same support but of opposite sense.

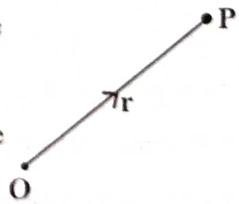
Note also that $-(-\overrightarrow{AB}) = \overrightarrow{AB}$

• **Subtraction:** Subtraction of **b** from **a** is the addition of the negative of **b** with **a**, i. e., $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

• **Scalar multiplication:** The product of a vector **a** by a scalar m, written as ma, or am, is a vector whose **(i)** length is $|m|$ times that of **a** **(ii)** support is same or parallel to that of **a** **(iii)** sense is same or opposite to that of **a** according as m is positive or negative.

## 5.3. Position vector of a point:

In order to specify the position of a point P in space we require to choose an arbitrary point O as origin. Then the vector directed from O to P i. e., $\overrightarrow{OP}$ is called the position vector of the point P relative to O. The point O is called the **origin** or the **origin of reference**.

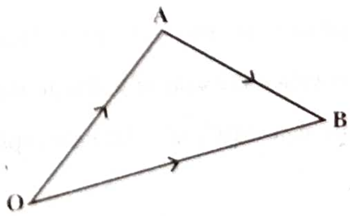



• Thus, the position vector of the point P with reference to O as origin is the vector whose initial point is O and whose terminal point is at P.

• Position vector of any point is **r** means $\overrightarrow{OP} = \mathbf{r}$, where O is the origin of reference.

• **Vector expressed in terms of position vectors of its end point:**

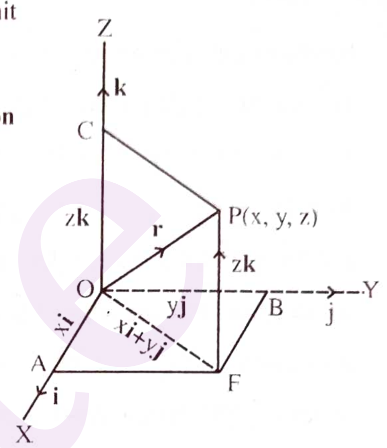
If the position vectors of the end points of vector $\overrightarrow{AB}$, with respect to O as the origin of reference, be **a** and **b**, then from figure, it is evident that $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.




Therefore, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.

## 5.4. Rectangular resolution of a vector:

Let OX, OY, OZ be three mutually perpendicular axes and form a right handed system of directions i. e., at O the rotation about OX, OY, OZ are from Y to Z, Z to X, X to Y respectively. Let **i**, **j**, **k** be the unit vectors along the three mutually perpendicular axes OX, OY, OZ.




If $\overrightarrow{OP} = \mathbf{r}$ and the co–ordinates of P be (x, y, z) then, $\overrightarrow{OA}$ = the **orthogonal projection** of **r** in OX direction $= \mathbf{i}x$.

Similarly, $\overrightarrow{OB} = \mathbf{j}y$, $\overrightarrow{OC} = \mathbf{k}z$.

Here, x, y, z are the rectangular components of **r**.

Now, $\overrightarrow{OP} = \overrightarrow{OF} + \overrightarrow{FP}$

$= (\overrightarrow{OA} + \overrightarrow{AF}) + \overrightarrow{FP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ or $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

or $OP^2 = OF^2 + FP^2 = (OA^2 + AF^2) + FP^2 = OA^2 + OB^2 + OC^2$

or $OP^2 = x^2 + y^2 + z^2$ or $OP = \sqrt{x^2 + y^2 + z^2}$ or $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

If OP makes angles α, β, γ with **i**, **j**, **k** directions, its direction cosines are cosα, cosβ, cosγ and are equal to $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$ respectively.

**Note: 1.** If x, y, z be positive, the respective directions of $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ are the same as those of **i**, **j**, **k**; if any one of them be negative then the corresponding direction will be opposite.

**Note: 2.** The vector **r** is called the resultant of the three vectors **i**x, **j**y, **k**z which are called the vector components.

### PROBLEM SET – I

1. (i) If the vertices of a triangle be the points **i** – **j** + 2**k**, 2**i** + 3**j** + 4**k** and 3**i** + 3**j** – 4**k**, what are the vectors determined by its sides? Also find the vectors represented by the **medians** of the triangle.

[**Ans.** **i** + 4**j** + 2**k**, **i** – 8**k**, – 2**i** – 4**j** + 6**k**; $\frac{3\mathbf{i}+8\mathbf{j}-4\mathbf{k}}{2}$, – 2**j** – 5**k**, $\frac{-3\mathbf{i}-4\mathbf{j}+14\mathbf{k}}{2}$]

*(ii) The position vectors of P, Q, R, S be $2\mathbf{i} + 4\mathbf{k}$, $5\mathbf{i} + 3\sqrt{3}\,\mathbf{j} + 4\mathbf{k}$, $-2\sqrt{3}\,\mathbf{j} + \mathbf{k}$, $2\mathbf{i} + \mathbf{k}$. Prove that RS is **parallel** to PQ and $\frac{2}{3}$rd of PR. **[WBSC – 90, 10]**

*(iii) Three points whose position vectors are A(2, 4, –1), B(4, 5, 1) and C(3, 6, –3) form a **triangle** ABC. Determine whether the triangle is **isosceles or right–angled** or both. Find also direction cosines of BC. **[WBSC – 09]**

(iv) Show that the vectors 2**i** – **j** + **k**, **i** – 3**j** – 5**k**, 3**i** – 4**j** – 4**k** form the sides of a **right angled** triangle.

(v) Show that the points whose position vectors are 4**i** + 5**j** + 6**k**, 5**i** + 6**j** + 4**k**, 6**i** + 4**j** + 5**k** form an **equilateral triangle**.

(vi) Show that the points whose position vectors are 2**i** + **j** – **k**, **i** + 2**j** + **k**, 2**i** + 2**j** – **k** and 3**i** + **j** – 3**k** respectively are the vertices of a **parallelogram**.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**1.(ii)** The position vectors of P, Q, R, S be $2\mathbf{i} + 4\mathbf{k}$, $5\mathbf{i} + 3\sqrt{3}\,\mathbf{j} + 4\mathbf{k}$, $-2\sqrt{3}\,\mathbf{j} + \mathbf{k}$, $2\mathbf{i} + \mathbf{k}$. Prove that RS is **parallel** to PQ and $\frac{2}{3}$ rd of PR. **[WBSC – 90, 10]**

**Solution:** Let with respect to O as origin position vectors of P, Q, R and S be respectively $2\mathbf{i} + 4\mathbf{k}$, $5\mathbf{i} + 3\sqrt{3}\,\mathbf{j} + 4\mathbf{k}$, $-2\sqrt{3}\,\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{k}$.

Therefore, $\overrightarrow{OP} = 2\mathbf{i} + 4\mathbf{k}$, $\overrightarrow{OQ} = 5\mathbf{i} + 3\sqrt{3}\,\mathbf{j} + 4\mathbf{k}$, $\overrightarrow{OR} = -2\sqrt{3}\,\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OS} = 2\mathbf{i} + \mathbf{k}$

Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5\mathbf{i} + 3\sqrt{3}\,\mathbf{j} + 4\mathbf{k}) - (2\mathbf{i} + 4\mathbf{k}) = 3\mathbf{i} + 3\sqrt{3}\,\mathbf{j}$

and $\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = (2\mathbf{i} + \mathbf{k}) - (-2\sqrt{3}\,\mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\sqrt{3}\,\mathbf{j} = \frac{2}{3}(3\mathbf{i} + 3\sqrt{3}\,\mathbf{j}) = \frac{2}{3}\overrightarrow{PQ}$ ....... (i)

which shows that the vector $\overrightarrow{RS}$ is parallel to the vector $\overrightarrow{PQ}$ and hence, RS is parallel to PQ.

Again from (i) we get, $|\overrightarrow{RS}| = \frac{2}{3}|\overrightarrow{PQ}|$

or $RS = \frac{2}{3}PQ$, which shows that RS is $\frac{2}{3}$ of PQ. **(Proved)**

**1.(iii)** Three points whose position vectors are A(2, 4, –1), B(4, 5, 1) and C(3, 6, –3) form a **triangle** ABC. Determine whether the triangle is **isosceles or right–angled** or both. Find also direction cosines of BC. **[WBSC – 09]**

**Solution:** With respect to O as origin let the position vectors of A(2, 4, –1), B(4, 5, 1) and C(3, 6, –3) be respectively $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$, where **i**, **j**, **k** are three unit vectors along three rectangular coordinate axes.

Then $\overrightarrow{OA} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) - (4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

And $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) - (3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$|\overrightarrow{AB}| = \sqrt{2^2 + 1^2 + 2^2} = 3$, $|\overrightarrow{BC}| = \sqrt{1^2 + 1^2 + 4^2} = 3\sqrt{2}$ and $|\overrightarrow{CA}| = \sqrt{1^2 + 2^2 + 2^2} = 3$

Therefore, $|\overrightarrow{AB}| = |\overrightarrow{CA}| \Rightarrow$ ABC is an isosceles triangle.

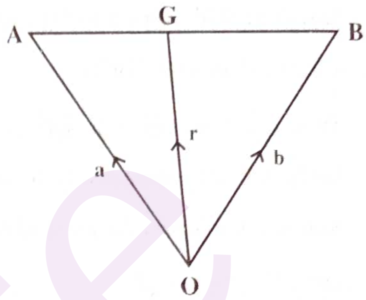
Again $|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = 3^2 + 3^2 = 18 = (3\sqrt{2})^2 = |\overrightarrow{BC}|^2$

or $|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{BC}|^2 \Rightarrow$ ABC is a right–angled triangle, whose $\angle A = 90^0$

Direction cosines of $\overrightarrow{BC}$ are $\frac{-1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{-4}{3\sqrt{2}}$. **(Ans)**

## 5.5. Ratio formula:

Let A, B be two given points and G a point in the line AB such that AG : GB = m : n. Let the position vectors of the points A, B and G with respect to any origin of reference O be **a**, **b** and **r** respectively.



Then, $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OR} = \mathbf{r}$.. We shall find **r** in terms of **a**, **b**.

We have $\frac{AG}{GB} = \frac{m}{n}$ or n. AG = m.GB. Therefore, $n.\overrightarrow{AG} = m.\overrightarrow{GB}$

or $n(\overrightarrow{OG} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OG})$ or $n(\mathbf{r} - \mathbf{a}) = m(\mathbf{b} - \mathbf{r})$

or $(m + n)\,\mathbf{r} = n\mathbf{a} + m\mathbf{b}$ or $\mathbf{r} = \frac{n\mathbf{a} + m\mathbf{b}}{m + n}$

- In particular, if m = n, $\overrightarrow{OG} = \frac{\mathbf{a} + \mathbf{b}}{2}$, and then G is the mid–point of AB.

**Note 1.**

- G divides **Internally** means G is the point on AB in between A and B

- G divides **Externally** means G is the point on the produced part of AB
- In the case of External division AG : GB = m : n

or, $\frac{AB + BG}{BG} = \frac{m}{n}$ or, $\frac{AB}{BG} + 1 = \frac{m}{n}$ or, $\frac{AB}{BG} = \frac{m}{n} - 1$ or, $\frac{AB}{BG} = \frac{m - n}{n}$ or AB : BG = (m – n) : n,

which shows that B divides AG Internally in the ratio (m – n) : n

- Hence, G divides the line segment AB Externally in the ratio m : n means B divides the line segment AG Internally in the ratio (m – n) : n.

**Note 2.** The ratio m : n is positive or negative according as the point G divides the segment AB **internally** or **externally** in case of AG : GB = m : n. Hence, if m : n be negative, then the point G divides AB externally and vice–versa.

### PROBLEM SET – II

2. *(i) Prove vectorially, the line joining the mid–points of the sides of a quadrilateral is a parallelogram.

(ii) If ABCD be a quadrilateral (plane or skew) show that the mid point of its sides are the vertices of a parallelogram and the joins of the mid–points of the opposite edges bisect each other. **[WBSC – 89]**

*(iii) In a parallelopiped show that by vector method that the four diagonals and the join of the mid–points of the opposite edges mit at a common point of bisection. **[WBSC – 88]**

*(iv) Prove that the medians of a triangle meet at a point which trisects them.

**OR** Prove that the medians of a triangle are concurrent.

(v) ABCD is a parallelogram; E, F are the mid–points of the sides AB and BC. Show that DE and DF trisect the diagonal AC.

*(vi) Show that the join of the mid–points of two sides of a triangle is parallel to the third side and half its length.

*(vii) Prove that the straight line joining the midpoints of the oblique sides of a trapezium is parallel and half the sum of parallel sides.

*(viii) Prove by the method of vectors that the internal bisectors of the angles of a triangle are concurrent.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

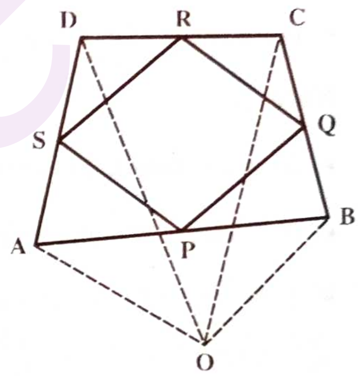
**2. (i)** Prove vectorially, the line joining the mid–points of the sides of a quadrilateral is a parallelogram.

**Solution:** With respect to O as origin, let **a**, **b**, **c**, **d** be respectively the position vectors of the vertices A, B, C and D of the quadrilateral ABCD.

Then $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, and $\overrightarrow{OD} = \mathbf{d}$.

Let P, Q, R, S be respectively the mid–points of the sides AB, BC, CD and DA of the rectangle ABCD.

Now since, P is the mid–point of AB, therefore, the position vector of P is given by




$\overrightarrow{OP} = \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2} = \frac{\mathbf{a}+\mathbf{b}}{2}$.

Similarly, $\overrightarrow{OQ} = \frac{\mathbf{b}+\mathbf{c}}{2}$, $\overrightarrow{OR} = \frac{\mathbf{c}+\mathbf{d}}{2}$ and $\overrightarrow{OS} = \frac{\mathbf{d}+\mathbf{a}}{2}$.

$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{\mathbf{b}+\mathbf{c}}{2} - \frac{\mathbf{a}+\mathbf{b}}{2} = \frac{\mathbf{c}-\mathbf{a}}{2}$ and $\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \frac{\mathbf{c}+\mathbf{d}}{2} - \frac{\mathbf{d}+\mathbf{a}}{2} = \frac{\mathbf{c}-\mathbf{a}}{2}$

Therefore, $\overrightarrow{PQ} = \overrightarrow{SR} \Rightarrow$ PQ and SR are parallel.

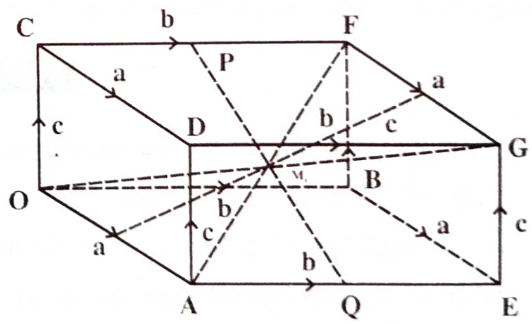
Again, $|\overrightarrow{PQ}| = |\overrightarrow{SR}| \Rightarrow PQ = SR$

Similarly, we can find PS and QR are parallel and PS = QR.

Hence, PQRS is a parallelogram. **(Proved)**

**2. (iii)** In a parallelopiped show that by vector method that the four diagonals and the join of the mid–points of the opposite edges meet at a common point of bisection. **[WBSC – 88]**

**Solution:** With respect to O as origin, let **a**, **b**, **c** be respectively the position vectors of A, B and C [as shown in the figure]. Then $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$




Now, $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \overrightarrow{OA} + \overrightarrow{OB} = \mathbf{a} + \mathbf{b}$

$\Rightarrow$ position vector of E is $\mathbf{a} + \mathbf{b}$

Similarly, position vector of F, $\overrightarrow{OF} = \mathbf{b} + \mathbf{c}$,

position vector of D, $\overrightarrow{OD} = \mathbf{a} + \mathbf{c}$ and the position vector of G,

$\overrightarrow{OG} = \overrightarrow{OE} + \overrightarrow{EG} = \overrightarrow{OE} + \overrightarrow{OC} = \mathbf{a} + \mathbf{b} + \mathbf{c}$.

Let $M_1$ be the mid–point of the diagonal OG. Then $\overrightarrow{OM_1} = \frac{\overrightarrow{OG}}{2} = \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$

Again, let $M_2$ be the mid–point of the diagonal AF, then $\overrightarrow{OM_2} = \frac{\overrightarrow{OA}+\overrightarrow{OF}}{2} = \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$

Similarly, we an find the position vectors of the mid–points of the diagonals CE and BD are $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$.

Hence, four diagonals meet at a common point whose position vector is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$.

Again let P, Q be the mid–points of CF and AE.

Then $\overrightarrow{OP} = \frac{\overrightarrow{OC}+\overrightarrow{OF}}{2} = \frac{\mathbf{c}+\mathbf{b}+\mathbf{c}}{2} = \frac{\mathbf{b}+2\mathbf{c}}{2}$, $\overrightarrow{OQ} = \frac{\overrightarrow{OA}+\overrightarrow{OE}}{2} = \frac{\mathbf{a}+\mathbf{a}+\mathbf{b}}{2} = \frac{2\mathbf{a}+\mathbf{b}}{2}$

Now $\overrightarrow{OP}+\overrightarrow{OQ}=\frac{b+2c}{2}+\frac{2a+b}{2}=\mathbf{a}+\mathbf{b}+\mathbf{c}$

Therefore, if $E_1$ be the mid–point of PQ, then $\overrightarrow{OE_1}=\frac{\overrightarrow{OP}+\overrightarrow{OQ}}{2}=\frac{a+b+c}{2}$

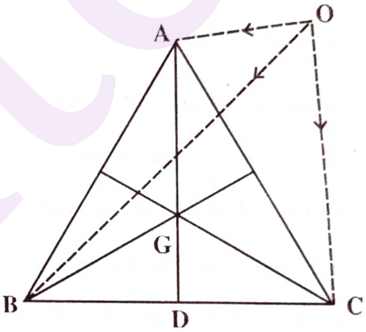
Similarly, we can find the position vectors of the mid–points of the join of the mid–points of the other opposite sides is $\frac{a+b+c}{2}$ [$M_1$ and $E_1$ are same point].

So, the join of the mid–points of the opposite edges meet at a point whose position vector is $\frac{a+b+c}{2}$.

Hence the theorem is **(Proved)**

**2. (iv)** Prove that the medians of a triangle meet at a point which trisects them.

**Solution:** With respect to O as origin, let **a**, **b**, **c** be respectively the position vectors of A, B and C [as shown in the figure]. Then $\overrightarrow{OA}=\mathbf{a}, \overrightarrow{OB}=\mathbf{b}, \overrightarrow{OC}=\mathbf{c}$.




Let D be the mid–point of BC.

Then the position vector of D, $\overrightarrow{OD}=\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}=\frac{b+c}{2}$

Let AD be divide at G in the ratio 2 : 1, i.e., AG : GD = 2 :1.

Then, the position vector of G, $\overrightarrow{OG}$

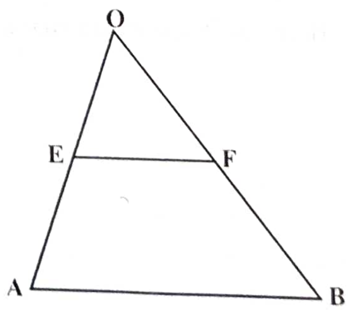
$$=\frac{2.\overrightarrow{OD}+1.\overrightarrow{OA}}{2+1}=\frac{2.\frac{\mathbf{b}+\mathbf{c}}{2}+1.\mathbf{a}}{3}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}.$$

Therefore, the position vector of G is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$. The symmetry of this result shows that each of the other two median is divide in the ratio 2 : 1 at a point whose position vector is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$, which is the centeroid of the triangle ABC.

Hence the theorem is proved.

**2. (vi)** Show that the join of the mid–points of two sides of a triangle is parallel to the third side and half its length.

**Solution:** Let OAB be the given triangle. With respect to O as origin, let **a**, **b** be respectively the position vectors of A, B [as shown in the figure]. Then $\overrightarrow{OA}=\mathbf{a}, \overrightarrow{OB}=\mathbf{b}$.




Let E, F be respectively the mid–points of OA and OB.

Then the position vectors of E and F are respevtively,

$\overrightarrow{OE}=\frac{\overrightarrow{OA}}{2}=\frac{a}{2}, \overrightarrow{OF}=\frac{\overrightarrow{OB}}{2}=\frac{b}{2}$

Therefore, $\overrightarrow{EF}=\overrightarrow{OF}-\overrightarrow{OE}=\frac{b}{2}-\frac{a}{2}=\frac{b-a}{2}=\frac{\overrightarrow{OB}-\overrightarrow{OA}}{2}=\frac{\overrightarrow{AB}}{2}$

Therefore, $\overrightarrow{EF}=\frac{1}{2}\overrightarrow{AB} \Rightarrow$ EF and AB are parallel.

Again, $\left|\overrightarrow{EF}\right|=\frac{1}{2}\left|\overrightarrow{AB}\right| \Rightarrow EF=\frac{1}{2}AB$

Hence, the join of the mid–points of two sides of a triangle is parallel to the third side and half its length. **(Proved)**

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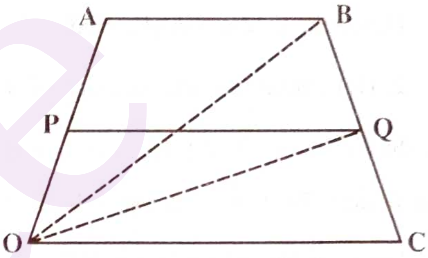
**2. (vii)** Prove that the straight line joining the midpoints of the oblique sides of a trapezium is parallel and half the sum of parallel sides.

**Solution:** Let OABC be a trapezium whose sides AB is parallel to OC. P and Q are the mid–points of the oblique sides OA and BC. PQ is joined. We have to prove that PQ is parallel to AB and OC and PQ = $\frac{1}{2}$(AB + OC).

**Proof:** Let with respect to O as origin the position vectors of A, B and C be respectively **a**, **b**, **c**.




Then, $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Therefore, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$,

$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}, \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\mathbf{c} - \mathbf{b}), \overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ} = \mathbf{b} + \frac{1}{2}(\mathbf{c} - \mathbf{b}) = \frac{1}{2}(\mathbf{b} + \mathbf{c})$

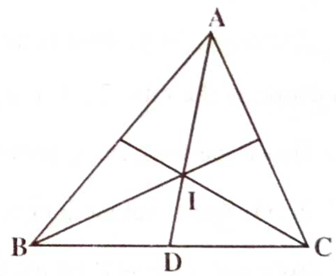
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}\mathbf{a} = \frac{1}{2}[(\mathbf{b} - \mathbf{a}) + \mathbf{c}] = \frac{1}{2}[\overrightarrow{AB} + \overrightarrow{OC}]$,

which shows that PQ is parallel to AB and OC and PQ = $\frac{1}{2}$ [AB + OC].

Hence the theorem is proved.

**2.(viii)** Prove by the method of vectors that the internal bisectors of the angles of a triangle are concurrent.

**Solution:** Let ABC be a triangle. We suppose AD bisect the ∠BAC internally and meets the opposite side BC at D. Let the length of the sides BC, CA and AB be respectively l, m, n. With respect to O as origin let **a**, **b**, **c** be the position vectors of A, B, C respectively.




Since AD divides BC in the ratio n : m, the position vector of D is $\frac{m\mathbf{b} + n\mathbf{c}}{m + n}$.

Let us take a point I on AD such that AI : ID = m + n : l

The position vector of I will then be $\dfrac{l\mathbf{a} + (m + n)\cdot\frac{m\mathbf{b} + n\mathbf{c}}{m + n}}{l + (m + n)} = \dfrac{l\mathbf{a} + m\mathbf{b} + n\mathbf{c}}{l + m + n}$ ....... (i)

By symmetry we see that this point must also be on the bisectors of the angles at B and C.

Hence the bisectors are concurrent and the point of concurrent is given by (i).

# 5.6. Linear combination of vectors:

Let **a**, **b**, **c**, ....... etc., be a given system of vectors and x, y, z, ......etc., a system of scalars such that a vector **r** can be expressed as **r** = x**a** + y**b** + z**c** + ........., then **r** is said to be a linear combination of the vectors **a**, **b**, **c**, ..... etc.

## Linearly dependent and independent system of vectors:

If the scalars x, y, z, ....., not all zero exists such that x**a** + y**b** + z**c** + .... = **0**, then the set **a**, **b**, **c**, .... of vectors is said to form a linearly dependent system of vectors.

If the system is not linearly dependent, then we call it linearly independent and in this case x = y = z = 0.

- A single vector **a** is linearly dependent only when it is the zero vector.

- For two collinear vectors **a** and **b** we may write **b** = x**a** or **b** – x**a** = **0**, where x is some non–zero scalar. Thus **a** and **b** are linearly dependent.

A necessary and sufficient condition that two proper vectors be linearly dependent is that they be collinear. [See Article 3.7]

- For three coplanar vectors **a**, **b**, **c** we may write **c** – x**a** – y**b** = **0** i.e., **a**, **b**, **c** are linearly dependent.

A necessary and sufficient condition that three proper vectors be linearly dependent is that they be coplanar. [See Article 3.7]

- For any four vectors, **a**, **b**, **c**, **d** we have always a relation of the form **d** – x**a** – y**b** – z**c** = **0**. Hence four vectors are always linearly dependent.

- Any four or more vectors form a linearly dependent sysem.

# PROBLEM SET – III

3. (i) Prove that the vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 6\mathbf{j} + 10\mathbf{k}$ are **linearly dependent**.

*(ii) Test whether the vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ are **linearly dependent.** **[WBSC – 93]**

(iii) Prove that the vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} = -\mathbf{j} + 2\mathbf{k}$ are **linearly dependent**.

(iv) Prove that the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 8\mathbf{j} + \mathbf{k}$ are **linearly dependent**.

(v) Prove that the vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ are **linearly independent**.

*(vi) Show that the vactors $\mathbf{a} = (2, -1, 2)$, $\mathbf{b} = (1, 3, -2)$, $\mathbf{c} = (2, 4, 1)$ and $\mathbf{d} = (5, 3, -3)$ form a **linearly dependent** system.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**3. (ii)** Test whether the vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ are linearly dependent. **[WBSC – 93]**

**Solution:** Let us consider two scalars x, y such that $x\mathbf{a} + y\mathbf{b} = \mathbf{c}$ ....... (i)

or $x(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + y(2\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Equating the coefficients of like vectors from both sides we get,

$x + 2y = 3$ .......(2) $-3x - 4y = 2$ .......(3) $2x - y = -1$ .......(4)

Solving (2) and (3) we get, $\frac{x}{-4-12} = \frac{y}{9+2} = \frac{1}{-4+6}$ or, $\frac{x}{-16} = \frac{y}{11} = \frac{1}{2}$ or, $x = -8$, $y = \frac{11}{2}$

Now when $x = -8$, $y = \frac{11}{2}$; from (4) $2x - y = 2(-8) - \frac{11}{2} = -16 - \frac{11}{2} = -\frac{43}{2} \neq -1$

Therefore, the given three vectors are not linearly dependent.

**(vi)** Show that the vactors $\mathbf{a} = (2, -1, 2)$, $\mathbf{b} = (1, 3, -2)$, $\mathbf{c} = (2, 4, 1)$ and $\mathbf{d} = (5, 3, -3)$ form a linearly dependent system.

**Solution:** Let us consider three scalars x, y, z such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$ .......(1)

or $x(2, -1, 2) + y(1, 3, -2) + z(2, 4, 1) = (5, 3, -3)$

Equating the coefficients of like components from both sides we get,

$2x + y + 2z = 5$ .......(2) $-x + 3y + 4z = 3$ .......(3) $2x - 2y + z = -3$ ....(4)

From (2) and (4) we get, $3y + z - 8 = 0$ .......(5) [(2) – (4)]

From(2) and (3) we get, $7y + 10z - 11 = 0$ .......(6) [$1\times(2) + 2\times(3)$]

From (5) and (6) we get by the rule of cross–multiplication,

$\frac{y}{-11+80} = \frac{z}{-56+33} = \frac{1}{30-7}$ or, $\frac{y}{69} = \frac{z}{-23} = \frac{1}{23}$ or, $y = 3$, $z = -1$

From (2) $2x + 3 - 2 = 5$ or $x = 2$. Therefore (1) becomes $2\mathbf{a} + 3\mathbf{b} - \mathbf{c} - \mathbf{d} = \mathbf{0}$.

Thus there are four scalars 2, 3, –1, –1 not all zero such that $2\mathbf{a} + 3\mathbf{b} - \mathbf{c} - \mathbf{d} = \mathbf{0}$.

Therefore, the given four vectors are linearly dependent. **(Proved)**

# 5.7. Collinear and Coplanar vectors:

**Collinear vectors:** Two vectors **a** and **b** are collinear (or like or parallel) when they have the same or parallel support [regardless of their magnitudes and sense of their direction].

From definition it follows that **a** and m**a** are two collinear (or like or parallel) vectors, their supports being same or parallel.

For two collinear vectors **a** and **b**, **b** = m**a**, m ($\neq 0$) is a scalar.

- **Theorem 1.** If **a**, **b** are two non–zero, non–collinear vectors and x, y are scalars such that x**a** + y**b** = **0**, then x = y = 0.

- **Theorem 2.** (Statement only): If **a** and **b** be two collinear vectors then either of them can be expressed as the product of the other by a suitable scalar, the numerical value of the scalar being the ratio of the lengths of **a** and **b**, and conversely.

- **Theorem 3.** (Statement): The necessary and sufficient condition for three distinct points A, B, C to lie on a straight line is that there should exist three scalars x, y, z, not all zero, such that x**a** + y**b** + z**c** = **0**; x + y + z = 0 where **a**, **b**, **c** are three position vectors of the points A, B, C respectively referred to a choosen origin.

**Coplanar vector:** A system of vectors is said to be coplanar if their supports are parallel to the same plane; otherwise they are non–coplanar. The vector (m**a** + n**b**) is coplanar with the vectors **a** and **b** whatever the scalars m and n may be.

- A plane parallel to the system of coplaner vectors is called the **plane of vectors**.

- **Theorem** (Statement only): The necessary and sufficient condition for four distinct points A, B, C, D, no three collinear, to be coplanar is that there should exist four scalars x, y, z, t, not all zero, such that x**a** + y**b** + z**c** + t**d** = **0**; x + y + z + t = 0 where **a**, **b**, **c**, **d** are the position vectors of the points A, B, C, D respectively referred to a choosen origin.

## PROBLEM SET – IV

4. (i) Show that the points $\mathbf{A} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{C} = 7\mathbf{i} - \mathbf{k}$ are collinear. **[WBSC – 85]**

(ii) Show that the points $\mathbf{A} = -4\mathbf{i} + 6\mathbf{j} + 10\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ and $\mathbf{C} = 8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ are collinear.

*(iii) Verify whether the three points $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$ and $-7\mathbf{b} + 10\mathbf{c}$ are collinear or not. **[WBSC – 87]**

(iv) Show that the three points $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $-2\mathbf{a} + 3\mathbf{b} + 2\mathbf{c}$, $-8\mathbf{a} + 13\mathbf{b}$ are collinear.

*(v) Determine the values of $l$ and m for which the vectors $-3\mathbf{i} + 4\mathbf{j} + l\mathbf{k}$ and $m\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ are collinear.

5. (i) Prove that the three points $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $-2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $-\mathbf{j} + 2\mathbf{k}$ are coplanar.

(ii) Prove that the four points $-6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $5\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$ and $-13\mathbf{i} + 17\mathbf{j} - \mathbf{k}$ are coplanar. **[WBSC – 92]**

(iii) Show that the four points whose position vectors are given by $-6\mathbf{a} + 3\mathbf{b} + 2\mathbf{c}$, $3\mathbf{a} - 2\mathbf{b} + 4\mathbf{c}$, $5\mathbf{a} + 7\mathbf{b} + 3\mathbf{c}$ and $-13\mathbf{a} + 17\mathbf{b} - \mathbf{c}$ are coplanar; $\mathbf{a}, \mathbf{b}, \mathbf{c}$ being three non–coplanar vectors.

(iv) Show that the four points whose position vectors are given by $-\mathbf{a} + 4\mathbf{b} - 3\mathbf{c}$, $3\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}$, $-3\mathbf{a} + 8\mathbf{b} - 5\mathbf{c}$ and $-3\mathbf{a} + 2\mathbf{b} + \mathbf{c}$ are coplanar; $\mathbf{a}, \mathbf{b}, \mathbf{c}$ being three non–coplanar vectors.

*(v) Show that the four points whose position vectors are $A(4, 8, 12)$; $B(2, 4, 6)$; $C(3, 5, 4)$; $D(5, 8, 5)$ are coplanar.

*(vi) Determine $\gamma$ such that $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \gamma\mathbf{j} + 3\mathbf{k}$ are coplanar.

*(vii) Show by vector method that the points $P(1, 5, -1)$, $Q(0, 4, 5)$, $R(-1, 5, 1)$ and $S(2, 4, 3)$ are coplanar.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**4.(iii)** Verify whether the three points $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$ and $-7\mathbf{b} + 10\mathbf{c}$ are collinear or not. **[WBSC – 87]**

**Solution:** Let us consider two scalars x, y such that or $x(\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}) + y(2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}) = -7\mathbf{b} + 10\mathbf{c}$ ..........(1)

Equating the coefficients of like vectors from both sides we get,

$x + 2y = 0$ .......... (2) $-2x + 3y = -7$ .......... (3) $3x - 4y = 10$ ......... (4)

From (2) and (3) we get, $-2(-2y) + 3y = -7$ or $y = -1$ and from (2) $x = -2y = -2(-1) = 2$

Now when $x = 2$, $y = -1$; from (4) $3x - 4y = 3 \times 2 - 4(-1) = 10$

Hence, $x = 2$, $y = -1$ satisfies equations (2), (3) and (4).

From (1) we get, $2(\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}) - 1(2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}) - (-7\mathbf{b} + 10\mathbf{c}) = 0$, where $2 + (-1) + (-1) = 0$.

Hence, the three points $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$ and $-7\mathbf{b} + 10\mathbf{c}$ are collinear.

**4.(v)** Determine the values of $l$ and m for which the vectors $-3\mathbf{i} + 4\mathbf{j} + l\mathbf{k}$ and $m\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ are collinear.

**Solution:** We know for two collinear vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b} = p\mathbf{a}$, p is a scalar.

Since, $m\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ and $-3\mathbf{i} + 4\mathbf{j} + l\mathbf{k}$ are collinear, therefore, $m\mathbf{i} + 8\mathbf{j} + 6\mathbf{k} = p(-3\mathbf{i} + 4\mathbf{j} + l\mathbf{k})$

Equating the coefficients of like vectors from both sides we get,

$m = -3p$ ...... (1) $8 = 4p$ ....... (2) $6 = pl$ ....... (3)

From (2), $p = 2$; From (1) $m = -3 \times 2 = -6$; From (3)

Therefore, $l = 3$, $m = -6$. **(Ans)**

**5.(v)** Show that the four points whose position vectors are A(4, 8, 12); B(2, 4, 6); C(3, 5, 4); D(5, 8, 5) are coplanar.

**Solution:** With respect to O as origin let the position vector of A(4, 8, 12), B(2, 4, 6), C(3, 5, 4) and D(5, 8, 5) be respectively $\overrightarrow{OA} = 4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ $\overrightarrow{OC} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ $\overrightarrow{OD} = 5\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$

Therefore $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) - (4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}) = -2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$

Similarly, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\mathbf{i} - 3\mathbf{j} - 8\mathbf{k}$ $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \mathbf{i} - 7\mathbf{k}$

Let us consider two scalars x and y such that, $x\overrightarrow{AB} + y\overrightarrow{AC} = \overrightarrow{AD}$ .........(1)

or $x(-2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}) + y(-\mathbf{i} - 3\mathbf{j} - 8\mathbf{k}) = \mathbf{i} - 7\mathbf{k}$

Equating the coefficients of like vectors from both sides we get,

$-2x - y = 1$ ....... (2) $-4x - 3y = 0$ ..... (3) $-6x - 8y = -7$ ........ (4)

From (2) $y = -2x - 1$; From (3), $-4x - 3(-2x - 1) = 0$ or $-4x + 6x = -3$ or $x = -\frac{3}{2}$. Therefore $y = 3 - 1 = 2$.

Now, for $x = -\frac{3}{2}$, $y = 2$, $-6x - 8y = -6\left(-\frac{3}{2}\right) - 16 = -7$

Therefore all three equations (2), (3) and (4) satisfied by $x = -\frac{3}{2}$, $y = 2$.

Therefore (1) becomes, $-\frac{3}{2}\overrightarrow{AB} + 2\overrightarrow{AC} = \overrightarrow{AD}$ or $-3(\overrightarrow{OB} - \overrightarrow{OA}) + 4(\overrightarrow{OC} - \overrightarrow{OA}) = 2(\overrightarrow{OD} - \overrightarrow{OA})$

or $1.\overrightarrow{OA} - 3\overrightarrow{OB} + 4\overrightarrow{OC} - 2\overrightarrow{OD} = \mathbf{0}$ or $x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC} + t\overrightarrow{OD} = \mathbf{0}$, where $x = 1$, $y = -3$, $z = 4$, $t = -2$

and $x + y + z + t = 1 - 3 + 4 - 2 = 0$. Hence, given four points are coplanar. **(Proved)**

**5.(vi)** Determine $\gamma$ such that $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \gamma\mathbf{j} + 3\mathbf{k}$ are coplanar.

**Solution:** Let us consider two scalars x and y such that, $x\,\mathbf{a} + y\,\mathbf{b} = \mathbf{c}$ ..........(1)

or $x(\mathbf{i} + \mathbf{j} + \mathbf{k}) + y(2\mathbf{i} - 4\mathbf{k}) = \mathbf{i} + \gamma\mathbf{j} + 3\mathbf{k}$

Equating the coefficients of like vectors from both sides we get,

$x + 2y = 1$ ...........(2) $x = \gamma$ ...........(3) $x - 4y = 3$ .........(4)

From (2) and (4) we get, $(x + 2y) - (x - 4y) = 1 - 3$ or $6y = -2$ or $y = -\frac{1}{3}$

Therefore (2) gives $x = 1 - 2y = 1 + \frac{2}{3} = \frac{5}{3}$ From (3) $\gamma = \frac{5}{3}$.

Therefore (1) becomes, $\frac{5}{3}\mathbf{a} - \frac{1}{3}\mathbf{b} = \mathbf{c}$ or $5\mathbf{a} - \mathbf{b} - 3\mathbf{c} = \mathbf{0}$, where $\gamma = \frac{5}{3}$, which shows that the vectors **a**, **b** and **c** are linearly dependent and hence coplanar and $\gamma = \frac{5}{3}$ **(Ans)**

**5.(vii)** Show by vector method that the points P(1, 5, –1), Q(0, 4, 5), R(–1, 5, 1) and S(2, 4, 3) are coplanar.

**Solution:** Let us consider three scalars x, y, z such that or $x(1, 5, -1) + y(0, 4, 5) + z(-1, 5, 1) = (2, 4, 3)$ ..........(1)

Equating the coefficients of like components from both sides we get,

$x - z = 2$ ..........(2) $5x + 4y + 5z = 4$ ..........(3) $-x + 5y + z = 3$ .........(4)

From(2) and (3) we get, $5(z + 2) + 4y + 5z = 4$ or $2y + 5z + 3 = 0$ ........(5)

From (2) and (4) we get, $-z - 2 + 5y + z = 3$ or $y = 1$

From (5) $2 + 5z + 3 = 0$ or $z = -1$ and from (2) $x = 2 + z = 2 - 1 = 1$.

$\therefore$ (1) can be written as $1(1, 5, -1) + 1(0, 4, 5) - 1(-1, 5, 1) - 1(2, 4, 3) = \mathbf{0}$ where $1 + 1 - 1 - 1 = 0$.

Hence the given four points are coplanar. **(Proved)**

## 5.8 Application in Mechanics:

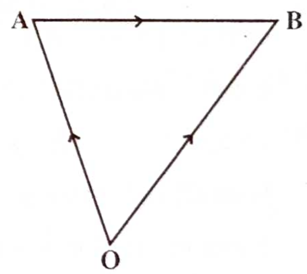
**Relative position:** When two points A and B both move, the relative position at any instant of the point A with respect to the position of B is the vector quantity $\overrightarrow{BA}$ in magnitude and direction i. e., position vector of A with respect to B as origin.

If we take into account a third point O with respect to which the relative positions of A and B are $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively then we shall have $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ or, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

**Relative displacement:** If A and B be positions of two moving points at the beginning of an interval and $A'$ and $B'$ be their positions at the end of the interval then the relative displacement which is the change in the relative position during the interval is the vector difference $(\overrightarrow{A'B'} - \overrightarrow{AB})$.

**Relative velocity:** Relative velocity of a moving point A with respect to another moving point B is the instantaneous rate of change of position of A relative to B. If however, two points A and B both move at a **uniform rate** then the relative velocity of A with respect to B is the change in their relative positions per unit of time.

**Theorem:** The relative velocity of A with respect to B is the vector difference of velocities **u** and **v** of A and B relative to a fixed point O, A and B being supposed to move at uniform rate.




**Proof:** Suppose A and B move to $A'$ and $B'$ respectively in one unit of time.

Then, by definition, relative velocity of A with respect to B

$= \overrightarrow{B'A'} - \overrightarrow{BA} = (\overrightarrow{OA'} - \overrightarrow{OB'}) - (\overrightarrow{OA} - \overrightarrow{OB})$

$= (\overrightarrow{OA'} - \overrightarrow{OA}) - (\overrightarrow{OB'} - \overrightarrow{OB}) = \overrightarrow{AA'} - \overrightarrow{BB'} = \mathbf{u} - \mathbf{v}$ **(Proved)**

### PROBLEM SET – V

6. (i) The velocity of a particle A relative to a particle B is $5\mathbf{i} + 3\mathbf{j}$ and the velocity of B relative to a third particle C is $6\mathbf{i} - 5\mathbf{j}$. Determine the magnitude and direction of velocity of A relative to C, **i** and **j** represents 1 km/hr. towards East and North respectively.where **i** and **j** denote unit velocity along East and North respectively. **[WBSC – 83]**

*(ii) The velocity of a boat relative to the water is represented by $3\mathbf{i} + 4\mathbf{j}$ and that of water relative to the earth is $\mathbf{i} - 7\mathbf{j}$. What is the velocity of the boat relative to the earth if **i** and **j** represent one km. an hour East and North respectively. **[WBSC – 87]**

*(iii) The velocity of a particle A relative to a particle B is $3\mathbf{i} + 4\mathbf{j}$ and that of B relative to C is $\mathbf{i} - 3\mathbf{j}$. Find the velocity of A relative to C; **i** and **j** represent velocity of one km/hr. along East and North respectively. **[WBSC – 85]**

*(iv) Velocity of a particle A relative to a particle B is $5\mathbf{i} + 3\mathbf{j}$ and velocity of B relative to a third particle C is $6\mathbf{i} - 5\mathbf{j}$. Determine the magnitude and direction of velocity of A if velocity of C is $2\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ denote unit velocity along East and North respectively. [WBSC – 90]

(v) P and Q are the two boats sailing on water flowting in a river. The velocity of P relative to water is $3\mathbf{i} + 4\mathbf{j}$, that of Q relative to water is $5\mathbf{i} - 12\mathbf{j}$ and velocity of water relative to ground is $3\mathbf{i}$. Find the velocity of P relative to Q and also velocities of P and Q with respect to ground, where $\mathbf{i}$ and $\mathbf{j}$ denote unit velocities towards East and North.

(vi) The velocity of a particle A relative to a particle B is $9\mathbf{i} - 6\mathbf{j}$ and the velocity of B relative to a third particle C is $9\mathbf{i} - 6\mathbf{j}$. Determine the magnitude and direction of velocity of A relative to C, $\mathbf{i}$ and $\mathbf{j}$ represents 1 km/hr. towards East and North respectively.where $\mathbf{i}$ and $\mathbf{j}$ denote unit velocity along East and North respectively. [WBSC – 83]

## ANSWERS

6.(i) Magnitude of velocity of A relative to C: $|\vec{V}_{AC}| = 5$ km/hr. $\theta = \tan^{-1}\frac{2}{11}$, South of East.

(ii) Velocity of Boat relative to Earth: $|\vec{V}_{AC}| = 5$ km/hr. $\theta = \tan^{-1}\frac{3}{4}$, South of East

(iii) The velocity of A relative to C: $|\vec{V}_{AC}| = \sqrt{17}$ km/hr. $\theta = \tan^{-1}\frac{1}{4}$, North of East.

(iv) Magnitude of the velocity of A is 11 units and its direction is towards East.

(v) Magnitude of velocity of P relative to Q: $\sqrt{260}$ km/h. $\theta = \tan^{-1}\frac{1}{8}$, West of North. Magnitude of velocity of P with respect to ground: $2\sqrt{13}$ km/h. $\theta = \tan^{-1}\frac{2}{3}$, North of East.

Magnitude of velocity of Q with respect to ground: $4\sqrt{13}$ km/h. $\theta = \tan^{-1}(1.5)$, South of East.

(vi) Magnitude of velocity of A relative to $C = 6\sqrt{13}$ Km/hr., $\theta = \tan^{-1}\frac{2}{3}$, South - East.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**6.**(ii) The velocity of a boat relative to the water is represented by $3\mathbf{i} + 4\mathbf{j}$ and that of water relative to the earth is $\mathbf{i} - 7\mathbf{j}$. What is the velocity of the boat relative to the earth if $\mathbf{i}$ and $\mathbf{j}$ represent one km. an hour East and North respectively. [WBSC – 87]

**Solution: (ii)** Let $\vec{V}_B$, $\vec{V}_W$, $\vec{V}_E$ denote the actual velocities of Boat, Water and Earth respectively.

Then the velocity of a Boat relative to Water i.e., $\vec{V}_B - \vec{V}_W = 3\mathbf{i} + 4\mathbf{j}$ ....(1)

Again, the velocity of Water relative to Earth i.e., $\vec{V}_W - \vec{V}_E = \mathbf{i} - 7\mathbf{j}$ ....(2)

Adding (1) and (2) we get, $(\vec{V}_B - \vec{V}_W) + (\vec{V}_W - \vec{V}_E) = (3\mathbf{i} + 4\mathbf{j}) + (\mathbf{i} - 7\mathbf{j})$

or, $\vec{V}_B - \vec{V}_E = 4\mathbf{i} - 3\mathbf{j} = \vec{V}_{BE}$ (say) ....(3)

which is the velocity of Boat relative to Earth.

Magnitude of the velocity of the Boat relative to the Earth, $|\vec{V}_{BE}| = \sqrt{16+9} = 5$ km/hr.

**Direction:** If $\theta$ be the angle made by $\vec{V}_{BE}$ with the direction of East, then $\theta$ lies some where in the South–East direction [since, x = 4, positive and y = –3, negative]. [Figure like : 7(iii)]

Hence the direction of $\vec{V}_{BE}$ is given by $\tan\theta$ = or $\theta = \tan^{-1}$ South of East. **(Ans)**

**6.(iv)** Velocity of a particle A relative to a particle B is 5**i** + 3**j** and velocity of B relative to a third particle C is 6**i** – 5**j**. Determine the magnitude and direction of velocity of A if velocity of C is 2**j**, where **i** and **j** denote unit velocity along East and North respectively. **[WBSC – 90]**

**Solution:** Let $\vec{V}_A$, $\vec{V}_B$, $\vec{V}_C$ denote the actual velocities of A, B and C respectively.

Then the velocity of A relative to B i.e., $\vec{V}_A - \vec{V}_B = 5\mathbf{i} + 3\mathbf{j}$ .........(1)

Again, the velocity of B relative to C i.e., $\vec{V}_B - \vec{V}_C = 6\mathbf{i} - 5\mathbf{j}$ .........(2)

Adding (1) and (2) we get, $(\vec{V}_A - \vec{V}_B) + (\vec{V}_B - \vec{V}_C) = (5\mathbf{i} + 3\mathbf{j}) + (6\mathbf{i} - 5\mathbf{j})$ or, $\vec{V}_A - \vec{V}_C = 11\mathbf{i} - 2\mathbf{j}$, .........(3)

which is the velocity of A relative to C. Now, given velocity of C, i.e., $\vec{V}_C = 2\mathbf{i}$

Therefore, from (3) we get, $\vec{V}_A - 2\mathbf{j} = 11\mathbf{i} - 2\mathbf{j}$ or, $\vec{V}_A = 11\mathbf{i}$

Therefore, the magnitude of the velocity of A is 11 unit and its direction is towards East [Since **i** denote unit vector along East]. **(Ans)**

## PROBLEM SET – VI

**7. *(i)** To a man walking due East at the rate of 6 km.hr, **wind** appears to blow from North East at the rate of $2\sqrt{2}$ km/hr. Show that the true magnitude of the **velocities** of the windis $2\sqrt{5}$ km.hr. Determine the direction. **[WBSC – 91]**

**(ii)** A person travelling Eastwords at a rate of 3 m.p.h finds that the **wind** seems to blow directly from the north; on **doubling** his speed it appears to come from the North–East. Find the vector wind velocity.

***(iii)** A man travelling east at 4 miles per hour finds that the **wind** seems to blow directly from the north. On **doubling** his speed he finds that it appears to come from North–East. Find the velocity of the wind.

***(iv)** A man is walking at the rate 3 km/hr, to him the **rain** seems to fall vertically; if he **increases** his speed to 5 km/hr in the same direction, it seems to fall at an angle $30^0$ to the vertical. Find the actual direction and velocity of the rain.

**(v)** A person travels due East at the rate of 6 miles per hour and observes that the **wind** seems to blow directly from the North; he then doubles his speed and the wind appears to come from the North–East. Determine the **direction** and the **velocity** of the wind.

**(vi)** A person travels due East at the rate of 8 miles per hour and observes that the **wind** seems to blow directly from the North. On doubling his speed he find that the wind appears to come from the North–East. Determine the **direction** and the **velocity** of the wind.

**(vii)** A man walks eastwards at the rate of 5 miles an hour, when the **wind** appears to come from the North. He then decreases his speed to two miles an hour and notices that the wind comes from the North–West. Show that the **velocity** of the wind is $\sqrt{34}$ miles per hour and its **direction** makes an angle of $\tan^{-1}\frac{5}{3}$ with the South line.

***(viii)** A school boy holding on umbrella runs with a velocity equal in magnitude to that of **rain** falling vertically as a consequence of which the rain strikes him on the face. At **what angle** should the boy hold the umbrella in order to protect him best? **[WBSC – 82]**

## ANSWERS

7. (i) Magnitude of the velocity of the wind = $2\sqrt{5}$ km/hr. $\theta = \tan^{-1}\frac{1}{2}$ South East.

(ii) Wind is blowing from the North–West with a velocity of $3\sqrt{2}$ m.p.h.

(iii) Magnitude of the velocity of the wind is = $4\sqrt{2}$ km/hr. and direction: $\theta = 45^0$ i.e., the wind is blowing from North–East towards South–East

(iv) Magnitude of the velocity of the rain is = $\sqrt{21}$ km/hr. **Direction:** $41^0$ with the vertical.

(v) Magnitude of the velocity of the rain is = $6\sqrt{2}$ km/hr. **Direction:** $45^0$ with the vertical, i.e., the wind is blowing from North–West towards South–East.

(vi) Magnitude of the velocity of the wind is = $8\sqrt{2}$ miles/hr. $\theta = 45^0$ i.e., the wind is blowing from North–West towards South–East.

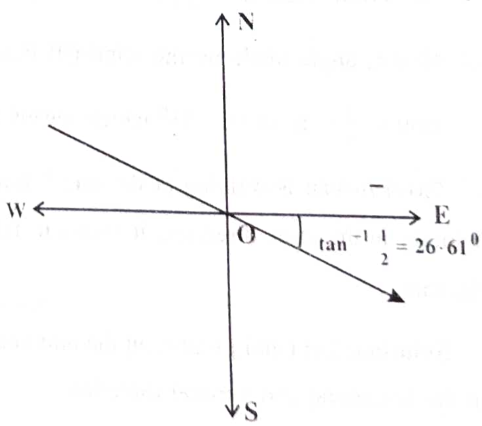
(viii) The boy hold the umbrella at an angle $45^0$ with the horizontal.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**7.(i)** To a man walking due East at the rate of 6 km./hr, **wind** appears to blow from North–East at the rate of 2 km/hr. Show that the true magnitude of the velocities of the wind is 2 km/hr. Determine the direction. **[WBSC – 91]**

**Solution:** Let **i** and **j** represent the unit velocity–vectors i.e., 1 km/hr. velocities along East and North respectively

The velocity of the man is $(\vec{V}_M) = 6i$.




Therefore, the velocity of the wind relative to the man is

$$(\vec{V}_{WM}) = \vec{V}_W - \vec{V}_M = -2(\cos 45^0 i + \sin 45^0 j) = -2(\mathbf{i} + \mathbf{j}).$$

$$\text{or, } \vec{V}_W = -2i - 2j + \vec{V}_M = -2i - 2j + 6i = 4i - 2j$$

The magnitude of the velocity of the wind

$$= \sqrt{4^2 + 2^2} = 2\sqrt{5} = 2 \text{ km/hr.}$$

**Direction:** Here x = 4, positive and y = – 2, negative.

Hence, angle made by the wind ($\theta$) lies some where in the South–East direction and is given by

$\tan\theta = \frac{2}{4} = \frac{1}{2}; \Rightarrow \theta = \tan^{-1}\frac{1}{2}$.

**7.(iii)** A man travelling east at 4 miles per hour finds that the **wind** seems to blow directly from the North. On doubling his speed he finds that it appears to come from North–East. Find the **velocity** of the wind.

**Solution:** Let **i** and **j** represent the unit velocity–vectors i.e., 1 km/hr. velocities along East and North respectively.

In the first case the velocity of the man is $(\vec{V}_M) = 4\mathbf{i}$.

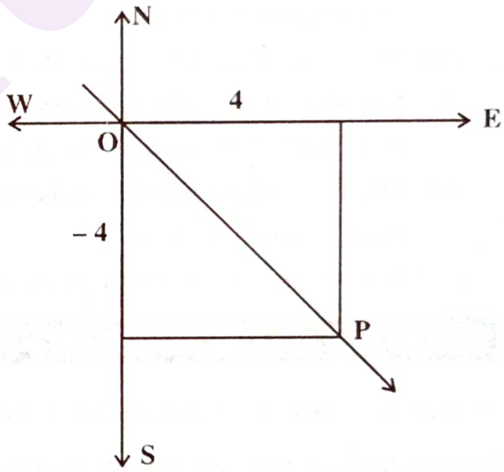
Let the true velocity of the wind be $(\vec{V}_W) = x\mathbf{i} + y\mathbf{j}$.

Therefore, the velocity of the wind relative to the man is $(\vec{V}_{WM}) = \vec{V}_W - \vec{V}_M = x\mathbf{i} + y\mathbf{j} - 4\mathbf{i} = (x-4)\mathbf{i} + y\mathbf{j}$.

But it is given that this velocity is directed from the North and hence the direction of $\vec{V}_{WM}$ is from North to South and therefore, coefficient of $\mathbf{i} = 0$.

This will give $x - 4 = 0$ or, $x = 4$.

In the second case, when the man doubles his speed the velocity of the man is $8\mathbf{i}$ and hence relative velocity is $x\mathbf{i} + y\mathbf{j} - 8\mathbf{i} = (x-8)\mathbf{i} + y\mathbf{j}$.




But this comes from North–East and hence this vector will be parallel to $k(\cos 45^0 \mathbf{i} + \sin 45^0 \mathbf{j}) = \frac{1}{\sqrt{2}}k(\mathbf{i} + \mathbf{j})$, k is a scalar.

$\therefore (x-8)\mathbf{i} + y\mathbf{j} = \frac{1}{\sqrt{2}}k(\mathbf{i}+\mathbf{j})$ which gives, $x - 8 = \frac{k}{\sqrt{2}}$ and $y = \frac{k}{\sqrt{2}}$

This gives $k = -4\sqrt{2}$ and hence $y = -4$, [ $\because x = 4$]

$\therefore$ the true velocity of the wind is $\overrightarrow{OP} = 4\mathbf{i} - 4\mathbf{j}$;

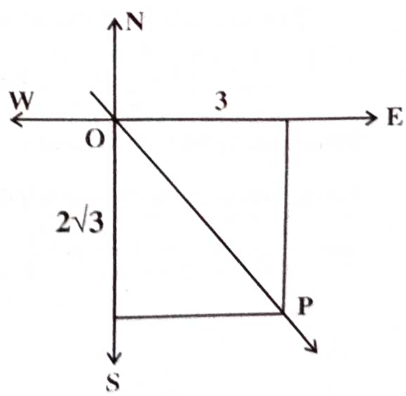
its magnitude $= \sqrt{4^2 + 4^2} = 4\sqrt{2}$ km/hr. and direction is from North–West to South–East.

**Direction:** Here $x = 4$, positive and $y = -4$, negative.

Hence, angle made by the wind (θ) lies some where in the South–West direction and is given by

$\tan\theta = \frac{4}{4} = 1; \Rightarrow \theta = 45^0$ which means that, the wind is blowing from North–West towards South–East.

**7.(iv)** A man is walking at the rate 3 km/hr, to him the **rain** seems to fall vertically; if he increases his speed to 5 km/hr in the same direction, it seems to fall at an angle $30^0$ to the vertical. Find the actual direction and velocity of the rain.




**Solution:** Let **i** and **j** represent the unit velocity–vectors i.e., 1 km/hr. velocities in the horizontal and vertical direction

In the first case the velocity of the man is $(\vec{V}_M) = 3\mathbf{i}$.

Let the true velocity of the rain be $(\vec{V}_R) = x\mathbf{i} + y\mathbf{j}$.

Therefore, the velocity of the rain relative to the man is

$(\vec{V}_{RM}) = \vec{V}_R - \vec{V}_M = x\mathbf{i} + y\mathbf{j} - 3\mathbf{i} = (x-3)\mathbf{i} + y\mathbf{j}$.

According to the problem, in the first case, the rain seems to fall vertically.

Therefore, coefficient of **i** = 0. This will give x – 3 = 0 or, x = 3.

In the second case, the velocity of the man is 5**i** and hence relative velocity is x**i** + y**j** – 5**i** = (x – 5)**i** + y**j**.

In the second case, the velocity of the man is 5**i** and hence relative velocity is x**i** + y**j** – 5**i** = (x – 5)**i** + y**j**.

In the second case direction of the rain seems to be $30^0$ to the vertical and hence this vector will be parallel to $k(\sin 30^0 \mathbf{i} + \cos 30^0 \mathbf{j}) = \frac{1}{2}k(\mathbf{i} + \sqrt{3}\,\mathbf{j})$, k is a scalar.

Therefore, $(x - 5)\mathbf{i} + y\mathbf{j} = \frac{k}{2}(\mathbf{i} + \sqrt{3}\,\mathbf{j})$ which gives, $x - 5 = \frac{k}{2}$ and $y = \frac{k\sqrt{3}}{2}$

This gives k = – 4 and hence $y = -2\sqrt{3}$, [ $\because$ x = 3]

$\therefore$ the true velocity of the rain is $\overrightarrow{OP} = 3\mathbf{i} - 2\sqrt{3}\,\mathbf{j}$ and its magnitude = $\sqrt{9+12} = \sqrt{21}$ km/hr

**Direction:** Here x = 3, positive and $y = -2\sqrt{3}$, negative.

Hence, angle made by the rain ($\theta$) with the horizontal is given by $\tan\theta = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}} \Rightarrow \theta = 49^0$.

Therefore, the angle with the vertical = $90^0 - 49^0 = 41^0$.

$\therefore$ direction of the rain is $41^0$ with the vertical hitting the man from behind.

**7.(viii)** A school boy holding on umbrella runs with a velocity equal in magnitude to that of rain falling vertically as a consequence of which the rain strikes him on the face. At what angle should the boy hold the umbrella in order to protect him best? **[WBSC – 82]**

**Solution:** Let **i** and **j** denote the unit velocity vector along the positive direction of x axis (horizontal) and y axis (vertically upwards).

Then velocity of boy ($\vec{V}_B$) = x**i**, velocity of rain ($\vec{V}_R$) = – x**j**, since, the boy runs with a velocity equal in magnitude to that of rain falling vertically.

Velocity of rain relative to boy $\vec{V}_{RB}) = \vec{V}_R - \vec{V}_B = -x\mathbf{j} - x\mathbf{i} = -x(\mathbf{i} + \mathbf{j})$.

Let the rain strike the boy at an angle $\theta$ with the horizontal.

Then $\tan\theta = \frac{x}{x} = 1 \Rightarrow \theta = 45^0$ South–West.

Therefore, the boy hold the umbrella at an angle $45^0$ with the horizontal in order to protect him best.

## PROBLEM SET – VII

**8.** ***(i)** A ship P is sailing with a velocity of 11 knots/hr. in the direction S–E. and a second ship is sailing with velocity of 13 knots/hr. in a direction $30^0$ E of N. Find the velocity of P relative to Q. **[WBSC – 88]**

**(ii)** There are two boats, first boat A is sailing towards North at 6 km/hr. and second boat B is sailing North–West at $6\sqrt{2}$ km/hr. Find the magnitude and the direction of the velocity of the second boat relative to first.

## ANSWERS

**8.** **(i)** Magnitude of relative velocity = 19.08 knots/hr. Direction: $\theta = 86.16^0$, South of East.

**(ii)** Magnitude of the velocity of B relative to A = 6 km./hr. Direction: Due West.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

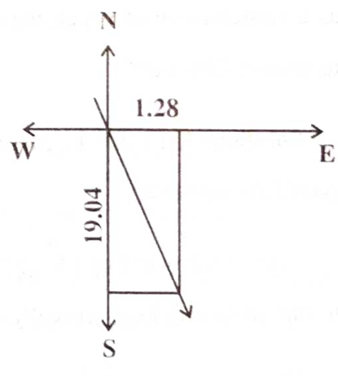
**8.(i)** A ship P is sailing with a velocity of 11 knots/hr. in the direction S–E. and a second ship is sailing with velocity of 13 knots/hr. in a direction $30^0$ E of N. Find the **velocity** of P relative to Q. **[WBSC – 88]**

**Solution:** Let **i** and **j** be the unit vectors of magnitude 1 knot./hr. towards East and North respectively.

Let $\vec{V}_P$ and $\vec{V}_Q$ be the actual velocities of P and Q respectively. The velocity of the first ship P is 11 knots/hr in the S–E direction.

Therefore $\vec{V}_P = 11\cos 45^0\,\mathbf{i} - 11\sin 45^0\,\mathbf{j}$ or, $\vec{V}_P = \frac{11}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

Again velocity of second ship Q is 13 knots/hr. in a direction $30^0$ E of N. That is the ship Q is sailing in a direction making $60^0$ with the East.




Therefore $\vec{V}_Q = 13\cos 60^0\,\mathbf{i} + 13\sin 60^0\,\mathbf{j}$ or $\vec{V}_Q = \frac{13}{2}(\mathbf{i} + \sqrt{3}\,\mathbf{j})$

So velocity of P relative to Q = $\vec{V}_P - \vec{V}_Q = \frac{11}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) - \frac{13}{2}(\mathbf{i} + \sqrt{3}\,\mathbf{j})$

$= \left(\frac{11}{\sqrt{2}} - \frac{13}{2}\right)\mathbf{i} - \left(\frac{11}{\sqrt{2}} + \frac{13}{2}\sqrt{3}\right)\mathbf{j} = 1.28\mathbf{i} - 19.04\mathbf{j} = \vec{V}_{PQ}$ (say).

So, magnitude of relative velocity, $\left|\vec{V}_{PQ}\right| = \sqrt{(1.28)^2 + (19.04)^2} = 19.08$ knots/hr.

**Direction:** Since, in $V_{PQ}$ coefficient of **i** is positive and coefficient of **j** is negative, it lies between some where in South–East.

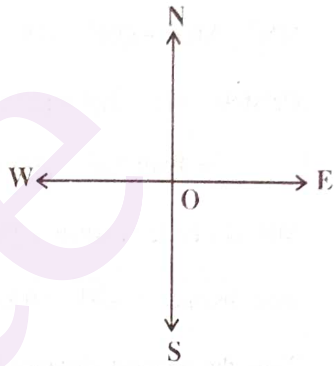
If $\theta$ be the angle made by the direction of $\vec{V}_{PQ}$ with the direction of East then

$\tan\theta = \frac{19.04}{1.28} = 14.895 \Rightarrow \theta = 86.16^0$, South of East **(Ans)**

**8.(ii)** There are two boats, first boat A is sailing towards North at 6 km/hr. and second boat B is sailing North–West at $6\sqrt{2}$ km/hr. Find the magnitude and the direction of the velocity of the second boat relative to first.

**Solution:** Let **i** and **j** be the unit velocity vectors towards East and North respectively.

Let $\vec{V}_A$ and $\vec{V}_B$ be the actual velocities of A and B respectively. The velocity of the first boat A is 6 km/hr. towards North.




Therefore $\vec{V}_A = 6\mathbf{j}$

Again velocity of second boat B is $6\sqrt{2}$ km/hr. in a direction North–West.

That is the boat B is sailing in a direction making $45^0$ with the West.

Therefore, $\vec{V}_B = 6\sqrt{2}\,[\cos 45^0\,(-\mathbf{i}) + \sin 45^0\mathbf{j}]$ or $\vec{V}_B = 6(-\mathbf{i} + \mathbf{j})$

So velocity of B relative to A = $\vec{V}_B - \vec{V}_A = 6(-\mathbf{i} + \mathbf{j}) - 6\mathbf{j} = -6\mathbf{i}$

Hence, required relative velocity = 6 km/hr and direction is due West.

## ■ MISCELLANEOUS

**9.(i)*** Two particles A and B are at same instant 15 km apart. Both starts at the same moment. A moves towards B with an uniform velocity of 5 km/h and B moves perpendicular to AB at 3.75 km/h. Find their **relative velocity** by vector method and time when they are nearest to each other. Also calculate the **shortest distance** between them.

**[WBSC – 89]**

## ANSWERS

**9.** (i) Shortest distance between A and B is 9 km. from the moment they started and the time required is 1.92 hrs.

## SOLUTION OF THE PROBLEM WITH '*' MARKS

**9.(i)** Two particles A and B are at same instant 15 km apart. Both starts at the same moment. A moves towards B with an uniform velocity of 5 km/h and B moves perpendicular to AB at 3.75 km/h. Find their **relative velocity** by vector method and time when they are nearest to each other. Also calculate the **shortest distance** between them.

**[WBSC – 89]**

**Solution:** Let P and Q are the respective positions of the particles A and B at any instant such that PQ = 15 km. A is moving towards B i.e., in the direction PQ and B is moving perpendicularly with the direction of PQ.

Let **i** and **j** are the unit vectors of magnitude 1 km/hr. along PQ and perpendicular to PQ respectively.

$\therefore$ actual velocity of A = $\vec{V}_A = 5\mathbf{i}$ and actual velocity of B = $\vec{V}_B = 3.75\mathbf{j}$

Hence velocity of A relative to B = $\vec{V}_A - \vec{V}_B = 5\mathbf{i} - 3.75\mathbf{j} = \vec{V}_{AB}$ (say).

$$\therefore \quad \left|\vec{V}_{AB}\right| = \sqrt{5^2 + (3.75)^2} = 6.25 \text{ km/hr.}$$

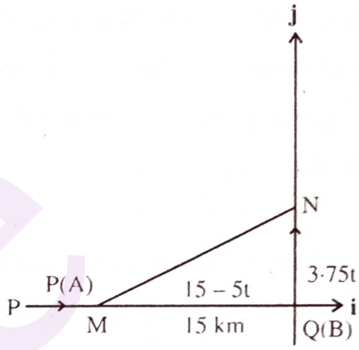
5.116

$\vec{V}_{AB}$ is making an angle of $\tan^{-1}\left(\frac{3.75}{5}\right) = \tan^{-1}\frac{3}{4}$ with the direction of PQ i.e., **i**.

Let M and N are the respective positions of A and B after t hours.

So, PM = 5t, i.e., MQ = PQ – PM = 15 – 5t and QN = 3.75t.

Hence the distance MN is expressed as:




$MN^2 = MQ^2 + QN^2 = (15 - 5t)^2 + (3.75t)^2$

Or $MN^2 = (15 - 5t)^2 + (3.75t)^2 = 225 - 150t + 25t^2 + 14.0625t^2$

$= 39.0625t^2 - 150t + 225 = (6.25t - 12)^2 + 81$

MN will be least when $6.25t - 12 = 0$ or, $t = \frac{12}{6.25} = 1.92$ hrs. and

least distance = $\sqrt{81} = 9$ km.

Thus the shortest distance between A and B is 9 km. from the moment they started and the time required is 1.92 hrs. **(Ans)**

N. Tute

# PRODUCT OF TWO VECTORS

## 6.1. Introduction:

The basic idea of ordinary multiplication of numbers in scalar algebra, by multiplying their magnitudes, cannot be applied to vector algebra because of directional properties of vectors. Two vector quantities can be combined in two distinct ways: one of them yields a **scalar quantity** and the other a **vector quantity**.

For example, in finding the work done by a constant force **F** acting at a point O when the point suffers a displacement **d** in some direction making an angle $\theta$ with the line of action of **F**, we multiply the magnitude of **F** i. e., $|\mathbf{F}|$, by $|\mathbf{d}|\cos\theta$ and the product thus obtained i. e., $|\mathbf{F}|.|\mathbf{d}|\cos\theta$ is called the work done by the force.

**Since work is not related to any direction in space, it is a scalar quantity.**

But, in finding the moment of the force **F** about the point which is the terminus of **d**, we multiply the magnitude of **F** i. e., $|\mathbf{F}|$, by $|\mathbf{d}|\sin\theta$ and the resulting quantity i. e., $|\mathbf{F}|.|\mathbf{d}|\sin\theta$ is a vector quantity, **since it is related to clock–wise or anti–clock wise sense of rotation in space.**

The two operations of combining the vector quantities **F** and **d** are called '**product**' as they have some properties in common with the product of numbers. We shall state the definitions of these operations in the following way:

## 6.2. Scalar (or Dot) product:

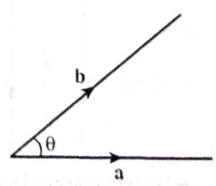
The dot (or scalar) product of two vectors **a** and **b** is defined as the product of their modulii (or lengths) $|\mathbf{a}|$ and $|\mathbf{b}|$ and the cosine of their included angle and is symbolically denoted as **a.b** (read as **a** dot **b**).

Thus $\mathbf{a}.\mathbf{b} = |\mathbf{a}|.|\mathbf{b}|\cos\theta$ where $|\mathbf{a}|$ and $|\mathbf{b}|$ represents the magnitudes of the vectors **a** and **b** respectively.

From the figure, $|\mathbf{b}|\cos\theta$ is the component of **b** along **a** and $|\mathbf{a}|\cos\theta$ is the component of **a** along **b**.

$|\mathbf{b}|\cos\theta$ or $|\mathbf{a}|\cos\theta$ is positive or negative according as $\theta$ is acute or obtuse.

- Dot (or Scalar) product is also called the "**inner product**" or the "**direct product**".




**• The scalar product is commutative:**

From the definition it follows that, $\mathbf{b}.\mathbf{a} = |\mathbf{b}|.|\mathbf{a}|.\cos(-\theta) = |\mathbf{a}|.|\mathbf{b}|.\cos\theta = \mathbf{a}.\mathbf{b}$

Thus, the commutative law holds for the scalar product of two vectors.

**• The associative law has no meaning for scalar product:**

Since, **a.b** is a scalar = p(say), therefore $(\mathbf{a}.\mathbf{b}).\mathbf{c} = p.\mathbf{c} = p\mathbf{c}$ is a vector.

Again, $(\mathbf{b}.\mathbf{c})$ is a scalar = q(say), therefore $\mathbf{a}.(\mathbf{b}.\mathbf{c}) = \mathbf{a}.q = q\mathbf{a}$ is also a vector. In general, $p\mathbf{c} \neq q\mathbf{a}$.

- If m, n be any scalars and **a**, **b** two vectors then, $(m\mathbf{a}) \cdot (n\mathbf{b}) = mn(\mathbf{a} \cdot \mathbf{b}) = mn\,\mathbf{a} \cdot \mathbf{b} = (n\mathbf{a}) \cdot (m\mathbf{b})$.

Since, $(m\mathbf{a}) \cdot (n\mathbf{b}) = mn(|\mathbf{a}|\,|\mathbf{b}|\cos\theta) = mn(\mathbf{a}.\mathbf{b})$.

The above result is evidently not altered by interchanging m and n.

- **Distributive laws:**

The distributive laws with respect to addition of vectors holds for dot product:

(i) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ (ii) $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}$

Cor. $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d}) = \mathbf{a}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{d}$

**Particular cases of dot product:**

If $\theta = 90^0$, then $\cos\theta = 0$; consequently, $\mathbf{a}\cdot\mathbf{b} = 0$.

- Thus the condition of perpendicularity of two vectors is that their dot product is zero.

Of course, $\mathbf{a}\cdot\mathbf{b} = 0$ gives three possibilities; either **a** may be a null vector or **b** may be a null vector or $\theta$ may be a right angle.

If $\theta = 0^0$, then $\cos\theta = 1$; consequently $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|$.

- Thus, if the two vectors are parallel, their scalar product is equal to the product of their modulii.

In particular, **a**.**a** which is written as $|\mathbf{a}|^2 = a^2$, where a is the module of **a**.

$(\mathbf{a}+\mathbf{b})^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2+2\,\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2$

$(\mathbf{a}-\mathbf{b})^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2-2\,\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2$

$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2-|\mathbf{b}|^2$

If **a** is an unit vector, then $|\mathbf{a}|^2 = 1$.

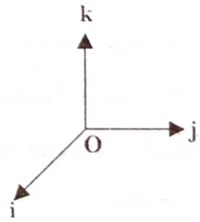
- For **i**, **j**, **k** three mutually perpendicular unit vectors, we have, $\mathbf{i}.\mathbf{i} = |\mathbf{i}|\cdot|\mathbf{i}|\cos 0^0 = 1$ and $\mathbf{i}\cdot\mathbf{j} = |\mathbf{i}|\cdot|\mathbf{j}|\cos 90^0 = 0$

Therefore, $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$ i.e., $|\mathbf{i}|^2 = |\mathbf{j}|^2 = |\mathbf{k}|^2 = 1$

and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$.

- These results can be written in the given table:

| · | **i** | **j** | **k** |
|---|---|---|---|
| **i** | 1 | 0 | 0 |
| **j** | 0 | 1 | 0 |
| **k** | 0 | 0 | 1 |




- **Scalar product in terms of components:**

Let $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, where **i**, **j**, **k** be the unit vectors along the three mutually perpendicular axes.

Then, $\mathbf{a}.\mathbf{b} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$

$= a_1b_1\mathbf{i}\cdot\mathbf{i}+a_1b_2\,\mathbf{i}\cdot\mathbf{j}+a_1b_3\,\mathbf{i}\cdot\mathbf{k}+a_2b_1\mathbf{j}\cdot\mathbf{i}+a_2b_2\,\mathbf{j}\cdot\mathbf{j}+a_2b_3\mathbf{j}\cdot\mathbf{k}+a_3b_1\mathbf{k}\cdot\mathbf{i}+a_3b_2\mathbf{k}\cdot\mathbf{j}+a_3b_3\mathbf{k}\cdot\mathbf{k}$

or $\mathbf{a}\cdot\mathbf{b} = a_1b_1+a_2b_2+a_3b_3$ [since, $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$]

Therefore, if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

- Angle between two proper vector **a** and **b**.

Let θ be the angle between the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

Then $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Again we know $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|.|\mathbf{b}|\cos\theta$

Therefore, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}$

In particular if **a** and **b** are both unit vectors, then $\cos\theta = \mathbf{a}\cdot\mathbf{b}$ i. e., the scalar product of two unit vectors is equal to the cosine of the angle between their directions.

- The scalar product of two proper vectors is positive, negative or zero according as the angle between them is acute, obtuse or right.

## PROBLEM SET – VIII

9. (i) If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, then show that $\mathbf{a}\cdot\mathbf{b} = 7$.

(ii) Show that the vectors $9\mathbf{i} + \mathbf{j} - 6\mathbf{k}$ and $4\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ are perpendicular to each other.

(iii) Prove that three vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{c} = 7\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ are at right angles to each other.

*(iv) The vectors $2\mathbf{i} + a\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ are perpendicular to each other. Show that $a = 3$. **[WBSC – 88, 87]**

10. (i) Find the angle between two vectors $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

*(ii) The position vector of two points A and B are respectively $3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ and $6\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}$. Calculate the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ where O is the origin.

(iii) The position vectors of the points A, B, C are respectively $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$. Find the greatest angle of the triangle ABC.

(iv) Find the angle between the diagonals of a cube.

(v) Find the value of λ such that the vectors $(3\mathbf{A} + 4\mathbf{B})$ and $(2\mathbf{A} + \mathbf{B})$ are perpendicular to each other, given that $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$. **[WBSC – 84]**

*(vi) If $\mathbf{A} = \mathbf{B} + \mathbf{C}$ where $\mathbf{B}.\mathbf{R} = 0$ and $\mathbf{C}$ is parallel to $\mathbf{R}$, show that $\mathbf{C} = \dfrac{(\mathbf{A}.\mathbf{R})\mathbf{R}}{R^2}$ **[WBSC – 88]**

(vii) If **a** and **b** are unit vectors and θ is the angle between them, show that $\sin\left(\frac{1}{2}\theta\right) = \frac{1}{2}|\mathbf{a} - \mathbf{b}|$ **[WBSC – 93]**

(viii) Show that if $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$ then, either vector is perpendicular to the other.

(ix) $\alpha = 4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\beta = \mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and $\gamma = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ are three vectors. Find a vector ρ which is perpendicular to both the vectors α and β and satisfy the relation $\rho.\gamma = 21$. **[WBSC – 03]**

*(x) Find a vector **d** which is perpendicular to both the vectors $\mathbf{a} = 4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and which satisfies the relation $\mathbf{d}.\mathbf{c} = 21$, where $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$. **[WBSC – 93]**

*(xi) Given $\alpha = 3\mathbf{i} - \mathbf{j}$, $\beta = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ where **i**, **j**, **k** have their usual meanings. Express $\beta$ in the form $\beta = \beta_1 + \beta_2$, where $\beta_1$ is parallel to $\alpha$ and $\beta_2$ is perpendicular to $\alpha$. **[WBSC – 87]**

*(xii) With usual notations for **i**, **j**, **k** show that the unit vector in the plane of the vector $\alpha = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\beta = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ that is perpendicular to the vector $\gamma = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ is $\pm\frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}$. **[WBSC – 89, 86]**

(xiii) If $\alpha = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\beta = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\gamma = 5\mathbf{i} + 8\mathbf{k}$ where **i**, **j**, **k** are unit vectors having the directions of positive x, y and z axes of three dimentional rectan–gular coordinate system, determine l amd m so that $\gamma - l\alpha - m\beta$ is orthogonal to both $\alpha$ and $\beta$. **[WBSC – 89]**

*(xiv) **a**, **b**, **c** are three mutually perpendicular vectors of equal magnitude. Show that **a** + **b** + **c** is equally inclined to **a**, **b**, **c**.

*(xv) If **a**, **b**, **c** be three non–coplanar vectors and $\mathbf{r}\cdot\mathbf{a} = \mathbf{r}\cdot\mathbf{b} = \mathbf{r}\cdot\mathbf{c} = 0$, show that **r** is a null vector.

**11.** (i) If $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ find **(a)** the projection of **a** in the direction of **b** and **(b)** the projection of **a** on x–axis. **[WBSC – 91]**

(ii) Find the projection of the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ on the vector $\mathbf{b} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$. **[WBSC – 93, 83]**

*(iii) If $\overrightarrow{AB} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{CD} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, find the projection of $\overrightarrow{CD}$ in the direction of $\overrightarrow{AB}$ and also on z–axis. **[WBSC – 90, 85]**

**12.** *(i) $\alpha, \beta, \gamma$ are vectors of lengths 3, 4, 5 respectively. It is also given that $\alpha$ is perpendicular to $\beta + \gamma$, $\beta$ to $\gamma + \alpha$ and $\gamma$ to $\alpha + \beta$. Show that the length of the vector $\alpha + \beta + \gamma$ is $5\sqrt{2}$. **[WBSC – 90]**

(ii) If $\alpha$, $\beta$ and $\gamma$ are three vectors satisfying the condition $\alpha + \beta + \gamma = \mathbf{0}$ and if $|\alpha| = 3$, $|\beta| = 4$ and $|\gamma| = 5$, then show that $\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha = -25$. **[WBSC – 84]**

*(iii) If $\alpha$, $\beta$ and $\gamma$ are three unit vectors satisfying the condition $\alpha + \beta + \gamma = \mathbf{0}$ then show that $\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha = -\frac{3}{2}$.

**13.** Prove by vector method:

**(i)(a)** $\cos(A + B) = \cos A \cos B - \sin A \sin B$ ***(b)** $\cos(A - B) = \cos A \cos B + \sin A \sin B$ **[WBSC – 92, 91]**

**(ii)** In any triangle ABC

***(a)** $a^2 = b^2 + c^2 - 2bc\cos A$ **(b)** $b^2 = c^2 + a^2 - 2ca\cos B$ **(c)** $c^2 = a^2 + b^2 - 2ab\cos C$

**(iii)** In any triangle ABC

**(a)** $a = b\cos C + c\cos B$ **(b)** $b = c\cos A + a\cos C$ ***(c)** $c = a\cos B + b\cos A$

## ANSWERS

10. (i) $\cos^{-1}\frac{12}{77}$ (ii) $\cos^{-1}\left[\frac{44}{\sqrt{74}\cdot\sqrt{184}}\right]$ (iii) $90^0$ (iv) $\cos^{-1}\frac{1}{3}$ (v) $\frac{-11\pm\sqrt{73}}{4}$

10. (ix) $\frac{21}{2}(\mathbf{i} - \mathbf{j} - \mathbf{k})$ (x) $7(\mathbf{i} - \mathbf{j} - \mathbf{k})$ (xi) $\beta_1 = \frac{3}{2}(3\mathbf{i} - \mathbf{j})$, $\beta_2 = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$

11. (i)(a) $\frac{32}{7}$ units. (b) 2 units. (ii) $\frac{19}{9}$ units. (iii) 3 units, 1 unit.

## PROBLEM SET – IX

14. *(i) Prove, using vector methods, the perpendiculars from vertices of a triangle to the opposite sides are concurrent. [WBSC – 93, 91]

(ii) In a tetrahedron, if two pairs of opposite edges are perpendiculars the third pair are also perpendicular to each other; and the sum of the squares on two opposite edges is the same for each pair.

(iii) Prove that the angle in a semi–circle is a right angle.

(iv) Prove that the median to the base of an isosceles triangle is perpendicular to the base.

(v) Show that the parallelogram whose diagonals are equal is a rectangle.

*(vi) Prove by vector method that the diagonals of a rhombus are at right angle.

(vii) Prove that the perpendicular bisector of the sides of a triangle are concurrent.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**9.(iv)** The vectors 2**i** + a**j** + **k** and 4**i** – 2**j** – 2**k** are perpendicular to each other. Show that a = 3.

**Solution:** Since the vectors 2**i** + a**j** + **k** and 4**i** – 2**j** – 2**k** are perpendicular to each other, therefore

$(2\mathbf{i} + a\mathbf{j} + \mathbf{k})\cdot(4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = 0$ or $8 - 2a - 2 = 0$ [since, $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{i} = 0$]

or $2a = 6$ or $a = 3$ **(Ans)**

**10.(ii)** The position vectors of two points A and B are respectively 3**i** + 7**j** – 4**k** and 6**i** – 2**j** + 12**k**. Calculate the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ where O is the origin.

**Solution:** By the problem $\overrightarrow{OA} = 3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ and $\overrightarrow{OB} = 6\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}$

Now $|\overrightarrow{OA}| = \sqrt{3^2 + 7^2 + 4^2} = \sqrt{74}$ and $|\overrightarrow{OB}| = \sqrt{6^2 + 2^2 + 12^2} = \sqrt{184}$

And $\overrightarrow{OA}.\overrightarrow{OB} = (3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}).(6\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}) = 18 - 14 - 48 = -44$ [since, $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$ and $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = \mathbf{k}.\mathbf{i} = 0$]

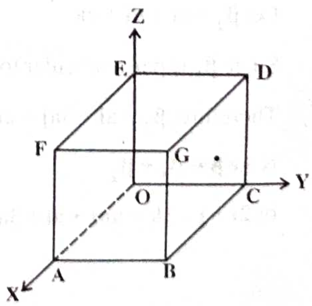
Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then $\cos\theta = \dfrac{\overrightarrow{OA}.\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|} = -\dfrac{44}{\sqrt{74}.\sqrt{184}}$

Therefore $\theta = \cos^{-1}\left[-\dfrac{44}{\sqrt{74}.\sqrt{184}}\right]$ **(Ans)**

**10.(iv)** Find the angle between the diagonals of a cube.

**Solution:** Let OABCDEFG be the cube having side a and one corner O is the origin. Let OX, OY, OZ be three rectangular axes along OA, $\overrightarrow{OC}$ and $\overrightarrow{OE}$.

Let **i**, **j**, **k** be three unit vectors along three rectangular axes OX, OY and OZ.




Then $\overrightarrow{OC} = a\mathbf{j}$, $\overrightarrow{OG} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$, $\overrightarrow{OF} = a\mathbf{i} + a\mathbf{k}$

Then $\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = (a\mathbf{i} + a\mathbf{k}) - a\mathbf{j} = a\mathbf{i} - a\mathbf{j} + a\mathbf{k}$

Now, $|\overrightarrow{OG}| = \sqrt{a^2 + a^2 + a^2} = a\sqrt{3}$ and $|\overrightarrow{CF}| = \sqrt{a^2 + a^2 + a^2} = a\sqrt{3}$

Let θ be the angle between the diaginals OG and CF,

then $\cos\theta = \dfrac{\overrightarrow{OG}.\overrightarrow{CF}}{|\overrightarrow{OG}||\overrightarrow{CF}|} = \dfrac{(ai+aj+ak)\cdot(ai-aj+ak)}{a\sqrt{3}\cdot a\sqrt{3}} = \dfrac{a^2-a^2+a^2}{3a^2} = \dfrac{1}{3}$

Therefore $\theta = \cos^{-1}\frac{1}{3}$.

Similarly, we can find the angle between the other diagonals is $\cos^{-1}\frac{1}{3}$. **(Ans)**

**10.(vi)** If **A** = **B** + **C** where **B.R** = 0 and **C** is parallel to **R**, show that $\mathbf{C} = \dfrac{(\mathbf{A.R})\mathbf{R}}{R^2}$.

**Solution:** Given, **A** = **B** + **C** or **A.R** = (**B** + **C**).**R** = **B.R** + **C.R** ............... (1)

Since **C** is parallel to **R**, let **C** = m**R**, m is a scalar and given **B.R** = 0

Therefore (1) becomes, **A.R** = m**R.R** = m$R^2$ or $m = \dfrac{\mathbf{A\cdot R}}{R^2}$

Therefore, **C** = m**R** or, $\mathbf{C} = \dfrac{(\mathbf{A.R})\mathbf{R}}{R^2}$ **(Proved)**

**10.(x)** Find a vector **d** which is perpendicular to both the vectors **a** = 4**i** + 5**j** – **k** and **b** = **i** – 4**j** + 5**k** and which satisfies the relation **d.c** = 21, where **c** = 3**i** + **j** – **k**. **[WBSC – 93]**

**Solution:** Let $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ ......(1)

By the problem, **d** is perpendicular to **a** = 4**i** + 5**j** – **k** and **b** = **i** – 4**j** + 5**k**

Therefore, **a.d** = 0 ⇒ $(4\mathbf{i}+5\mathbf{j}-\mathbf{k}).(d_1\mathbf{i}+d_2\mathbf{j}+d_3\mathbf{k}) = 0$ or $4d_1 + 5d_2 - d_3 = 0$ ......(2)

And **b.d** = 0 ⇒ $(\mathbf{i}-4\mathbf{j}+5\mathbf{k}).(d_1\mathbf{i}+d_2\mathbf{j}+d_3\mathbf{k}) = 0$ or $d_1 - 4d_2 + 5d_3 = 0$ ......(3)

From (2) and (3) we get, $\dfrac{d_1}{25-4} = \dfrac{d_2}{-1-20} = \dfrac{d_3}{-16-5}$ or, $\dfrac{d_1}{21} = \dfrac{d_2}{-21} = \dfrac{d_3}{-21} = p$ (say)

Therefore, $d_1 = p, d_2 = -p, d_3 = -p$ ......(4)

Again **d.c** = 21

or $(d_1\mathbf{i}+d_2\mathbf{j}+d_3\mathbf{k}).(3\mathbf{i}+\mathbf{j}-\mathbf{k}) = 21$ or $3d_1 + d_2 - d_3 = 21$

or $3p - p + p = 21 \Rightarrow p = 7$. Therefore, $d_1 = 7, d_2 = -7, d_3 = -7$

Therefore **d** = 7(**i** – **j** + **k**) **(Ans)**

**10.(xi)** Given α = 3**i** – **j**, β = 2**i** + **j** – 3**k** where **i**, **j**, **k** have their usual meanings. Express β in the form $\beta = \beta_1 + \beta_2$, where $\beta_1$ is parallel to α and $\beta_2$ is perpendicular to α. **[WBSC – 87]**

**Solution:** Given α = 3**i** – **j**, β = 2**i** + **j** – 3**k**

Now, $\beta_1$ is parallel to α therefore, $\beta_1 = m\alpha$, m is any scalar.

Let $\beta_2$ = a**i** + b**j** + c**k**

Since $\beta_2$ is perpendicular to α therefore, $\beta_2.\alpha = 0$ or (a**i** + b**j** + c**k**).(3**i** – **j**) = 0 or 3a – b = 0 or b = 3a

Therefore, $\beta_2$ = a**i** + 3a**j** + c**k**

Now $\beta = \beta_1 + \beta_2$

or 2**i** + **j** – 3**k** = mα + a**i** + 3a**j** + c**k** or 2**i** + **j** – 3**k** = m(3**i** – **j**) + a**i** + 3a**j** + c**k** .....(1)

or $2\mathbf{i}+\mathbf{j}-3\mathbf{k}=(3m+a)\mathbf{i}+(3a-m)\mathbf{j}+c\mathbf{k}$

Equating the coefficients of like vectors from both sides we get,

$3m+a=2$ ......(2) $\quad 3a-m=1$ ......(3) $\quad c=-3$ ......(4)

From (2) $a=2-3m$ therefore $3(2-3m)-m=1$ [from (3)] or $6-9m-m=1$ or $10m=5$ or $m=\frac{1}{2}$.

From (1) $a=2-3m=2-\frac{3}{2}=\frac{1}{2}$

Therefore, (1) becomes,

$2\mathbf{i}+\mathbf{j}-3\mathbf{k}=\frac{1}{2}(3\mathbf{i}-\mathbf{j})+\frac{1}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-3\mathbf{k}$ or $\beta=\beta_1+\beta_2$ where $\beta_1=\frac{3}{2}(3\mathbf{i}-\mathbf{j})$ and $\beta_2=\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-3\mathbf{k}$ **(Ans)**

**10. (xii)** With usual notations for **i**, **j**, **k** show that the unit vector in the plane of the vector $\alpha=\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\beta=\mathbf{i}+\mathbf{j}-2\mathbf{k}$ that is perpendicular to the vector $\gamma=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ is $\pm\frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}$ **[WBSC – 89, 86]**

**Solution:** Any vector in the plane of the vector $\alpha=\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\beta=\mathbf{i}+\mathbf{j}-2\mathbf{k}$ is $a\alpha+b\beta$, where a and b are scalars.

By the problem, $a\alpha+b\beta$ and $\gamma$ are perpendicular.

Therefore, $(a\alpha+b\beta)\cdot\gamma=0$ or $a\alpha\cdot\gamma+b\beta\cdot\gamma=0$ or $a(\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})+b(\mathbf{i}+\mathbf{j}-2\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=0$

or $a(2-2-1)+b(2-1-2)=0$ or $-a-b=0$ or $b=-a$

Therefore, $a\alpha+b\beta=a\alpha-a\beta=a(\alpha-\beta)=a\{(\mathbf{i}+2\mathbf{j}-\mathbf{k})-(\mathbf{i}+\mathbf{j}-2\mathbf{k})\}=a(\mathbf{j}+\mathbf{k})$

Therefore the required unit vector $=\pm\frac{a(\mathbf{j}+\mathbf{k})}{\sqrt{a^2+a^2}}=\pm\frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}$ **(Proved)**

**10.(xiv)** **a**, **b**, **c** are three mutually perpendicular veetors of equal magnitude. Show that $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is equally inclined to **a**, **b**, **c**.

**Solution:** $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}=a^2$ [since **a**, **b**, **c** are mutually perpendicular vectors, $\mathbf{b}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{a}=0$.]

$(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{b}=b^2$, since $\mathbf{a}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{b}=0$

$(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{c}=c^2$, since $\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}=0$

Since, **a**, **b**, **c** are of equal magnitude, therefore $a^2=b^2=c^2$ or $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{b}=(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{c}$

If A, B, C be the angles made by $\mathbf{a}+\mathbf{b}+\mathbf{c}$ with **a**, **b**, **c** respectively, then form (1) we get

$|\mathbf{a}+\mathbf{b}+\mathbf{c}||\mathbf{a}|\cos A=|\mathbf{a}+\mathbf{b}+\mathbf{c}||\mathbf{b}|\cos B=|\mathbf{a}+\mathbf{b}+\mathbf{c}||\mathbf{c}|\cos C$

Therefore, $\cos A=\cos B=\cos C$ [since, $|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|$ or $A=B=C$ Hence proved.

**10.(xv)** If **a**, **b**, **c** be three non–coplanar vectors and $\mathbf{r}\cdot\mathbf{a}=\mathbf{r}\cdot\mathbf{b}=\mathbf{r}\cdot\mathbf{c}=0$, show that **r** is a null vector.

**Solution:** Given $\mathbf{r}\cdot\mathbf{a}=\mathbf{r}\cdot\mathbf{b}=\mathbf{r}\cdot\mathbf{c}=0$

$\mathbf{r}\cdot\mathbf{a}=0$ gives either $\mathbf{r}=\mathbf{0}$ or **r** is perpendicular to **a**.

Similarly from $\mathbf{r}\cdot\mathbf{b}=0$ and $\mathbf{r}\cdot\mathbf{c}=0$ we get $\mathbf{r}=\mathbf{0}$ or **r** is perpendicular to **b** and **c**.

Therefore either $\mathbf{r}=\mathbf{0}$ and **r** is perpendicular to **a**, **b**, **c**.

Since **a**, **b**, **c** are non–coplanar vector therefore **r** cannot be perpendicular to **a**, **b**, **c** simultaneously

Therefore $\mathbf{r}=0$ i.e., **r** is a null vector.

**11. (iii)** If $\overrightarrow{AB} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{CD} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, find the projection of $\overrightarrow{CD}$ in the direction of $\overrightarrow{AB}$ and also on z–axis.

**[WBSC – 90, 85]**

**Solution:** Projection of $\overrightarrow{CD}$ in the direction of $\overrightarrow{AB} = \overrightarrow{CD}.\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \dfrac{\overrightarrow{CD}.\overrightarrow{AB}}{|\overrightarrow{AB}|} = \dfrac{(5\mathbf{i} - 3\mathbf{j} + \mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{|2\mathbf{i} + \mathbf{j} + 2\mathbf{k}|} = \dfrac{10 - 3 + 2}{\sqrt{2^2 + 1^2 + 2^2}} = \dfrac{9}{3}$

$= 3$ unit.

Projection of $\overrightarrow{CD}$ on z–axis $= \overrightarrow{CD}\cdot\dfrac{\mathbf{k}}{|\mathbf{k}|} = (5\mathbf{i} - 3\mathbf{j} + \mathbf{k})\cdot\mathbf{k} = 1$ unit. **(Ans)**

**12.(i)** $\alpha$, $\beta$, $\gamma$ are vectors of lengths 3, 4, 5 respectively. It is also given that $\alpha$ is perpendicular to $\beta + \gamma$, $\beta$ to $\gamma + \alpha$ and $\gamma$ to $\alpha + \beta$. Show that the length of the vector $\alpha + \beta + \gamma$ is $5\sqrt{2}$. **[WBSC – 90]**

**Solution:** By the problem, $|\alpha| = 3$, $|\beta| = 4$, $|\gamma| = 5$.

Again, $\alpha\cdot(\beta + \gamma) = 0 \Rightarrow \alpha\cdot\beta + \alpha\cdot\gamma = 0$

Similarly, $\beta\cdot(\gamma + \alpha) = 0 \Rightarrow \beta\cdot\gamma + \beta\cdot\alpha = 0$ and, $\gamma\cdot(\alpha + \beta) = 0 \Rightarrow \gamma\cdot\alpha + \gamma\cdot\beta = 0$

Now $|\alpha + \beta + \gamma|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + 2(\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha)$

$= |\alpha|^2 + |\beta|^2 + |\gamma|^2 + (\alpha\cdot\beta + \alpha\cdot\gamma) + (\beta\cdot\gamma + \beta\cdot\alpha) + (\gamma\cdot\alpha + \gamma\cdot\beta) = 32 + 42 + 52 = 50$

Therefore, $|\alpha + \beta + \gamma| = 5\sqrt{2}$ **(Proved)**

**12.(iii)** If $\alpha$, $\beta$ and $\gamma$ are three unit vectors satisfying the condition $\alpha + \beta + \gamma = \mathbf{0}$ then show that $\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha = -\frac{3}{2}$

**Solution:** By the problem, $|\alpha| = 1$, $|\beta| = 1$, $|\gamma| = 1$.

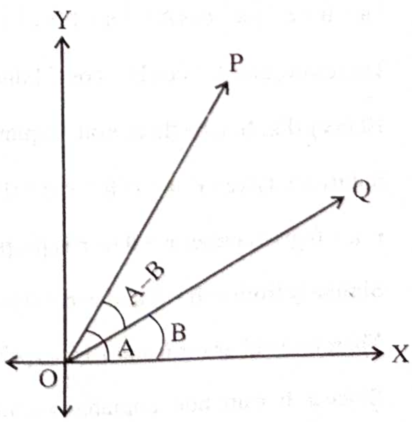
Now $\alpha + \beta + \gamma = \mathbf{0}$ gives $|\alpha + \beta + \gamma|^2 = \mathbf{0}$. or $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + 2(\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha) = 0$

or $1 + 1 + 1 + 2(\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha) = 0$

Therefore, $\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha = -\frac{3}{2}$ **(Proved)**

**13.(i)(b)** $\cos(A - B) = \cos A \cos B + \sin A \sin B$ **[WBSC – 92, 91]**

**Solution:** Let O be the origin and OX and OY are perpendicular axes. **i** and **j** denote unit vectors along OX and OY respectively.




Let OP and OQ makes angles A and B with OX.

Let **p** and **q** denote the unit vectors along $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

Then $\mathbf{p} = \mathbf{i}\cos A + \mathbf{j}\sin A$ and $\mathbf{q} = \mathbf{i}\cos B + \mathbf{j}\sin B$ and $|\mathbf{p}| = 1$, $|\mathbf{q}| = 1$.

Now angle between the vectors **p** and **q** is $(A - B)$.

Therefore, $\mathbf{p}.\mathbf{q} = |\mathbf{p}|.|\mathbf{q}|\cos(A - B) = \cos(A - B)$

or $\cos(A - B) = \mathbf{p}.\mathbf{q} = (\mathbf{i}\cos A + \mathbf{j}\sin A).(\mathbf{i}\cos B + \mathbf{j}\sin B)$

or $\cos(A - B) = \mathbf{i}.\mathbf{i}\cos A\cos B + \mathbf{i}.\mathbf{j}\cos A\sin B + \mathbf{j}.\mathbf{i}\sin A\cos B + \mathbf{j}.\mathbf{j}\sin A\sin B$

or $\cos(A - B) = \cos A\cos B + \sin A\sin B$ [since, $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = 1$ and $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$] **(Proved)**

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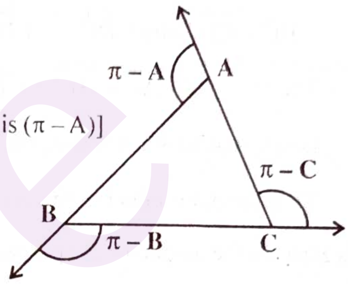
**13. (ii) (a)** Prove by vector method: $a^2 = b^2 + c^2 - 2bc\cos A$

**Solution:** Let ABC be a triangle in which $\angle BAC = A$, $\angle ABC = B$, $\angle BCA = C$ and $BC = a$, $CA = b$, $AB = c$ and hence, $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{c}$




Now, in triangle ABC, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$

or $\overrightarrow{BC} = -(\overrightarrow{AB} + \overrightarrow{CA})$ or $BC^2 = (\overrightarrow{AB} + \overrightarrow{CA})^2$ or $BC^2 = AB^2 + CA^2 + 2\,\overrightarrow{AB}.\overrightarrow{CA}$

or $BC^2 = AB^2 + CA^2 + 2\,|\overrightarrow{AB}|.|\overrightarrow{CA}|\cos(\pi - A)$ [since angle between $\overrightarrow{AB}$ and $\overrightarrow{CA}$ is $(\pi - A)$]

or $AB^2 = BC^2 + CA^2 - 2\,BC.CA\cos A$

or $a^2 = b^2 + c^2 - 2bc\cos A$ **(Proved)**

**13.(iii)(c)** In any triangle ABC prove that $a = b\cos C + c\cos B$

**Solution:** Let ABC be a triangle [ In the above figure of 13(ii)(a) ] in which $\angle BAC = A$, $\angle ABC = B$, $\angle BCA = C$ and $BC = a$, $CA = b$, $AB = c$ and hence, $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{c}$

Now, in triangle ABC, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$

or $\overrightarrow{AB} = -(\overrightarrow{BC} + \overrightarrow{CA})$ or $\overrightarrow{AB}.\overrightarrow{AB} = -\overrightarrow{AB}.(\overrightarrow{BC} + \overrightarrow{CA}) = -\overrightarrow{AB}.\overrightarrow{BC} - \overrightarrow{AB}.\overrightarrow{CA}$

or $AB^2 = -|\overrightarrow{AB}|\,|\overrightarrow{BC}|\cos(\pi - B) - |\overrightarrow{AB}|\,|\overrightarrow{CA}|\cos(\pi - A)$ or $AB^2 = -AB\cdot BC(-\cos B) - AB\cdot CA(-\cos A)$

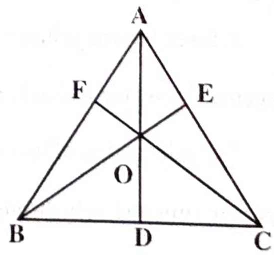
or $AB^2 = AB\cdot BC\cos B + AB\cdot CA\cos A$ or $AB = BC\cos B + CA\cos A$

or $c = a\cos B + b\cos A$ **(Proved)**

**14. (i)** Prove, using vector methods, the perpendiculars from vertices of a triangle to the opposite sides are concurrent.

**[WBSC – 93, 91]**

**Solution:** Let ABC be a triangle. The perpendiculars BE and CF drawn from the vertices B and C upon the respective opposite sides AC and AB meet at O. AO is joined and produced to D to meet BC. We have to prove that AD is perpendicular to BC.




With respect to O as origin let the position vectors of the vertices A, B and C of triangle ABC are respectively **a, b, c**. Therefore, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$

Now it is evident that, $\overrightarrow{BE} = p\,\overrightarrow{BO}$, $\overrightarrow{CF} = q\,\overrightarrow{CO}$ and $\overrightarrow{AD} = r\,\overrightarrow{AO}$, where p, q, r are non–zero scalars. Therefore, $\overrightarrow{BE} = -p\mathbf{b}$, $\overrightarrow{CF} = -q\mathbf{c}$ and $\overrightarrow{AD} = -r\mathbf{a}$

Again, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{c} - \mathbf{b}$ and $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \mathbf{a} - \mathbf{c}$

Since, BE is perpendicular to CA, therefore, $\overrightarrow{BE}.\overrightarrow{CA} = 0$ or $-p\mathbf{b}.(\mathbf{a} - \mathbf{c}) = 0$ or $\mathbf{b}.\mathbf{a} = \mathbf{b}.\mathbf{c}$ or $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{c}$ ................(i)

Again, CF is perpendicular to AB, therefore, $\overrightarrow{CF}.\overrightarrow{AB} = 0$ or $-q\mathbf{c}.(\mathbf{b} - \mathbf{a}) = 0$ or $\mathbf{c}.\mathbf{b} = \mathbf{c}.\mathbf{a}$ or $\mathbf{b}.\mathbf{c} = \mathbf{c}.\mathbf{a}$ ................(ii)

From (i) and (ii) we get, $\mathbf{a}.\mathbf{b} = \mathbf{c}.\mathbf{a}$ or $\mathbf{a}.\mathbf{c} = \mathbf{a}.\mathbf{b}$ or $\mathbf{a}.(\mathbf{c} - \mathbf{b}) = 0$ or $-r\mathbf{a}.(\mathbf{c} - \mathbf{b}) = 0$ or $\overrightarrow{AD}.\overrightarrow{BC} = 0$

Therefore, AD is perpendicular to BC.

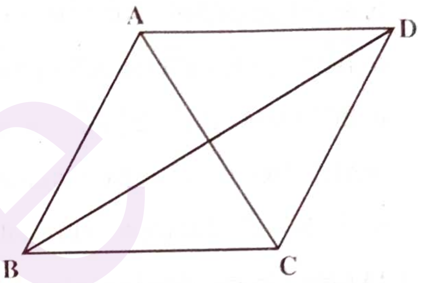
Hence the perpendiculars from the vertices upon the opposite sides are concurrent. **(Proved)**

**14. (vii)** Prove by vector method that the diagonals of a rhombus are at right angle.

**Solution:** Let ABCD be a rhombus. From triangular law, we get from triangle BCD, $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

From triangle ABC, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{DC}$ [since, $\overrightarrow{AB} = \overrightarrow{DC}$]

$= \overrightarrow{BC} - \overrightarrow{CD}$ [since, $\overrightarrow{DC} = -\overrightarrow{CD}$]




Therefore, $\overrightarrow{BD}.\overrightarrow{AC} = (\overrightarrow{BC} + \overrightarrow{CD}).(\overrightarrow{BC} - \overrightarrow{CD}) = |\overrightarrow{BC}|^2 - |\overrightarrow{CD}|^2 = 0$

[sincs, ABCD is a rhombus, BC = CD $\Rightarrow |\overrightarrow{BC}|^2 = |\overrightarrow{CD}|^2$]

Thus the dot product of the vectors representing the diagonals of the rhombus is zero. So the diagonals are perpendicular to each other. **(Proved)**

## 6.3. Elementary application in Mechanics:

**Displacement:** A displacement of a point from its position at A to its position at B is specified in magnitude and direction by the vector $\overrightarrow{AB}$. Displacement being a vector quantity, two or more displacements can be compounded according to the vector law of addition. Two successive displacements, one from A to B followed by another from B to C will produce the effect of the resultant displacement from A to C and will be The work done by a force acting on a particle is the scalar quantity proportional to the product of the force and the resolved part of the displacement in the direction of the force.

Thus if the vectors **F** and **d** represent the force and displacement respectively and θ be the angle between them, then the magnitude of the **work done** is given by $|\mathbf{F}|\,|\mathbf{d}|\cos\theta = \mathbf{F}.\mathbf{d}$

No work is done by the force, if the displacement be in the direction perpendicular to the direction of the force. In this case $\theta = \frac{\pi}{2}$, and hence $\cos\theta = 0$.

A **force** has magnitude, direction as well as a definite line of action. To represent a force we need the concept of a **Line vector** – a vector which is restricted to lie in a definite line. A force acting on a body can be represented by a line vector.

The joint action of two concurrent forces produces the same dynamical effect as that of a single force, equivalent to their **vector sum** and acts through the point of concurrence.

Again the joint action of seveal forces represented by the vectors $\mathbf{F}_1, \mathbf{F}_2, \ldots\ldots\ldots, \mathbf{F}_n$ acting at a point P, will be the same as the action of a single force represented by a vector **R** where $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots\ldots\ldots + \mathbf{F}_n = \Sigma\mathbf{F}$ acting through the same point P.

**R** is called the resultant of the system of forces.

## PROBLEM SET–X

15. *(i) Show that the force acting in the direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$ will be required to have a component of 20 kgs in the direction of $\mathbf{i}+8\mathbf{j}+\mathbf{k}$ is $6\sqrt{22}$ kg. [WBSC – 88]

*(ii) Forces of magnitudes 5, 3 and 2 units each act in directions $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $3\mathbf{i}-2\mathbf{j}+6\mathbf{k}$ and $2\mathbf{i}-3\mathbf{j}-6\mathbf{k}$ respectively on a particle which is displaced from the point (2, –1, –3) to the point (5, –1, 1). Find the work done by the forces. [WBSC–93]

(iii) A force whose components are 4, 3, 2 lbs moves from the point (1, 2, 3) to the point (4, 3, 4). The unit for measuring the distance in the co–ordinate system being 1 ft. Use vector method to calculate the work done.

(iv) Forces of magnitudes 1, 2, 3 kgf act on a particle in the direction of (6, 2, 3), (3, –2, 6) and (2, –3, 6) respectively. Find the amount of work done in displacing the particle from P(2, 3, 4) to Q(3, 4, 5). Take unit of distance as metre.

*(v) The point of application of the force (–2, 4, 2) is displaced from the point (3, –5, 1) to the point (5, 9, 7), but the force is suddenly halved when the point of application moves half the distance. Find the work done by the force. [WBSC – 92]

(vi) The point of application of a force, $\mathbf{F}=12\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ moves from A = (2, –1, 1) to B(5, 8, 2) when other force $\mathbf{G}=6\mathbf{i}+2\mathbf{j}+\mathbf{k}$ is suddenly added to $\mathbf{F}$ and then the point of application moves from B to C(7, 12, 3). Fine the total work done.

(vii) A particle acted on by the constant forces $3\mathbf{i}+2\mathbf{j}+5\mathbf{k}$ and $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ is displaced from the point $2\mathbf{i}-\mathbf{j}-3\mathbf{k}$ to the point $4\mathbf{i}-3\mathbf{j}+7\mathbf{k}$. Calculate the total work done. [WBSC – 04, 07, 08, 09]

*(viii) Forces $2\mathbf{i}-5\mathbf{j}+6\mathbf{k}$, $-\mathbf{i}-2\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}+7\mathbf{j}$ act on a particle is displaced from the point $4\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ to the point $6\mathbf{i}+\mathbf{j}-3\mathbf{k}$. Find the work done. [WBSC – 91]

(ix) Forces $2\mathbf{i}+7\mathbf{j}$, $2\mathbf{i}-5\mathbf{j}+6\mathbf{k}$, $-\mathbf{i}+2\mathbf{j}-\mathbf{k}$ act on a point P whose position vector is $4\mathbf{i}-3\mathbf{j}-2\mathbf{k}$. Determine the **work done** if the point of application is displaced to the point Q whose position vector is $6\mathbf{i}+\mathbf{j}-3\mathbf{k}$. [WBSC – 88]

(x) The point of application of the force $\mathbf{F}=6\mathbf{i}+8\mathbf{k}$ is changed from P(1, –1, 2) to Q(–1, 1, 2). Find the **work done** in the process. [WBSC – 87]

(xi) Two forces $\mathbf{P}=2\mathbf{i}-5\mathbf{j}+6\mathbf{k}$ and $\mathbf{Q}=-\mathbf{i}+2\mathbf{j}-\mathbf{k}$ act on a particle. Determine the **work done** when the particle is displaced from A to B, position vectors of A and B being $4\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ and $6\mathbf{i}+\mathbf{j}-3\mathbf{k}$ respectively.

## ANSWERS

15. (ii) $\frac{213}{7}$ unit. (iii) 17 ft-lb. (iv) $\frac{4}{7}$ kg-m (v) 48 unit. (vi) 141 unit. (vii) 26 unit. (viii) 1 unit. (ix) 17 unit. (x) 12 unit, negative sign indicates that work is done against the force. (xi) 15 unit, negative sign indicates that work is done against the force.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**15.(i)** Show that the force acting in the direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$ will be required to have a component of 20 kgs in the direction of $\mathbf{i}+8\mathbf{j}+\mathbf{k}$ is $6\sqrt{22}$ kg. **[WBSC – 88]**

**Solution:** Let the force $\mathbf{F}$ of magnitude P kg. acts in the direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$. Then $\mathbf{F}=\frac{P(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{1^2+1^2+1^2}}=\frac{P(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{3}}$

Now, the projection of $\mathbf{F}$ in the direction $\mathbf{i}+8\mathbf{j}+\mathbf{k}$

$$=\frac{\mathbf{F}\cdot(\mathbf{i}+8\mathbf{j}+\mathbf{k})}{\sqrt{1^2+8^2+1^2}}=\frac{\mathbf{F}\cdot(\mathbf{i}+8\mathbf{j}+\mathbf{k})}{\sqrt{66}}=\frac{P(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{3}}\cdot\frac{(\mathbf{i}+8\mathbf{j}+\mathbf{k})}{\sqrt{66}}=\frac{P(1+8+1)}{\sqrt{198}}=\frac{10P}{\sqrt{198}}$$

By the problem, $\frac{10P}{\sqrt{198}}=20$ or $P=2\sqrt{198}=6\sqrt{22}$ kg. **(Proved)**

**15.(ii)** Forces of magnitudes 5, 3 and 2 units each act in directions $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $3\mathbf{i}-2\mathbf{j}+6\mathbf{k}$ and $2\mathbf{i}-3\mathbf{j}-6\mathbf{k}$ respectively on a particle which is displaced from the point (2, –1, –3) to the point (5, –1, 1). Find the work done by the forces.

**[WBSC – 93]**

**Solution:** Let $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ be the forces having magnitudes 5, 3 and 2 units respectively.

Then $\mathbf{F}_1=\frac{5(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})}{\sqrt{6^2+2^2+3^2}}=\frac{5}{7}(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})$, $\mathbf{F}_2=\frac{3(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})}{\sqrt{3^2+2^2+6^2}}=\frac{3}{7}(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})$, $\mathbf{F}_3=\frac{2(2\mathbf{i}-3\mathbf{j}-6\mathbf{k})}{\sqrt{2^2+3^2+6^2}}=\frac{2}{7}(2\mathbf{i}-3\mathbf{j}-6\mathbf{k})$

Therefore, total force: $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3=\frac{5}{7}(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})+\frac{3}{7}(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})+\frac{2}{7}(2\mathbf{i}-3\mathbf{j}-6\mathbf{k})=\frac{43\mathbf{i}-2\mathbf{j}+21\mathbf{k}}{7}$

The position vectors of the given two points (2, –1, –3) and ( 5, –1, 1) are respectively $2\mathbf{i}-\mathbf{j}-3\mathbf{k}$ and $5\mathbf{i}-\mathbf{j}+\mathbf{k}$

Therefore displacement: $\mathbf{d}=(5\mathbf{i}-\mathbf{j}+\mathbf{k})-(2\mathbf{i}-\mathbf{j}-3\mathbf{k})=3\mathbf{i}+4\mathbf{k}$

Hence work done by the forces $=\mathbf{F}\,.\,\mathbf{d}=\frac{43\mathbf{i}-2\mathbf{j}+21\mathbf{k}}{7}\cdot(3\mathbf{i}+4\mathbf{k})=\frac{129+84}{7}=\frac{213}{7}$ unit. **(Ans)**

**15.(v)** The point of application of the force (–2, 4, 2) is displaced from the point (3, –5, 1) to the point (5, 9, 7), but the force is suddenly halved when the point of application moves half the distance. Find the work done by the force. **[WBSC – 92]**

**Solution:** Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are three unit vectors along three rectangular axes OX, OY and OZ respectively.

Then the force vector $\mathbf{F}=-2\mathbf{i}+4\mathbf{j}+2\mathbf{k}$

Position vector of the given two points are $3\mathbf{i}-5\mathbf{j}+\mathbf{k}$ and $5\mathbf{i}+9\mathbf{j}+7\mathbf{k}$

The displacement $\mathbf{d}=(5\mathbf{i}+9\mathbf{j}+7\mathbf{k})-(3\mathbf{i}-5\mathbf{j}+\mathbf{k})=2\mathbf{i}+14\mathbf{j}+6\mathbf{k}$

Therefore, by the problem, the total work done

$$=\mathbf{F}\cdot\frac{\mathbf{d}}{2}+\frac{\mathbf{F}}{2}\cdot\frac{\mathbf{d}}{2}=\left(\frac{1}{2}+\frac{1}{4}\right)\mathbf{F}\cdot\mathbf{d}=\frac{3}{4}(\mathbf{F}\cdot\mathbf{d})$$

$$=\frac{3}{4}(-2\mathbf{i}+4\mathbf{j}+2\mathbf{k})\,.\,(2\mathbf{i}+14\mathbf{j}+6\mathbf{k})$$

$$=\frac{3}{4}(-4+56+12)=48 \text{ unit. } \textbf{(Ans)}$$

**15.(viii)** Forces $2i - 5j + 6k$, $-i - 2j - k$ and $2i + 7j$ act on a particle is displaced from the point $4i - 3j - 2k$ to the point $6i + j - 3k$. Find the work done. **[WBSC – 91]**

**Solution:**

Let $F_1, F_2, F_3$ be the given three forces. Then $F_1 = 2i - 5j + 6k$, $F_2 = -i - 2j - k$, $F_3 = 2i + 7j$

The resultant force $F = F_1 + F_2 + F_3 = (2i - 5j + 6k) + (-i - 2j - k) + (2i + 7j) = 3i + 5k$

Given two points are $4i - 3j - 2k$ and $6i + j - 3k$.

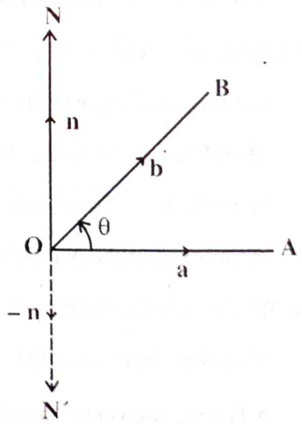
Therefore displacement: $d = (6i + j - 3k) - (4i - 3j - 2k) = 2i + 4j - k$

Hence work done by the forces $= F \cdot d = (3i + 5k) \cdot (2i + 4j - k) = 6 - 5 = 1$ unit. **(Ans)**

## 6.4. Vector (or Cross ) product:

**Definition:** The vector product (or cross product) of two vectors **a** and **b**, written as $\mathbf{a} \times \mathbf{b}$ and read as 'a cross b', is defined as the vector $|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\mathbf{n}$ where $|\mathbf{a}|$, $|\mathbf{b}|$ are the modulus of **a** and **b**, $\theta$ is the angle between them and **n** is the unit vector perpendicular to the plane of the vectors **a** and **b** in the direction of the translation of a right–handed screw due to a rotation from **a** to **b**.

Therefore, $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\mathbf{n}$




Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\angle AOB = \theta$. We draw ON perpendicular to the plane containing $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Consider the direction of $\overrightarrow{ON}$ such that as $\overrightarrow{OA}$ is turned into $\overrightarrow{OB}$ through the angle $\theta$, $\overrightarrow{ON}$ points in the direction in which a right handed screw would progress if turned in the same manner. Then the vector product of **a** and **b** is a vector of modulus $|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\mathbf{n}$ in the direction of $\overrightarrow{ON}$. Hence, if we take the length ON equal to $|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$, then $\overrightarrow{ON}$ shall represent the vector product of **a** and **b**.

- Vector (or Cross ) product is also called the **"outer product"**.

### Properties of vector product:

- **The vector product is not commutative** i. e., $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

Since by definition, $\mathbf{b} \times \mathbf{a}$ implies rotation in the plane from **b** to **a**, so the direction of the vector $\mathbf{b} \times \mathbf{a}$ is just opposite to the direction of $\mathbf{a} \times \mathbf{b}$; but their modules are equal, each being equal to $|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$.

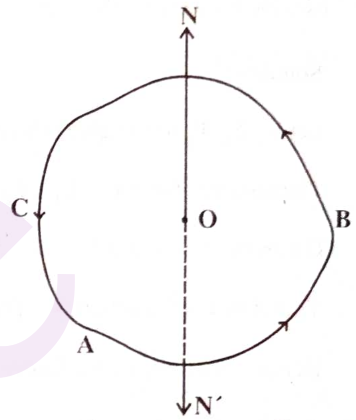
Therefore $\mathbf{b} \times \mathbf{a} = -\,\mathbf{a} \times \mathbf{b}$. Thus the vector product is not commutative.

- **The vector product is not associative** i. e., $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
- The distributive laws with respect to addition holds good for the vector product of two vectors, namely,

(i) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (ii) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

- If m, n be any scalars and **a**, **b** two vectors then $(m\,\mathbf{a}) \times (n\,\mathbf{b}) = mn(\mathbf{a} \times \mathbf{b}) = (n\,\mathbf{a}) \times (m\,\mathbf{b})$.

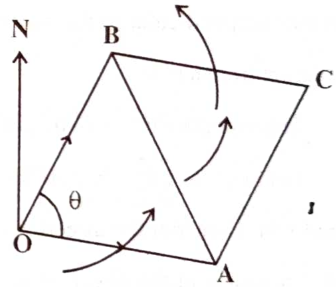
• **Vector area:** Vector area is a vector quantity whose modulus is the plane area bounded by a closed figure traced in a definite sense and whose direction is normal to the plane of the figure, its sense and the direction of description of the boundary corresponding to the translation and rotation of a right–handed screw. It may be represented by a vector drawn along the normal to the plane of the curve in the direction relative to which it is positive and with magnitude equal to the measure of the area. Thus if the length ON be equal to the measure of the area ABC, $\overrightarrow{ON}$ will represent the vector area ABC.




The vector areas being vectors can be added according to the vector law of addition.

• **Vector product as a vector area:**

Consider a triangle OAB. Suppose that the side OA has to be turned through an angle $\theta$ in the direction indicated by the arrows, so that it may coincide with the side OB. For this sense of rotation the positive direction of the normal to the area OAB is along $\overrightarrow{ON}$.




If $\mathbf{n}$ be the unit vector along the normal $\overrightarrow{ON}$, using the definition of vector product, we have, $\overrightarrow{OA} \times \overrightarrow{OB} = |\overrightarrow{OA}|\,|\overrightarrow{OB}| \sin\theta\, \mathbf{n}$

But area of triangle OAB is $\frac{1}{2}$OA.OB $\sin\theta$.

Therefore, we can write, $\Delta OAB = \frac{1}{2} |\overrightarrow{OA}|\,|\overrightarrow{OB}| \sin\theta\, \mathbf{n}$

or $2\,\Delta OAB = |\overrightarrow{OA}|\,|\overrightarrow{OB}| \sin\theta\, \mathbf{n} = \overrightarrow{OA} \times \overrightarrow{OB}$

Now consider the parallelogram OACB on OA and OB as its two adjacent sides. Then, with the same sense of rotation as for the $\Delta OAB$, we have $2\,\Delta OAB$ = Parallelogram OACB.

Thus, Parallelogram OACB = $\overrightarrow{OA} \times \overrightarrow{OB}$

• Hence, **the vector product of two vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ may be interpreted as the vector area of the parallelogram completed on OA and OB as its adjacent sides.**

**Particular cases:** If the two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, then the angle between them is zero; i. e., $\theta = 0$ and so $\sin\theta = 0$.

• Therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ is the condition for $\mathbf{a}$ and $\mathbf{b}$ to be parallel.

The vector product of a vector with itself is a null vector, i. e., $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

• If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\theta = 90^0$ and so $\sin\theta = 1$.

In that case $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\mathbf{n}$.

• If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors in the directions of the three axes forming a right–handed system mutually perpendicular, then $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

And $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$; $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

These results can be written in the following table:

| × | i | j | k |
|---|---|---|---|
| i | 0 | k | − j |
| j | − k | 0 | i |
| k | j | − i | 0 |

- **Vector product in terms of components:**

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the three mutually perpendicular axes.

Then, $\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$= a_1b_1\, \mathbf{i}\times\mathbf{i} + a_1b_2\, \mathbf{i}\times\mathbf{j} + a_1b_3\, \mathbf{i}\times\mathbf{k} + a_2b_1\, \mathbf{j}\times\mathbf{i} + a_2b_2\, \mathbf{j}\times\mathbf{j} + a_2b_3\, \mathbf{j}\times\mathbf{k} + a_3b_1\, \mathbf{k}\times\mathbf{i} + a_3b_2\, \mathbf{k}\times\mathbf{j} + a_3b_3\, \mathbf{k}\times\mathbf{k}$

or $\mathbf{a} \times \mathbf{b} = a_1b_2\, \mathbf{i}\times\mathbf{j} + a_1b_3\mathbf{i}\times\mathbf{k} + a_2b_1\, \mathbf{j}\times\mathbf{i} + a_2b_3\, \mathbf{j}\times\mathbf{k} + a_3b_1\, \mathbf{k}\times\mathbf{i} + a_3b_2\, \mathbf{k}\times\mathbf{j}$

$= a_1b_2\, \mathbf{k} + a_1b_3\, (-\mathbf{j}) + a_2b_1\, (-\mathbf{k}) + a_2b_3\, \mathbf{i} + a_3b_1\, \mathbf{j} + a_3b_2\, (-\mathbf{i})$

or $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ [ since, $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$ and $\mathbf{i}\times\mathbf{j} = \mathbf{k}, \mathbf{j}\times\mathbf{k} = \mathbf{i}, \mathbf{k}\times\mathbf{i} = \mathbf{j}$ etc]

- In the determinant form we may write this as $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- **Angle between two proper vectors:**

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the three mutually perpendicular axes.

Then, $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

Again $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ , $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$ and

$|\mathbf{a} \times \mathbf{b}| = \sqrt{(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2}$

Let $\theta$ be the angle between the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then,

$$\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{a}|} = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2}}{\sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

If $\theta = 0$ then, $(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 = 0$ or $a_2b_3 - a_3b_2 = 0,\ a_3b_1 - a_1b_3 = 0,\ a_1b_2 - a_2b_1 = 0$

or, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$. Hence the two vectors **a** and **b** are parallel.

**Note:** If we express **a**, **b** as a linear combination of three non–coplanar vectors **l**, **m**, **n** i.e., as $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}$ and $\mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$, then we can show that

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{m}\times\mathbf{n} & \mathbf{n}\times\mathbf{l} & \mathbf{l}\times\mathbf{m} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

# PROBLEM SET–XI

**16.** (i) The vectors **a** and **b** are given by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$; $\mathbf{b} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$; obtain $\mathbf{a} \times \mathbf{b}$.

(ii) Show that the vector product of two vectors $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{j} + 4\mathbf{k}$ is $-9\mathbf{i} - 12\mathbf{j} + 3\mathbf{k}$.

*(iii) Show that the vector product of two vectors $4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is $7\mathbf{i} - 9\mathbf{j} + \mathbf{k}$.

(iv) Find the **length** of the vector $(3\mathbf{i} + 4\mathbf{j}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k})$.

**17.** (i) Two vectors **a** and **b** are expressed in terms of unit vectors as follows: $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. What is the **unit vector perpendicular** to each of the vectors? **[WBSC – 83]**

*(ii) What is the **unit vector perpendicular** to each of the vector $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$. Calculate the sine of the angle between these vectors. **[WBSC – 84]**

(iii) Find a **unit vector perpendicular** to each of the vector $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(iv) If the position vectors of the three points A, B, C be $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ respectively, then find a **vector perpendicular** to the plane ABC.

*(v) The vectors joining the origin to the points A, B, C are $(2, -1, 1)$, $(3, 0, 1)$, $(1, -2, 3)$ respectively. Find a **unit vector perpendicular** to the plane ABC.

**18.** (i) Show that the **vector area** of the triangle the position vectors of whose vertices are $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + \mathbf{j} - \mathbf{k}$ is $\frac{1}{2}(18\mathbf{i} - 11\mathbf{j} - \mathbf{k})$.

*(ii) Show that the **vector area** of the triangle the position vectors of whose vertices are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ is $\frac{1}{2}(-4\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

(iii) Position vector of A and B referred to the origin O are $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $-3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$. Find the **scalar area** of the triangle OAB.

*(iv) Prove that the **vector area** of a triangle whose vertices have position vector **a**, **b** and **c** is $\frac{1}{2}(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$

(v) Prove that three points **a**, **b**, **c** are **collinear** if $\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

(vi) If **a**, **b**, **c** are vectors from the origin to the points A, B, C; show that $\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}$ is perpendicular to the plane ABC.

**19.** (i) Prove that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times (\mathbf{c} + \mathbf{a}) + \mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{0}$.

*(ii) Show that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$. Interpret the result geometrically.

(iii) Interpret geometrically the vector equation $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$, where **a** and **b** are constants vector and **r** is any arbitrary vector.

*(iv) If **a** is an arbitrary vector, find the value of: $\frac{1}{2}[\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k})]$. **[WBSC – 87]**

(v) If $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ are the two diagnals of a parallelogram, find its area. **[WBSC – 91]**

(vi) Find the **area** of the parallelogram having diagonals $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

(vii) The vector from the origin to the points A, B, C, D are: $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{c} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{d} = \mathbf{k} - \mathbf{j}$. Using suitable products of vector concept, show that the line AB and CD are parallel to each other.

**20.** (i) Prove with usual notations the following formulae:

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$ *(b) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

*(ii) In any triangle ABC show that, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

# ANSWERS

**16.** (i) $42\mathbf{i}+14\mathbf{j}-2\mathbf{k}$ (iv) $\sqrt{74}$ unit. **17.** (i) $\pm\frac{8}{\sqrt{3}}(\mathbf{i}-\mathbf{j}-\mathbf{k})$ (ii) $\pm\frac{1}{\sqrt{35}}(-3\mathbf{i}-\mathbf{j}+5\mathbf{k})$ (iii) $\pm\frac{1}{3}(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})$ (iv) $3(2\mathbf{i}-\mathbf{j}-2\mathbf{k})$ (v) $\pm\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$ **18.** (iii) $-\mathbf{i}-\mathbf{j}-\mathbf{k}$ or $\sqrt{3}$ sq. unit.

**19.** (iii) **r**, **a**, **b** are coplanar. (iv) **a** (v) $5\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ or $\sqrt{30}$ sq. units. (vi) $\mathbf{i}+7\mathbf{j}+5\mathbf{k}$ or $5\sqrt{3}$ sq. unit.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**16.(iii)** The vector product of two vectors $4\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = \mathbf{i}(6+1)-\mathbf{j}(8+1)+\mathbf{k}(4-30 = 7\mathbf{i}-9\mathbf{j}+\mathbf{k} \textbf{ (Proved)}$$

**17.(ii)** The vector perpendicular to the vectors $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=-3\mathbf{i}+4\mathbf{j}-\mathbf{k}$ is given by

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = \mathbf{i}(1-4)-\mathbf{j}(-2+3)+\mathbf{k}(8-3) = -3\mathbf{i}-\mathbf{j}+5\mathbf{k}$$

Therefore, the required unit vector $= \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,\mathbf{a}|} = \frac{-3\mathbf{i}-\mathbf{j}+5\mathbf{k}}{\sqrt{(-3)^2+(-1)^2+5^2}} = \frac{-3\mathbf{i}-\mathbf{j}+5\mathbf{k}}{\sqrt{35}}$ **(Ans)**

**17.(v)** With respect to O as origin let the position vectors of A, B, C are respectively given by $\overrightarrow{OA}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\overrightarrow{OB}=3\mathbf{i}+\mathbf{k}$, $\overrightarrow{OC}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

Therefore, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(3\mathbf{i}+\mathbf{k})-(2\mathbf{i}-\mathbf{j}+\mathbf{k})=\mathbf{i}+\mathbf{j}$ and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(\mathbf{i}-2\mathbf{j}+3\mathbf{k})-(2\mathbf{i}-\mathbf{j}+\mathbf{k})=-\mathbf{i}-\mathbf{j}+2\mathbf{k}$

Therefore, vector perpendicular to the plane ABC is given by .

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ -1 & -1 & 2 \end{vmatrix} = \mathbf{i}(2+0)-\mathbf{j}(2+0)+\mathbf{k}(-1+1) = 2\mathbf{i}-2\mathbf{j}$$

Therefore the required unit vector $= \frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|} = \pm\frac{2\mathbf{i}-2\mathbf{j}}{\sqrt{2^2+(-2)^2}} = \pm\frac{2\mathbf{i}-2\mathbf{j}}{2\sqrt{2}} = \pm\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$ **(Ans)**

**18.(ii)** With respect to O as origin let the position vectors of the vertices A, B, C of triangle ABC are respectively $\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

Then $\overrightarrow{OA}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\overrightarrow{OB}=\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $\overrightarrow{OC}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$

Therefore, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(\mathbf{i}+2\mathbf{j}+3\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})=\mathbf{j}+2\mathbf{k}$ and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(2\mathbf{i}+3\mathbf{j}+\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})=\mathbf{i}+2\mathbf{j}$

Therefore, the area of the triangle ABC

$$\frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{AB}\right)=\frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \frac{1}{2}[\mathbf{i}(-4)-\mathbf{j}(-2)+\mathbf{k}(-1)=\frac{1}{2}(-4\mathbf{i}+2\mathbf{j}-\mathbf{k})$$

**18.(iv)** With respect to O as origin, let the position vectors of the vertices A, B, C of triangle ABC are respectively **a, b, c.**

Therefore $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$

Then, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\mathbf{b}-\mathbf{a}$ and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\mathbf{c}-\mathbf{a}$

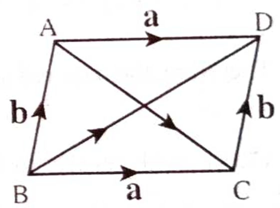
Therefore the required vector area of triangle ABC $=\frac{1}{2}\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\frac{1}{2}\{(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\}$

$$=\frac{1}{2}(\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{a})=\frac{1}{2}(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})=[\text{since, }\mathbf{a}\times\mathbf{a}=0]\ \textbf{(Proved)}$$

**19.(ii)** L.H.S $=(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=\mathbf{a}\times\mathbf{a}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{b}$

$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{b}$ $[\text{since, }\mathbf{a}\times\mathbf{a}=\mathbf{0}]=2(\mathbf{a}\times\mathbf{b})=$ R.H.S **(Proved)**

**2nd Part:**




Let ABCD be a parallelogram, in which $\overrightarrow{AD}=\overrightarrow{BC}=\mathbf{a}$ and $\overrightarrow{BA}=\overrightarrow{CD}=\mathbf{b}$.

Now in triangle ADC, $\overrightarrow{AC}+\overrightarrow{CD}+\overrightarrow{DA}=0$ or $\overrightarrow{AC}=-\overrightarrow{CD}-\overrightarrow{DA}=\overrightarrow{AD}-\overrightarrow{CD}=\mathbf{a}-\mathbf{b}$

Again in Triangle BDC, $\overrightarrow{BD}+\overrightarrow{DC}+\overrightarrow{CB}=0$ or $\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}=\mathbf{a}+\mathbf{b}$

Therefore, $\overrightarrow{AC}\times\overrightarrow{BD}=(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2(\overrightarrow{BC}\times\overrightarrow{BA})$ or $\overrightarrow{BC}\times\overrightarrow{BA}=\frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{BD}\right)$

$\Rightarrow$ Area of the parallelogram ABCD is half of the area of the parallelogram formed by the diagonals of the parallelogram ABCD. This is the required geometrical interpretation. **(Ans)**

**19.(iv)** Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$

Therefore, $\mathbf{a}\times\mathbf{i}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times\mathbf{i}=a_2(\mathbf{j}\times\mathbf{i})+a_3(\mathbf{k}\times\mathbf{i})==-a_2\mathbf{k}+a_3\mathbf{j}$

Therefore, $\mathbf{i}\times(\mathbf{a}\times\mathbf{i})=\mathbf{i}\times(-a_2\mathbf{k}+a_3\mathbf{j})=-a_2(\mathbf{i}\times\mathbf{k})+a_3(\mathbf{i}\times\mathbf{j})=a_2\mathbf{j}+a_3\mathbf{k}$

Similarly, $\mathbf{j}\times(\mathbf{a}\times\mathbf{j})=a_3\mathbf{k}+a_1\mathbf{i}$ and $\mathbf{k}\times(\mathbf{a}\times\mathbf{k})=a_1\mathbf{i}+a_2\mathbf{j}$

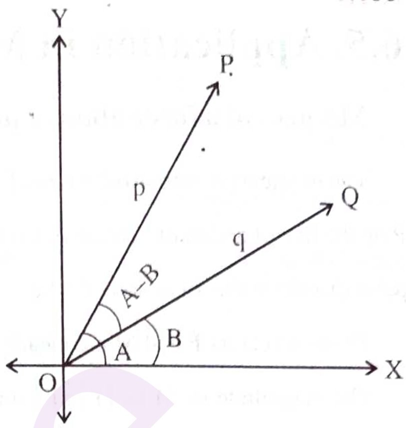
Therefore, $\frac{1}{2}[\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k})]=\frac{1}{2}[a_2\mathbf{j}+a_3\mathbf{k}+a_3\mathbf{k}+a_1\mathbf{i}+a_1\mathbf{i}+a_2\mathbf{j}]$

$=\frac{1}{2}[2a_1\mathbf{i}+2a_2\mathbf{j}+2a_3\mathbf{k}]=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}$ **(Ans)**

**20.(i)(b) Solution:**

Let O be the origin and OX and OY are perpendicular axes. **i** and **j** denote unit vectors along OX and OY respectively. Let **p** and **q** denote the unit vectors along $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in the XY plane and let these vector make angles A and B with x–axis. Since **p** and **q** lies on XY plane, their **k** component of **p** and **q** will be zero.

Therefore, $\mathbf{p} = \mathbf{i}\cos A + \mathbf{j}\sin A$ and $\mathbf{q} = \mathbf{i}\cos B + \mathbf{j}\sin B$ and $|\mathbf{p}| = 1$, $|\mathbf{q}| = 1$.




Now angle between the vectors $\mathbf{p}$ and $\mathbf{q}$ is $(A - B)$.

Now, $\mathbf{q} \times \mathbf{p} = (\mathbf{i}\cos B + \mathbf{j}\sin B) \times (\mathbf{i}\cos A + \mathbf{j}\sin A)$

or $\sin(A - B)\mathbf{k} = \mathbf{i} \times \mathbf{i}\cos B\cos A + \mathbf{i} \times \mathbf{j}\cos B\sin A$

$+ \mathbf{j} \times \mathbf{i}\sin B\cos A + \mathbf{j} \times \mathbf{j}\sin B\sin A$

or $\sin(A - B)\mathbf{k} = \mathbf{k}\cos B\sin A + (-\mathbf{k})\sin B\cos A$

[since, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$]

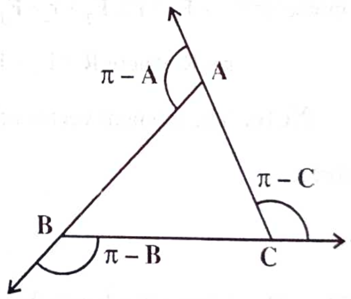
or $\sin(A - B)\mathbf{k} = (\sin A\cos B - \cos A\sin B)\mathbf{k}$.

Therefore, $\sin(A - B) = (\sin A\cos B - \cos A\sin B)$ **(Proved)**

**20.(ii)** Using vector method, prove that in any triangle ABC show that, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

**Solution:** Let the vectors denoting the sides of the triangle ABC are given by, $BC = \mathbf{a}$, $CA = \mathbf{b}$ and $AB = \mathbf{c}$.

Now according to vector addition, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$




Hence $\mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$ or $\mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$

or $\mathbf{a} \times \mathbf{b} = -\mathbf{a} \times \mathbf{c}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$ ................(i)

Again $\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$ or $\mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$

or $-\mathbf{b} \times \mathbf{a} = \mathbf{b} \times \mathbf{c}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$ ................(ii)

From (i) and (ii) we get, $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$

or $\frac{1}{2}(\mathbf{a} \times \mathbf{b}) = \frac{1}{2}(\mathbf{b} \times \mathbf{c}) = \frac{1}{2}(\mathbf{c} \times \mathbf{a})$

or $|\mathbf{a}||\mathbf{b}|\sin(\pi - C) = |\mathbf{b}||\mathbf{c}|\sin(\pi - A) = |\mathbf{c}||\mathbf{a}|\sin(\pi - B)$

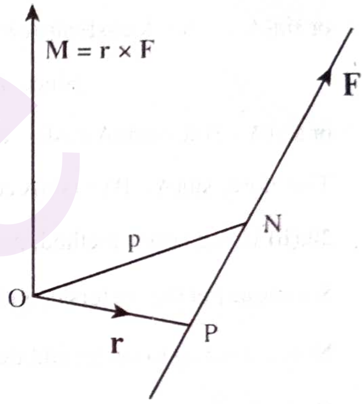
or $ab\sin C = bc\sin A = ca\sin B$ or $\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

Hence, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ **(Proved)**

# 6.5. Application in Mechanics:

### Moment of a force about a point:

The moment (or torque) of a force **F** about the point O is the vector $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, where **r** is the position vector of any point P on the line of action of F, relative to O. **M** being perpendicular to both **r** and **F**, is perpendicular to the plane in which the point O and the line of action of **F** lie.




The two vectors **F** and **M** are clearly perpendicular to each other so that $\mathbf{F}.\mathbf{M} = 0$.

The magnitude of **M** is $|\mathbf{r}||\mathbf{F}|\sin\angle OPN = p|\mathbf{F}|$, where p is the length of the perpendicular ON from O on the line of action of **F**. This gives the magnitude, direction and the line of action of **F** such that a rotation from OP to F is positive with respect to the direction of **M**.

For a number of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots\ldots, \mathbf{F}_n$ passing through P, we have the sum of the moments $= \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \mathbf{r} \times \mathbf{F}_3 + \ldots\ldots\ldots + \mathbf{r} \times \mathbf{F}_n = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots\ldots + \mathbf{F}_n)$

$= \mathbf{r} \times \mathbf{R}$, where $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots\ldots + \mathbf{F}_n$. **R** is called the resultant force.

**Note:** The moment vector of a non–zero force about a point O is zero, iff the point O lies on the line of action of the force.

## PROBLEM SET – XII

**21.** **(i)** A force $3\mathbf{i} + \mathbf{k}$ acts through the point $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Find the **torque** ($\mathbf{M} = \mathbf{r} \times \mathbf{F}$) about the point $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. **[WBSC – 85]**

**(ii)** A force $7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ acts through the point (–1, 1, 2). Find the **torque** ($\mathbf{M} = \mathbf{r} \times \mathbf{F}$) about the point (2, 1, 4).

**(iii)** Forces $2\mathbf{i} + 7\mathbf{j}$, $2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ act on a point P whose position vector is $4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. Find the vector–oment of the resultant forces acting at P about the point Q whose position vector is $6\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

***(iv)** A force of 6 units acts through the point (4, –1, 7) in the direction of the vector $9\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$. Find the **moment** of the force about the point (1, –3, 2) and **moments** about the z–axis. **[WBSC – 91]**

**(v)** A force of magnitude 35 grams act through a point having position A(1, –2, 3) and has a direction $6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find the moment of the force about origin and also its moment about the axes of coordinates, take distancs in centimeter.

**(vi)** A force of 15 units acts in the direction of the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and passes through a point $2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Find the moment of the force about the point $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

***(vii)** Given that the position vectors of P and Q are (1, 3, 1) and (3, 5, 2) respectively. Show that the **moment** of the force PQ about an axis through A(3, 1, 0) in the direction of the vector (3, 4, 12) is **[WBSC – 89]**

*(viii) Find the **moment** about a line through the origin having direction $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, due to 30 kg force acting at a point $(-4, 2, 5)$ in the direction of $12\mathbf{i}-4\mathbf{j}-3\mathbf{k}$. [WBSC – 87]

*(ix) Find the **moment** about the point $(-2, 4, -6)$ of the force represented in magnitude and positions by AB, where the points A and B have the co–ordinates $(1, 2, -3)$ and $(3, -4, 2)$ respectively. [WBSC – 83]

# ANSWERS

18. (i) $\sqrt{211}$ units. (ii) $\sqrt{38}$ units. (iii) $\sqrt{761}$ units. (iv) $\frac{102\sqrt{13}}{11}$ units.

(v) $-60\mathbf{i}+75\mathbf{j}+70\mathbf{k}$, $M_x=60$, $M_y=75$, $M_z=70$ gm-cm. (vi) $5\sqrt{117}$ units. (viii) $\frac{-760}{13}$ units. (ix) $\sqrt{341}$

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**21.(iv)** A force of 6 units acts through the point $(4, -1, 7)$ in the direction of the vector $9\mathbf{i}+6\mathbf{j}-2\mathbf{k}$. Find the moment of the force about the point $(1, -3, 2)$ and moments about the z–axis. [WBSC – 91]

**Solution:**

Let $\mathbf{F}$ be the force of magnitude 6 units and acting in the direction $9\mathbf{i}+6\mathbf{j}-2\mathbf{k}$.

Unit vector in the direction of $9\mathbf{i}+6\mathbf{j}-2\mathbf{k}$ is $=\dfrac{9\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{9^2+6^2+2^2}}=\dfrac{1}{11}(9\mathbf{i}+6\mathbf{j}-2\mathbf{k})$

Then $\mathbf{F}=\dfrac{6(9\mathbf{i}+6\mathbf{j}-2\mathbf{k})}{\sqrt{9^2+6^2+2^2}}=\dfrac{6}{11}(9\mathbf{i}+6\mathbf{j}-2\mathbf{k})$

Let $\overrightarrow{OP}=4\mathbf{i}-\mathbf{j}+7\mathbf{k}$ and $\overrightarrow{OA}=\mathbf{i}-3\mathbf{j}+2\mathbf{k}$

Therefore, $\mathbf{r}=\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OP}=(4\mathbf{i}-\mathbf{j}+7\mathbf{k})-(\mathbf{i}-3\mathbf{j}+2\mathbf{k})=3\mathbf{i}+2\mathbf{j}+5\mathbf{k}$

Hence moment of the force $\mathbf{F}$ about the point $(1, -3, 2)$ is

$$\mathbf{M}=\mathbf{r}\times\mathbf{F}=\frac{6}{11}\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & 5\\ 9 & 6 & -2\end{vmatrix}=\frac{6}{11}(-34\mathbf{i}+51\mathbf{j})$$

Therefore, magnitude $=|\mathbf{M}|=\dfrac{6}{11}\sqrt{34^2+51^2}=\dfrac{102\sqrt{13}}{11}$ unit. **(Ans)**

In order to find the moment about z–axis, we shall have to find out the projection of $\mathbf{M}$ upon z–axis. But in $\mathbf{M}$, the coefficient of $\mathbf{k}$ is zero. Hence the moment about z–axis = 0.

**21.(vii)** Given that the position vectors of P and Q are $(1, 3, 1)$ and $(3, 5, 2)$ respectively. Show that the moment of the force $\overrightarrow{PQ}$ about an axis through $A(3, 1, 0)$ in the direction of the vector $(3, 4, 12)$ is $-\frac{80}{13}$. [WBSC – 89]

**Solution:** We know that moment about an axis is a scalar and it will be obtained by taking projection of moment vector on the axis $3\mathbf{i}+4\mathbf{j}+12\mathbf{k}$.

With respect to the origin O the position vector of P and Q are respectively $\overrightarrow{OP}=\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $\overrightarrow{OQ}=3\mathbf{i}+5\mathbf{j}+2\mathbf{k}$

Therefore, Force vector $\mathbf{F} = \overrightarrow{OQ} - \overrightarrow{OP} = (3\mathbf{i}+5\mathbf{j}+2\mathbf{k})-(\mathbf{i}+3\mathbf{j}+\mathbf{k}) = 2\mathbf{i}+2\mathbf{j}+\mathbf{k}$

Again $\overrightarrow{OA} = 3\mathbf{i}+\mathbf{j}$

Therefore, $\mathbf{r} = \overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = (3\mathbf{i}+5\mathbf{j}+2\mathbf{k})-(3\mathbf{i}+\mathbf{j}) = 0\mathbf{i}+4\mathbf{j}+2\mathbf{k}$

Hence moment of the force $\mathbf{F}$ about A(3, 1, 0) is

$$\mathbf{M} = \mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 4\mathbf{j} - 8\mathbf{k}$$

Hence, the moment of the force $\mathbf{F}$ about axis, the line $3\mathbf{i}+4\mathbf{j}+12\mathbf{k}$ is

$$\frac{(3\mathbf{i}+4\mathbf{j}-12\mathbf{k})\cdot(4\mathbf{j}-8\mathbf{k})}{\sqrt{3^2+4^2+12^2}} = -\frac{80}{30} = \text{unit of moment.}$$

Negative sign indicates sense of turning. **(Proved)**

**21.(viii)** Find the moment about a line through the origin having direction $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, due to 30 kg force acting at a point (–4, 2, 5) in the direction of $12\mathbf{i}-4\mathbf{j}-3\mathbf{k}$. **[WBSC – 87]**

**Solution:** Let $\mathbf{F}$ be the force of magnitude 30 kg. acting in the direction of $12\mathbf{i}-4\mathbf{j}-3\mathbf{k}$.

Then $\mathbf{F} = \dfrac{30(12\mathbf{i}-4\mathbf{j}-3\mathbf{k})}{\sqrt{12^2+4^2+3^2}} = \dfrac{30}{13}(12\mathbf{i}-4\mathbf{j}-3\mathbf{k})$ The force $\mathbf{F}$ is acting at the point (–4, 2, 5)

Let $\overrightarrow{OP} = -4\mathbf{i}+2\mathbf{j}+5\mathbf{k} = \mathbf{r}$

Hence moment of the force $\mathbf{F}$ about the origin is

$$\mathbf{M} = \mathbf{r}\times\mathbf{F} = \frac{30}{13}\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & 5 \\ 12 & -6 & -3 \end{vmatrix} = \frac{30}{13}(14\mathbf{i}+48\mathbf{j}-8\mathbf{k})$$

Hence, the moment of the force $\mathbf{F}$ about the line $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is

$$\frac{30}{13}\times\frac{(14\mathbf{i}-48\mathbf{j}-8\mathbf{k})\cdot(2\mathbf{i}-2\mathbf{j}+\mathbf{k})}{\sqrt{2^2+2^2+1^2}} = \frac{10}{13}(28-96-8) = -\frac{760}{13} \text{ unit. } \textbf{(Ans)}$$

**21.(ix)** Find the moment about the point (–2, 4, –6) of the force represented in magnitude and positions by AB, where the points A and B have the co-ordinates (1, 2, –3) and (3, –4, 2) respectively. **[WBSC – 83]**

**Solution:** Let O be the origin and **i**, **j**, **k** are unit vectors along three rectangular coordinate axes respectively.

Then, $\overrightarrow{OA} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{i}-4\mathbf{j}+2\mathbf{k}$

Therefore, the force vector $\mathbf{F} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\mathbf{i}-4\mathbf{j}+2\mathbf{k})-(\mathbf{i}+2\mathbf{j}-3\mathbf{k}) = 2\mathbf{i}-6\mathbf{j}+5\mathbf{k}$

Let the point (–2, 4, –6) be P. Then $\overrightarrow{OP} = -2\mathbf{i}+4\mathbf{j}-6\mathbf{k}$

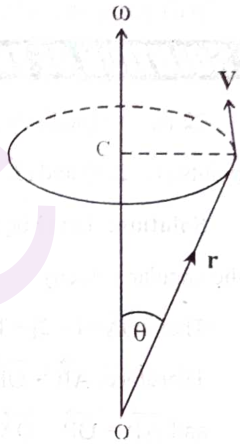
and $\mathbf{r} = \overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = (\mathbf{i}+2\mathbf{j}-3\mathbf{k})-(-2\mathbf{i}+4\mathbf{j}-6\mathbf{k}) = 3\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

$$\text{Therefore, moment vector } \mathbf{M} = \mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 3 \\ 2 & -1 & 5 \end{vmatrix} = 3\mathbf{i}-9\mathbf{j}-14\mathbf{k}$$

Therefore, magnitude $= |\mathbf{M}| = \sqrt{3^2+9^2+14^2} = \sqrt{341}$ unit. **(Ans)**

## Velocity at any point of a rotating rigid body:

If one point of a rigid body be fixed, the motion of the body is one of rotation about an axis through that point, every point on the axis being instantaneously at rest.




Let OC be the axis of rotation along which the angular velocity vector is $\boldsymbol{\omega}$. Let A be the fixed point in the body, $\mathbf{r}$ be the position vector of A relative to O. The particle at A is moving in a circular orbit with radius AC and centre C on the axis through O. Its velocity $\mathbf{V}$ is therefore perpendicular to the plane of OA and OC and is of magnitude

$|\boldsymbol{\omega}| AC = |\boldsymbol{\omega}| |\mathbf{r}| \sin\angle AOC \therefore \mathbf{V} = \boldsymbol{\omega} \times \mathbf{r}$

Thus the velocity of any particle of a body revolving about a fixed axis is equal to the vector product of the angular velocity and the position vector of the particle referred to an origin on the axis of rotation.

## PROBLEM SET–XIII

**22.** *(i) A rigid body is rotating with an angular speed of 2.5 radians per second about an axis AB where A and B are the points (1, –2, 1) and (3, –4, 2). Find the velocity of the point P(5, –1, –1) of the body. **[WBSC – 93]**

*(ii) A rigid body is spinning with an angular speed of 4 radians per second about an axis through O(1, 3, –1) in the direction of the vector A(0, 3, –1). Find the velocity of any point P(4, –2, 1) on the body. **[WBSC – 92, 89, 85]**

*(iii) The angular velocity of a rotating rigid body about an axis of rotation is given by $2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$. Find the linear velocity of a point P on the body whose position vector relative to a point on the axis of rotation is $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. **[WBSC – 90]**

(iv) The angular velocity of a rotating rigid body about an axis of rotation is given by $\boldsymbol{\omega} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find the linear velocity of a point P on the body whose position vector relative to a point on the axis of rotation is $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

*(v) A rigid body is spinning with a speed of 45 rpm about an axis through (4, –3, 9). If the direction cosines of the axis of rotation are proportional to (3, –4, 12); find the linear velocity of the particle at the point P(2, –1, 1) of the rigid body.

*(vi) A rigid body is spinning with angular speed of 27 radians per second about an axis parallel to the direction of the vector (2, 1, –2) and passing through the point (1, 3, –1). Show that the velocity of a point of the body whose position vector is (4, 8, 1) is 9(12, –10, 7). **[WBSC – 86]**

(vii) A rigid body is rotating with an angular velocity 6 radians per second about an axis parallel to $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and passing through the point $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Find the velocity of the particle at the point $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(viii) A rigid body is rotating with an angular velocity 5 radians per second about an axis in the direction of the vector (0, 3, –4) passing through the point A(1, –3, 2). Find the velocity of the particle at the point P(4, –1, 7).

# ANSWERS

22. (i) $\frac{2.5\sqrt{173}}{3}$ unit/sec. (ii) $\frac{2\sqrt{910}}{5}$ unit/sec. (iii) $\sqrt{230}$ unit. (iv) $\sqrt{285}$ unit of velocity. (v) $\frac{3\pi\sqrt{17}}{13}$ unit/sec.

(vii) $2\sqrt{306}$ unit of velocity. (viii) $\sqrt{754}$ unit/sec.

## SOLUTION OF THE PROBLEMS WITH "*" MARKS

**22.(i)** A rigid body is rotating with an angular speed of 2.5 radians per second about an axis AB where A and B are the points (1, –2, 1) and (3, –4, 2). Find the velocity of the point P(5, –1, –1) of the body. **[WBSC – 93]**

**Solution:** Let O be the origin and **i**, **j**, **k** are unit vectors along three rectangular coordinate axes respectively and w be the angular velocity.

Then, $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $OP = 5\mathbf{i} - \mathbf{j} - \mathbf{k}$

Therefore, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

and $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (5\mathbf{i} - \mathbf{j} - \mathbf{k}) - (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\text{Then } \omega = \frac{2.5(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2.5}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

Therefore, velocity of the point

$$P = \omega \times \overrightarrow{AP} = \frac{2.5}{3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 4 & 1 & -2 \end{vmatrix} = \frac{2.5}{3}(3\mathbf{i} + 8\mathbf{j} + 10\mathbf{k})$$

Therefore, magnitude $= |\mathbf{V}| = \frac{2.5}{3}\sqrt{3^2 + 8^2 + 10^2} = \frac{2.5\sqrt{173}}{3}$ = unit per sec. **(Ans)**

**22.(ii)** A rigid body is spinning with an angular speed of 4 radians per second about an axis through O(1, 3, –1) in the direction of the vector A(0, 3, –1). Find the velocity of any point P(4, –2, 1) on the body. **[WBSC – 92, 89, 85]**

**Solution:** Let O be the origin and **i**, **j**, **k** are unit vectors along three rectangular coordinate axes respectively and ω be the vector form of the angular speed of 4 radians per seconds in the direction of the vector (0. 3, –1).

$$\text{Then } \omega = \frac{4(3\mathbf{j} - \mathbf{k})}{\sqrt{3^2 + 1^2}} = \frac{4}{\sqrt{10}}(3\mathbf{j} - \mathbf{k})$$

Again, $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\overrightarrow{OP} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ Therefore, $\mathbf{r} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (4\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$

Therefore, velocity of the point
$$P = \omega \times \mathbf{r} = \frac{4}{\sqrt{10}} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -1 \\ 3 & -5 & 2 \end{vmatrix} = \frac{4}{\sqrt{10}}(\mathbf{i} - 3\mathbf{j} - 9\mathbf{k})$$

Therefore, magnitude $= |\mathbf{V}| = \frac{4}{\sqrt{10}}\sqrt{1^2 + 3^2 + 9^2} = \frac{4\sqrt{91}}{\sqrt{10}} = \frac{2\sqrt{910}}{5}$ unit per sec. **(Ans)**

N.Tute

**22.(iii)** The angular velocity of a rotating rigid body about an axis of rotation is given by $2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$. Find the linear velocity of a point P on the body whose position vector relative to a point on the axis of rotation is $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

**[WBSC – 90]**

**Solution:** Let ω be the angular velocity and **V** be the linear velocity.

Then by the problem, $\omega = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Therefore, the required linear velocity $= \omega \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix} = 10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$

Therefore, magnitude $= |\mathbf{V}| = \sqrt{10^2 + 3^2 + 11^2} = \sqrt{230}$ unit. **(Ans)**

**22.(v)** A rigid body is spinning with a speed of 45 rpm about an axis through (4, –3, 9). If the direction cosines of the axis of rotation are proportional to (3, –4, 12); find the linear velocity of the particle at the point P(2, –1, 1) of the rigid body.

**Solution:** Here the body is spinning with a speed of 45 rpm = 45 × 2π radians per minute

$$= \frac{90\pi}{60} \text{ redian per sec.} = \frac{3\pi}{2} \text{ radian/sec.}$$

Direction of angular velocity is having direction cosines proprtional to (3, –4, 12)

Therefore, the direction cosines are $\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}$

Therefroe unit vector in this direction $= \frac{1}{13}(3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k})$ Therefore ω = (3p/26)( 3i – 4j + 12k)

Again $\mathbf{r} = \overrightarrow{OP} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) - (4\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}) = -2\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$

Therefore, velocity of the point P: $v = w \times r = \frac{2\pi}{26} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 12 \\ -2 & 2 & -8 \end{vmatrix}$

$$= \frac{2\pi}{26}(8\mathbf{i} - 2\mathbf{k}) \text{ Modulus of velocity} = \frac{3\pi\sqrt{17}}{13} \textbf{ (Ans)}$$

**22.(vi)** A rigid body is spinning with angular speed of 27 radians per second about an axis parallel to the direction of the vector (2, 1, –2) and passing through the point (1, 3, –1). Show that the velocity of a point of the body whose position vector is (4, 8, 1) is 9(12, –10, 7).

**[WBSC – 86]**

**Solution:** Let O be the origin and **i, j, k** are unit vectors along three rectangular coordinate axes respectively and ω be the vector form of the angular speed of 27 radians per seconds about an axis parallel to the direction of the vector (2, 1, –2) i.e., $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Then $\omega = \dfrac{27(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})}{\sqrt{2^2 + 1^2 + 2^2}} = 9(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

Again, $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\overrightarrow{OP} = 4\mathbf{i} + 8\mathbf{j} + \mathbf{k}$ Therefore, $\mathbf{r} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (4\mathbf{i} + 8\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$

Therefore, velocity of the point $P = \omega \times \mathbf{r} = 9 \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 3 & 5 & 2 \end{vmatrix} = 9(12\mathbf{i} - 10\mathbf{j} + 7\mathbf{k}) = 9(12, -10, 7)$ **(Proved)**

# MISCELLANEOUS

## OBJECTIVE MULTIPLE CHOICE:

1. (i) Point A is $\vec{a}+2\vec{b}$, P is $\vec{a}$ and P divides AB in the **ratio** 2 : 3. The position vector of B is

(a) $2\vec{a}-\vec{b}$ (b) $\vec{b}-2\vec{a}$ (c) $\vec{a}-3\vec{b}$ (d) $\vec{b}$ **[WBSC – 03]**

(ii) The vectors $\vec{a}$ and $\vec{b}$ represents two adjacent sides AB and BC of a parallelogram ABCD then vector $\overrightarrow{AC}$ representing its diagonal is given by (a) $\vec{a}-\vec{b}$ (b) $\vec{a}+\vec{b}$ (c) $-(\vec{a}+\vec{b})$ (d) none of these. **[WBSC – 11]**

(iii) Position vectors of two points A and B are respectively A(2, 3, 4) and B(2, 5, 6). The position vector of the **middle point** joining AB is – (a) 2i + 4j + 5k (b) –2j – 2k (c) 4i + 8j + 10k (d) None of these

**[WBSC – 10]**

2. (i) Find the **unit vector** along the vector 2i + 3j – k. (i, j, k have usual meanings) **[WBSC – 05]**

(ii) Find the unit vector in the direction of the vector 6i – 3j – 2k. **[WBSC – 11]**

(iii) Find the unit vector in the direction of the vector $\vec{a}$ = 2i – 2j + k. **[WBSC – 10]**

(iv) Find the unit vector in the direction of the vector $\vec{a}$ = 2i – 6j + 3k. **[WBSC – 12]**

(v) Find the **unit vector** in the direction of the vector obtained by joining A(1, 2, –3) and B(3, 5, 3).

**[WBSC – 09]**

3. (i) If vectors = 2i – 3j + k and 4i + mj + 2k are **collinear** then m is – (a) 0 (b) 3 (c) –3 (d) None of these.

**[WBSC – 10]**

(ii) Show that the three points whose position vectors are i – 2j + k, 2i + 3j – 4k and –7j + 10k are **collinear.**

**[WBSC – 11]**

4. (i) If $\vec{a}=4\vec{i}+2\vec{j}-5\vec{k}$, $\vec{b}=-12\vec{i}+15\vec{k}-6\vec{j}$, then the vectors $\vec{a}$ and $\vec{b}$ are

(a) orthogonal (b) parallel (c) equal (d) None of these. **[WBSC – 03]**

(ii) $\vec{a}$ = 2i – 2j + k and $\vec{b}$ = 3i + 2j + k, their dot product $\vec{a}\cdot\vec{b}$ is (a) 11 (b) 3 (c) 5 (d) none of these.

**[WBSC – 11]**

(iii) $\vec{a}$ = 2i – 2j + k and $\vec{b}$ = 2i – 3j – k, then $\vec{a}\cdot\vec{b}$ is (a) 11 (b) 9 (c) 10 (d) none of these. **[WBSC – 12]**

(iv) If $\vec{a}\cdot\vec{b}=0$ then (a) $\vec{a}\perp\vec{b}$ (b) $\vec{a}\parallel\vec{b}$ (c) $\vec{a}=\vec{b}$ (d) None. **[WBSC – 04]**

(v) Find m if (3i – 2j + mk).(mi + 2j + k) = 0 **[WBSC – 09]**

(vi) If (mi – 3j + 4k).(2i + 4j + mk) = 0 then the value of m is – (a) 3 (b) –2 (c) 2 (d) None of these.

**[WBSC – 09, 11]**

(vii) If the vector $2\vec{i}+a\vec{j}+\vec{k}$ and $4\vec{i}-2\vec{j}-2\vec{k}$ are **perpendicular** to each other then the value of a is –

(a) 2, (b) 4, (c) 3, (d) None of these. **[WBSC – 06, 08]**

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(viii) The vectors $\lambda\vec{i}+\lambda\vec{j}+3\vec{k}$ and $3\vec{i}+3\vec{j}-2\lambda\vec{k}$ are **perpendicular** to each other if $\lambda$ is equal to

(a) 0 (b) −2 (c) all real values (d) None. [WBSC – 04, 07, 10]

(ix) Show that the vectors corresponding to the positions of the points (3, −2, 1) and (2, 3, 0) are at **right angles.** [WBSC – 05]

(x) The **angle** between the vectors $2\vec{i}+4\vec{j}-7\vec{k}$ and $3\vec{i}+2\vec{j}+2\vec{k}$ is (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) None. [WBSC – 04]

(xi) The **angle** between the vectors (2i + 3j + k) and (2i − j − k) is – (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) 0 [WBSC – 08, 10]

(xii) If $\vec{A}=2\vec{i}+\vec{j}+\vec{k}$, $\vec{B}=\vec{i}-2\vec{j}+2\vec{k}$ and $\vec{C}=3\vec{i}-4\vec{j}+2\vec{k}$, then the **projection** of $(\vec{A}+\vec{C})$ in the direction of $\vec{B}$ is (a) $\frac{9}{19}$ (b) $\frac{17}{3}$ (c) $\frac{2}{3}$ (d) None. [WBSC – 04, 07]

(xiii) If $\vec{a}$ = 2i + 2j + k, $\vec{b}$ = i + j − k then the **projection** of $\vec{a}$ on $\vec{b}$ is (a) 1 (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) none.

5. (i) Find a vector **perpendicular** to both : 2i + 3j − k and 3i − j + 2k. [WBSC – 05]

(ii) If $\vec{A}$ = 4i − i + 3k and $\vec{B}$ = −2i + j − 2k; find the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ [WBSC – 11]

(iii) If $\vec{a}=2\vec{i}-3\vec{j}-k$ and $\vec{b}=\vec{i}-4\vec{j}-2\vec{k}$ find their cross product. [WBSC – 06]

(iv) If (2i + 6j + 27k) × (i + 3j + pk) = 0 then the value of p will be –

(a) 1 (b) 27 (c) $\frac{27}{2}$ (d) None of these. [WBSC – 09]

## SUBJECTIVE :

1. (i) Position vector of P, Q, R and S be 2i + 4k, 5i + 3$\sqrt{3}$ j + 4k, −2$\sqrt{3}$ j + k, 2i + k prove that RS is parallel to PQ and is $\frac{2}{3}$ of PQ. [WBSC – 10]

(ii) The position vector of vertices of a triangle are given by 2i + 4j − k, 4i + 5j + k and 3i + 6j − 3k respectively. Show that the triangle is a **right angled** triangle. [WBSC – 09]

(iii) Show that the vectors 2i − j + k, i − 3j − 5k, 3i − 4j − 4k form the sides of a right angled triangle. [WBSC – 15]

2. (i) $\vec{\alpha}=4i+5j-k$, $\vec{\beta}=i-4j+5k$ and $\vec{\gamma}=2i+j-k$ are three vectors. Find a vector $\vec{p}$ which is perpendicular to both the vectors $\vec{\alpha}$ and $\vec{\beta}$ and satisfies the relation $\vec{p}\cdot\vec{\gamma}=21$. [WBSC – 03]

(ii) If $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are three vectors such that $\vec{\alpha}+\vec{\beta}+\vec{\gamma}=0$ and $|\vec{\alpha}|=2$, $|\vec{\beta}|=4$, $|\vec{\gamma}|=6$ then show that $\vec{\alpha}.\vec{\beta}+\vec{\beta}.\vec{\gamma}+\vec{\gamma}.\vec{\alpha}=-28$ [WBSC – 17]

(iii) If α, β and γ are three unit vectors satisfying the condition α + β + γ = **0** then show that $\alpha\cdot\beta+\beta\cdot\gamma+\gamma\cdot\alpha=-\frac{3}{2}$ [WBSC – 14]

(iv) If a and b are unit vectors and θ be the angle between them, show that $\sin\frac{\theta}{2}=\frac{1}{2}|a-b|$ [WBSC – 18, 19]

(v) Find the **unit vector** perpendicular to both the vectors (3i + j + 2k) and (2i − 2j + 4k) and find the angle between them. [WBSC – 04, 15, 16]

(vi) Find the **unit vector** perpendicular to both the vectors $3i + j + k$ and $2i - 2j + 4k$ and find the angle between the given vectors. [WBSC – 08]

(vii) If $\vec{a} = 4i - j + 3k$ and $\vec{b} = -2i + j - 2k$; find the **unit vector** perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. [WBSC – 09]

(viii) Find $\vec{\alpha} \times \vec{\beta}$, where $\vec{\alpha} = 3\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{\beta} = \vec{j} - 3\vec{k}$ [WBSC – 17]

(ix) Find the area of the triangle, position vectors whose vertices are $3i + j$, $5i + 2j + k$, $i - 2j + 3k$. [WBSC – 10, 12]

(x) Find the scalar area of the triangle the position vectors of whose vertices are $i + j + k$, $i + 2j + 3k$, $2i + 3j + k$. [WBSC – 14, 18]

(x) The position vectors of three angular points of a triangle ABC are given by $A = i + 2j + 4k$, $B = -2i + 2j + k$ and $C = 3i + j - 2k$. Find the angles of the triangle. [WBSC – 10]

(xi) If the vectors $2i + 3j + 4k$ and $3i + 4j + 7k$ are two diagonals of a parallelogram, find its **area**. [WBSC – 06, 09]

(xii) If $3i + j - 2k$ and $i - 3j + 4k$ are two diagonals of a parallelogram, find its **area.** [WBSC – 10]

4. (i) A particle acted on by forces $(4i + j - 3k)$ and $(3i + j - k)$ is displaced from the point A(1, 2, 3) to the point B(5, 4, 1). Find the **work done** by the forces on the particle. [WBSC – 04, 07]

(ii) A particle acted on by two forces $3i + 2j + 5k$ and $2i + j - 3k$ is displaced from the point $2i - j - 3k$ to the point $4i - 3j + 7k$. Calculate the total work done by the forces. [WBSC – 12, 17]

(iii) A particle acted on by constant forces of $2i + 2j + k$ and $3i + 4j - 2k$ and is displaced from the point (1, 2, 3) to (5, 4, 1). Find the **work done** by the forces. [WBSC – 06, 08, 09]

(iv) A particle acted on by a constant force of 22 kgf in the direction of the vector $9i + 6j - 2k$ is displaced from the point (1, 2, 1) to the point (5, 4, 2); find the **work done** if the displacement is in metres. [WBSC – 05, 11]

(v) A praticle is acted on by forces 9 and 10 units at direction whose direction cosines are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ and $0, \frac{4}{5}, \frac{3}{5}$ respectively displaces a particle from the point $2\vec{i} - \vec{j} - 3\vec{k}$ to the point $4\vec{i} - 3\vec{j} + 7\vec{k}$, Calculate the total **work done** by the forces. [WBSC – 06]

(vi) Find the work done by the forces $2i + 2j + k$ and $3i + 4j - 2k$ acting on a particle displaced from the point (4, –3, –2) to (6, 1, –3). [WBSC – 16]

(vii) A force of 15 units acts in the directions of the vector $i - 2j + 2k$ and passes through a points $2i - 2j + 2k$. Find the moment of the force about the point $i + j + k$. [WBSC – 15]

(viii) Find the moment of the force $7i - j + 3k$, acting at the point (–1, 1, 2), about the point (2, 1, 4) [WBSC – 18]

5. Using vector method show that the quadrilateral whose diagonals bisect each other is a parallelogram. [WBSC – 15]

# UNIT – 3

## TRIGONOMETRY PAGE NO

## Chapter 1

# Measurement of Trigonometrical Angles and Trigonometrical Ratios

**1.1** Trigonometry is a very important branch of mathematics. It deals with the angles of a triangle and the measure of all sorts of positive and negative angles which falls within the perview of the subject.

## Trigonometric Angle:

### Positive and Negative Angle :

Let on the plane of the paper O be a fixed point and $\overrightarrow{OX}$ be a fixed line and $\overrightarrow{OP}$ be a revolving line whose initial position coincides with $\overrightarrow{OX}$. Now if the line $\overrightarrow{OP}$ comes from its initial position $\overrightarrow{OX}$ to the final position $\overrightarrow{OP}$ by revolving about O (either in clockwise or in anti-clockwise sence) then the angle made by $\overrightarrow{OP}$ with $\overrightarrow{OX}$ i.e., $\angle XOP$ is known as **Trigonometrical Angle.**

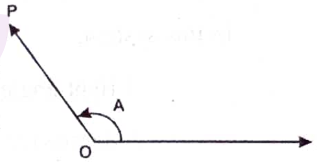



Fig. - 1

If the line $\overrightarrow{OP}$ comes from its initial position $\overrightarrow{OX}$ to the final position $\overrightarrow{OP}$ by revolving about O in the **anti-clockwise** sense then the angle made by $\overrightarrow{OP}$ with $\overrightarrow{OX}$ i.e., $\angle XOP = A$ (say) [Fig. – 1] is known as **Trigonometrical Positive Angle.**

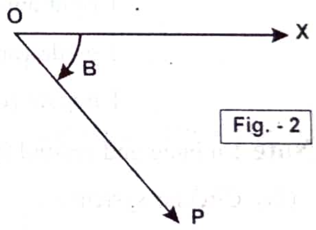



Fig. - 2

If the line $\overrightarrow{OP}$ comes from its initial position $\overrightarrow{OX}$ to the final position $\overrightarrow{OP}$ by revolving about O in the **clockwise** sense then the angle made by $\overrightarrow{OP}$ with $\overrightarrow{OX}$ i.e., $\angle XOP = B$ (say) [Fig. – 2] is known as **Trigonometrical Negative Angle.**

Hence, **Trigonometrical Angle may have positive or negative value.**

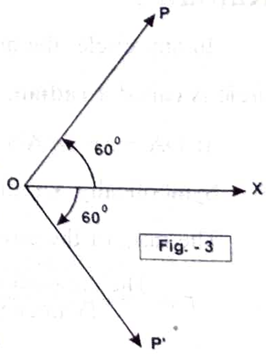



Fig. - 3

As per the above discussion, if the revolving line comes to the position $\overrightarrow{OP}$ by revolving through an angle $60^0$ in the anti-clockwise sense from its initial position $\overrightarrow{OX}$ about O then $\angle XOP = 60^0$ and if it comes to $\overrightarrow{OP'}$ by revolving through an angle $60^0$ in the clockwise sense from its initial position $\overrightarrow{OX}$ about O then $\angle XOP' = -60^0$ [Fig. – 3].

**Note:** In the above discussion, the fixed point O is called vertex of trigonometrical angles, $\overrightarrow{OX}$ is the initial arm, $\overrightarrow{OP}$, $\overrightarrow{OP'}$ are the final arms and the revolving line is called **generating line.**

Since, the revolving line can revolve any number of times in both anti-clockwise or clockwise direction so trigonometrical angles may have any positive or negative value.

Hence, **trigonometrical angle has no definite limit.**

### Geometrical Angle and Trigonometrical Angle :

| Geometrical Angle | Trigonometrical Angle |
|---|---|
| 1. In geometry an angle is supposed to be formed by the intersection of two straight lines. | 1. In trigonometry an angle is formed by the revolution of a straight line about a fixed point. |
| 2. Geometrical angle always varies from $0^{\circ}$ to $360^{\circ}$ | 2. Trigonometrical angle has no definite limit.. |
| 3. Geometrical angles are always positive. | 3. Trigonometrical angles may have any positive or negative value. |

### 1.2 Systems of measuring Angles :

(i) Sexagesimal system (or English system) :

In this system,

1 right angle $= 90$ degrees (or $90^{\circ}$)

1 degree (or $1^{\circ}$) $= 60$ minutes (or $60'$)

1 minute (or $1'$) $= 60$ seconds (or $60''$)

(ii) Centesimal system (or French system) :

1 right angle $= 100$ grades (or $100^{g}$)

1 grade (or $1^{g}$) $= 100$ minutes (or 100)

1 minute (or 1) $= 100$ seconds (or 100)

**Note :** minute and second in sexagesimal and centesimal systems are different.

(iii) Circular system :

In this system a **radian** is considered as the unit for the measurement of angles.

## Radian :

In any circle, the angle subtended at its centre by an arc of the circle whose length is equal to the radius of the circle is called a **radian.**

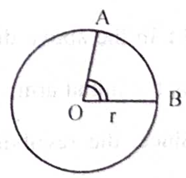
If $\overline{OA} = r = \text{arc}AB$ = radius of the circle, then $\angle AOB$ = one radian $= 1^{c}$

Symbolically, x radian $= x^{c}$.

The ratio of the circumference to the diameter is constant and is denoted by the Greak letter $\pi$.

$$\therefore \pi = \frac{\text{The circumference of any circle}}{\text{Diameter of the circle}}$$

An approximate value of is $\pi$ is $\frac{22}{7}$. A more accurate value of $\pi$ is $\frac{355}{133}$.



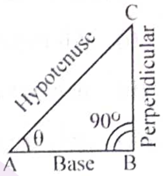

Radian is a constant angle.

1 radian $= \frac{2}{\pi}$ right angle. $= \frac{2}{\pi} \times 90^{\circ}$. or, $1^{c} = \frac{180^{\circ}}{\pi}$ or, $180^{\circ} = \pi^{c}$.

## 1.3 Definition of Trigonometrical Ratios :

Let $\Delta ABC$ is a right angle triangle.

Then the six trigonometrical ratios of the angle $\theta$ are defined as follows :




$$\frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{\overline{BC}}{\overline{AC}} = \sin\theta \qquad \frac{\text{Base}}{\text{Hypotenuse}} = \frac{\overline{AB}}{\overline{AC}} = \cos\theta \qquad \frac{\text{Perpendicular}}{\text{Base}} = \frac{\overline{BC}}{\overline{AB}} = \tan\theta$$

$$\frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{\overline{AC}}{\overline{BC}} = \text{cosec}\theta \qquad \frac{\text{Hypotenuse}}{\text{Base}} = \frac{\overline{AC}}{\overline{AB}} = \sec\theta \qquad \frac{\text{Base}}{\text{Perpendicular}} = \frac{\overline{AB}}{\overline{BC}} = \cot\theta$$

**Note :** (i) Trigonometrical ratios are **pure number.**

(ii) $\sin\theta$ does not imply $\sin \times \theta$, $\cos\theta$ does not imply $\cos \times \theta$.etc.

### Relations among the Trigonometrical Ratios :

(i) $\sin\theta \,.\, \text{cosec}\theta = 1$ or, $\sin\theta = \frac{1}{\text{cosec}\theta}$ or, $\text{cosec}\theta = \frac{1}{\sin\theta}$

(ii) $\cos\theta \,.\, \sec\theta = 1$ or, $\cos\theta = \frac{1}{\sec\theta}$ or, $\sec\theta = \frac{1}{\cos\theta}$

(iii) $\tan\theta.\cot\theta = 1$ or, $\tan\theta = \frac{1}{\cot\theta}$ or, $\cot\theta = \frac{1}{\tan\theta}$

(iv) $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(v) $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$ , $1 + \cot^2\theta = \text{cosec}^2\theta$

Note : (i) $\sin^2\theta = (\sin\theta)^2 = \sin\theta \times \sin\theta$ (ii) $\sin^2\theta \neq \sin\theta^2$ etc.

## Trigonometrl Ratios of some standard Angles :

By using some elementary knowledge of Geometry we can find the values of the Trigonometrl Ratios of some standard Angles such as $0^0$, $30^0$, $45^0$, $60^0$, $90^0$.

Following table gives the values of the Trigonometrl Ratios of $0^0$, $30^0$, $45^0$, $60^0$ and $90^0$.

| $\theta$ → / Trigonometrical Ratios ↓ | $0^0$ | $30^0$ | $45^0$ | $60^0$ | $90^0$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | Not defined |
| $\cot\theta$ | Not defined | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 |
| $\sec\theta$ | 1 | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | 2 | Not define |
| $\text{cosec}\theta$ | Not define | 2 | $\sqrt{2}$ | $\frac{2}{\sqrt{3}}$ | 1 |

## 1.4 Signs of Trigonometrical Ratios :

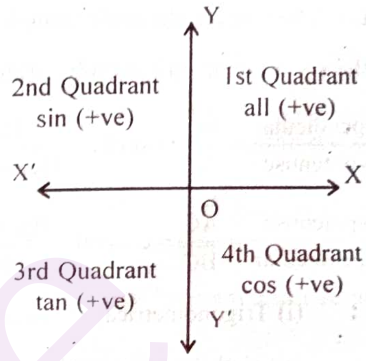
The rule regarding the signs of trigonometrical ratios is :

(i) All trigonometrical ratios are positive in the first quadrant,

(ii) only sin (and cosec) is positive in the second quadrant,

(iii) only tan (and cot) is positive in the third quadrant, and

(iv) only cos (and sec) is positive in the fourth quadrant.




The above rule is called "**all, sin, tan, cos**" rule.

### Limits to the values of Trigonometrical Ratios :

**(i)** $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$

**(ii)** $\sec \geq 1$ or $\sec\theta \leq -1$ and $\operatorname{cosec}\theta \geq 1$ or $\operatorname{cosec}\theta \leq -1$.

**(iii)** $\tan\theta$ and $\cot\theta$ can have any real values.

### Trigonometrical Ratios of $(-\theta)$ :

| | | |
|---|---|---|
| $\sin(-\theta) = -\sin\theta$ | $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$ | $\tan(-\theta) = -\tan\theta$ |
| $\cot(-\theta) = -\cot\theta$ | $\cos(-\theta) = \cos\theta$ | $\sec(-\theta) = \sec\theta$ |

- Trigonometrical Ratios of angles $(n \times 90^\circ \pm \theta)$, where n is an integer and $\theta$ is the positive acute angle.

**Step I :** Find the quadrant in which the angle $(n \times 90^\circ + \theta)$ or $(n \times 90^\circ - \theta)$ lies.

Then by using "all, sin, tan, cos" rule find the sign of the particular trigonometrical ratio.

**Step II :** see whether n is even or odd integer.

(i) If n is an even integer, the ratio remains same.

(ii) If n is an odd integer, the trigonometrical ratios altered :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| sin | changes | to | cos | cos | changes | to | sin |
| tan | " | " | cot | cot | " | " | tan |
| sec | " | " | cosec | cosec | " | " | sec |

### As examples :

$\sec(-945^\circ) = \sec 945^\circ$ [$\because \sec(-\theta) = \sec\theta$] $= \sec(10 \times 90^\circ + 45^\circ) = -\sec 45^\circ = \sqrt{2}$ **Ans.**

[$\because$ n = 10 is even, sec remains same and $10 \times 90^\circ + 45^\circ$ is in the 3rd quadrant, sign is –ve]

Again $\cot(-870^\circ) = -\cot 870^\circ = -\cot(9 \times 90^\circ + 60^\circ) = -(-\tan 60^\circ)$

$= \tan 60^\circ = \sqrt{3}$ Ans. [cot ratio is –ve, since $9 \times 90^\circ + 60^\circ$ lies in the 2nd quadrant. and n = 9 = odd integer, so cot ratio has changed to tan]

# PROBLEM SET

[Problems with '*' marks are solved at the end of the problem set]

1. Prove that :

(i) $(1 + \sec A + \tan A)(1 - \operatorname{cosec} A + \cot A) = 2$

(ii) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta.\operatorname{cosec}^2\theta$

2. *(i) If $\sin\alpha + \operatorname{cosec}\alpha = 2$, show that, $\sin^n\alpha + \operatorname{cosec}^n\alpha = 2$

(ii) If $\tan\alpha + \cot\alpha = 2$, show that, $\tan^{29}\alpha + \cot^{29}\alpha = 2$

(iii) If $V_n = \cos^n\alpha + \sec^n\alpha$ and $V_1 = 2$, show that $V_n = 2$

*(iv) If $V_n = \sin^n\alpha + \cos^n\alpha$, show that the value of $6V_4 - 4V_6$ is independent of $\alpha$.

3. (i) If $\cos\theta + \sec\theta = \sqrt{3}$, show that, $\cos^3\theta + \sec^3\theta = 0$

*(ii) If $\operatorname{cosec}\theta + \sin\theta = \sqrt{3}$, show that $\operatorname{cosec}^3\theta + \sin^3\theta = 0$

4. (i) If $a\cos^2\theta + b\sin^2\theta = c$, show that, $\tan\theta = \pm\sqrt{\frac{c-a}{b-c}}$

*(ii) If $7\sin^2\theta + 3\cos^2\theta = 4$, show that, $\tan\theta = \pm\frac{1}{\sqrt{3}}$ **[WBSC – 82]**

(iii) If $a^2\sec^2\alpha - b^2\tan^2\alpha = c^2$, show that, $\sin\alpha = \pm\sqrt{\frac{c^2-a^2}{c^2-b^2}}$.

5. (i) If $a\cos\theta - b\sin\theta = c$, prove that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$

*(ii) If $p\cos\theta - q\sin\theta + r = 0$, find the value of $p\sin\theta + q\cos\theta$ in terms of p, q, r, **[Ans.** $\pm\sqrt{p^2+q^2-r^2}$ **]**

**[WBSC – 87]**

6. (i) If $\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, show that $\sin\alpha + \cos\alpha = \pm\sqrt{2}\cos\theta$

*(ii) If $4x\sec A = 1 + 4x^2$, prove that, $\sec A + \tan A = 2x$ or $\frac{1}{2x}$.

(iii) Show that $\frac{\sec\theta - 1}{\tan\theta} = \frac{1}{x}$, if $\frac{\sec\theta + 1}{\tan\theta} = x$

(iv) Show that, $\sec^2\theta \,.\, \tan\theta + 2\sec\theta \,.\, \operatorname{cosec}\theta + \operatorname{cosec}^2\theta \,.\, \cot\theta = \sec^3\theta \,.\, \operatorname{cosec}^3\theta$

7. *(i) If $\cos A + \sin A = \sqrt{2}\cos A$, prove that $\cos A - \sin A = \sqrt{2}\sin A$

(ii) If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, prove that $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

8. (i) If a and b be two unequal real numbers, show that the equation $\sec^2\theta = \frac{4ab}{(a+b)^2}$ cannot be solved. **[HS – 90]**

*(ii) For real values of x and y only, show that the equation, $\sec^2\theta = \frac{4xy}{(x+y)^2}$, is possible only when x = y.

**[WBSC – 87]**

(iii) If x and y are two distinct real positive numbers, can the relation $\sec\theta = \frac{2xy}{x^2+y^2}$ be true ?

9. (i) If $\cos^4\theta + \cos^2\theta = 1$, prove that, $\tan^4\theta + \tan^2\theta = 1$

*(ii) If $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$, show that, $\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = 1$

(iii) If $\tan^4\theta + \tan^2\theta = 1$, prove that, $\cos^4\theta + \cos^2\theta = 1$

*(iv) If $\sin^4 x + \sin^2 x = 1$, show that $\cot^4 x + \cot^2 x = 1$ **[WBSC – 05, 06, 07]**

10. (i) If $(1 + \sin\alpha)(1 + \sin\beta)(1 + \sin\gamma) = (1 - \sin\alpha)(1 - \sin\beta)(1 - \sin\gamma)$, show that each expression $= \pm \cos\alpha \cos\beta \cos\gamma$.

*(ii) If $(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma) = (1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma)$, show that each expression $= \pm \sin\alpha \sin\beta \sin\gamma$.

*11. If $\cos^2\theta - \sin^2\theta = \tan^2\alpha$, show that $\cos^2\alpha - \sin^2\alpha = \tan^2\theta$. **[WBSC – 96]**

*12. If $\tan k\theta = k \tan\theta$, prove that, $\frac{\sin^2 k\theta}{\sin^2\theta} = \frac{k^2}{1 + (k^2 - 1)\sin^2\theta}$ **[WBSC – 95]**

*13. If $\frac{\sin^4\alpha}{a} + \frac{\cos^4\alpha}{b} = \frac{1}{a + b}$ show that, $\frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3} = \frac{1}{(a + b)^3}$

14. *(i) Show that, $\cos^2\theta + \cos^2(\alpha + \theta) - 2\cos\alpha \,.\, \cos\theta \cos(\alpha + \theta)$ is independent of $\theta$.

*(ii) Show that the value of, $\sin^2(x + \alpha) + \sin^2(x + \beta) - 2\cos(\alpha - \beta) \,.\, \sin(x + \alpha) \,.\, \sin(x + \beta)$ is independent of x.

15. *(i) If $\sec\alpha = \sec\beta \sec\gamma + \tan\beta \tan\gamma$, show that, $\sec\beta = \sec\gamma \sec\alpha \pm \tan\gamma \tan\alpha$.

(ii) If $\operatorname{cosec} x = \operatorname{cosec} y \operatorname{cosec} z + \cot y \cot z$, prove that $\operatorname{cosec} y = \operatorname{cosec} z \operatorname{cosec} x \pm \cot z \cot x$.

*(iii) If $\sec x \sec y + \tan x \tan y = \sec z$, show that, $\sec x \tan y + \tan x \sec y = \pm \tan z$

16. Find the values of

*(i) $\cos(-1170^\circ)$ **[WBSC – 97, 03]** **[Ans 0]**

*(ii) $\sin 1755^\circ$ **[WBSC – 95]** $\left[\text{Ans. } \frac{1}{\sqrt{2}}\right]$

*(iii) $\cos(-1575^\circ)$ **[WBSC – 99]** **[Ans** $\frac{-1}{\sqrt{2}}$ **]**

(iv) $\cot(-870^\circ)$ **[HS – 94]** $\left[\text{Ans. } \frac{1}{\sqrt{3}}\right]$

17. *(i) If $\tan x \tan 3x = 1$; find $\tan 2x$. **[Ans. 1]** **[WBSC – 95, 98, 06]**

*(ii) If $\tan x \tan 3x = 1$, find $\tan x$. **[Ans.** $\sqrt{2} - 1$ **]** **[WBSC – 07]**

(iii) If $\cos 2\theta = -\frac{1}{2}$, find $\cos\theta$. **[Ans.** $\frac{1}{2}$ **]** **[WBSC – 03]**

18. Prove that,

(i) $\cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0$

*(ii) $\sin 420^\circ \cos 390^\circ - \cos(-300^\circ).\sin(-330^\circ) = \frac{1}{2}$ **[WBSC – 04]**

(iii) $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8} = 2$

*(iv) $\cos^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\frac{5\pi}{4} + \sin^2\frac{7\pi}{4} = 2$

(v) $\tan\frac{\pi}{12} \,.\, \tan\frac{5\pi}{20} \,.\, \tan\frac{7\pi}{12} \,.\, \tan\frac{11\pi}{12} = 1$

*(vi) $\tan\frac{3\pi}{20} \,.\, \tan\frac{5\pi}{20} \,.\, \tan\frac{7\pi}{20} = 1$ **[WBSC – 96, 04]**

(vii) $\tan\frac{\pi}{8} \,.\, \tan\frac{3\pi}{8} \,.\, \tan\frac{5\pi}{8} \,.\, \tan\frac{7\pi}{8} = 1$

*(viii) $\cos 1^0 \cos 2^0 \cos 3^0 \ldots\ldots\ldots \cos 179^0 = 0$ **[WBSC – 05]**

*(ix) If $10\alpha = \frac{\pi}{2}$, then $\tan 3\alpha \tan 5\alpha \tan 7\alpha = 1$ **[WBSC – 06]**

*(x) $\tan 1^0 \tan 2^0 \tan 3^0 \ldots\ldots\ldots \tan 89^0 = 1$ **[WBSC – 06]**

19. *(i) If $\tan 25^\circ = a$, prove that, $\frac{\tan 155^\circ - \tan 115^\circ}{1+\tan 155^\circ \tan 115^\circ} = \frac{1-a^2}{2a}$

*(ii) Find the value of $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ **[Ans. 1]**

(iii) If $\tan\theta = \frac{15}{8}$ and $\cos\theta$ is negative, find $\frac{\sin(-\theta)-\cos\theta}{\tan(-\theta)+\sec(-\theta)}$ $\left[\textbf{Ans. } -\frac{23}{68}\right]$

*(iv) If $\cot\theta = \frac{3}{4}$ and $\sin\theta$ is negative find $\frac{\cos(-\theta)+\operatorname{cosec}\theta}{\cos\theta+\sin(-\theta)}$ $\left[\textbf{Ans. } -\frac{37}{4}\right]$

20. (i) If a and b be two unequal real numbers, show that the equation $\sec^2\theta = \frac{4ab}{(a+b)^2}$ cannot be solved.

*(ii) For real values of x and y if the relation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ be true find the relation between x and y.

(iii) If x and y are two real positive numbers, can the relation $\sec\theta = \frac{2xy}{x^2+y^2}$ be true? Justify.

*(iv) If x, y and z are three unequal positive real numbers, can the relation $\cos\theta = \frac{x^2+y^2+z^2}{xy+yz+zx}$ be true? Justify.

(v) If $a \neq 1$ be a real positive number, can the relation can the relation $\sin\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$ be true? Justify.

(vi) If x and y are two unequal real positive numbers, can the relation $\sec\theta = \frac{2xy}{(x+y)^2}$ be true? Justify. **[HS – 97]**

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**2.** (i) **Solution :** Given, $\sin\alpha + \operatorname{cosec}\alpha = 2$ or, $\sin\alpha + \frac{1}{\sin\alpha} = 2$ or, $\sin^2\alpha + 1 = 2\sin\alpha$

or, $\sin^2\alpha - 2\sin\alpha + 1 = 0$ or, $(\sin\alpha - 1)^2 = 0 \Rightarrow \sin\alpha = 1$ $\therefore$ $\operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = \frac{1}{1} = 1$

$\therefore \sin^n\alpha + \operatorname{cosec}^n\alpha = (1)^n + (1)^n = 1 + 1 = 2$ **(Proved)**

**2.** (iv) **Solution :** $\because V_n = \sin^n\alpha + \cos^n\alpha$ $\therefore$ $V_4 = \sin^4\alpha + \cos^4\alpha$ and $V_6 = \sin^6\alpha + \cos^6\alpha$

$\therefore 6V_4 - 4V_6 = 6(\sin^4\alpha + \cos^4\alpha)\} - 4(\sin^6\alpha + \cos^6\alpha)$

$= 6\{(\sin^2\alpha + \cos^2\alpha)^2 - 2\sin^2\alpha.\cos^2\alpha) - 4\{(\sin^2\alpha + \cos^2\alpha)^3 - 3\sin^2\alpha\cos^2\alpha(\sin^2\alpha + \cos^2\alpha)\}$

$= 6(1 - 2\sin^2\alpha\cos^2\alpha) - 4(1 - 3\sin^2\alpha\cos^2\alpha)$ $[\because \sin^2\alpha + \cos^2\alpha = 1]$

$= 6 - 12\sin^2\alpha\cos^2\alpha - 4 + 12\sin^2\alpha\cos^2\alpha = 6 - 4 = 2$, which is independent of $\alpha$. **(Proved)**

**3.** (ii) **Solution :** Given $\operatorname{cosec}\theta + \sin\theta = \sqrt{3}$ ---------- (1)

$(\operatorname{cosesc}\theta + \sin\theta)^3 = (\sqrt{3})^3$ [cubing both sides]

or, $\operatorname{cosec}^3\theta + \sin^3\theta + 3\operatorname{cosec}\theta.\sin\theta(\operatorname{cosec}\theta + \sin\theta) = 3\sqrt{3}$

or, $\operatorname{cosec}^3\theta + \sin^3\theta + 3.\sqrt{3} = 3\sqrt{3}$ [from (1)]

or, $\operatorname{cosec}^3\theta + \sin^3\theta = 0$ **(Proved)**

**4.** (ii) **Solution :** Given, $7\sin^2\theta + 3\cos^2\theta = 4$

or, $7\sin^2\theta + 3\cos^2\theta = 4(\sin^2\theta + \cos^2\theta)$ $[\because \sin^2\theta + \cos^2\theta = 1]$

or, $7\sin^2\theta - 4\sin^2\theta = 4\cos^2\theta - 3\cos^2\theta$ or, $3\sin^2\theta = \cos^2\theta$

or, $\frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{3}$ or, $\tan^2\theta = \frac{1}{3}$ $\therefore$ $\tan\theta = \pm\frac{1}{\sqrt{3}}$ **(Proved)**

**5.** (ii) **Solution :** Given, $p\cos\theta - q\sin\theta + r = 0$ or, $p\cos\theta - q\sin\theta = -r$

or, $(p\cos\theta - q\sin\theta)^2 = (-r)^2$ [squaring both sides]

or, $p^2\cos^2\theta + q^2\sin^2\theta - 2pq\sin\theta\cos\theta = r^2$

or, $p^2(1 - \sin^2\theta) + q^2(1 - \cos^2\theta) - 2pq\sin\theta\cos\theta = r^2$

or, $p^2 - p^2\sin^2\theta + q^2 - q^2\cos^2\theta - 2pq\sin\theta\cos\theta = r^2$

or, $p^2\sin^2\theta + 2pq\cos\theta\sin\theta + q^2\cos^2\theta = p^2 + q^2 - r^2$

or, $(p\sin\theta + q\cos\theta)^2 = p^2 + q^2 - r^2$

or, $p\sin\theta + q\cos\theta = \pm\sqrt{p^2 + q^2 - r^2}$ **(Ans)**

**6.** (ii) **Solution :** Given, $4x\sec A = 1 + 4x^2$ or, $\sec A = \frac{1 + 4x^2}{4x}$ ----------(1)

$\therefore \sec^2 A = \frac{(1 + 4x^2)^2}{16x^2}$ or, $\sec^2 A - 1 = \frac{(1 + 4x^2)^2}{16x^2} - 1$

or, $\tan^2 A = \dfrac{(1+4x^2)^2 - 16x^2}{16x^2} = \dfrac{1+8x^2+16x^4-16x^2}{16x^2}$

$= \dfrac{1-8x^2+16x^4}{16x^2} = \dfrac{(1-4x^2)^2}{16x^2}$

$\Rightarrow \tan A = \pm\dfrac{1-4x^2}{4x}$ ----------(2)

Adding (1) and (2) we get,

$\sec A + \tan A = \dfrac{1+4x^2}{4x} \pm \dfrac{1-4x^2}{4x}$

$= \dfrac{1+4x^2+1-4x^2}{4x}$ or, $\dfrac{1+4x^2-1+4x^2}{4x} = \dfrac{2}{4x}$ or $\dfrac{8x^2}{4x} = \dfrac{1}{2x}$ or $2x$

$\therefore$ $\sec A + \tan A = 2x$ or, $\dfrac{1}{2x}$ **(Proved)**

**7. (i) Solution :** Given, $\cos A + \sin A = \sqrt{2}\cos A$

or, $\sin A = \sqrt{2}\cos A - \cos A = (\sqrt{2}-1)\cos A$

$= \dfrac{(\sqrt{2}+1)(\sqrt{2}-1)}{\sqrt{2}+1}.\cos A = \dfrac{1}{\sqrt{2}+1}.\cos A$

or, $\cos A = (\sqrt{2}+1)\sin A = \sqrt{2}\sin A + \sin A$

$\therefore$ $\cos A - \sin A = \sqrt{2}\sin A$ **(Proved)**

**8. (ii) Solution :** The numerical value of $\sec\theta$ is always greater than or equal to 1.

$\therefore$ $\sec^2\theta \geq 1$ or, $\dfrac{4xy}{(x+y)^2} \geq 1$ $\left[\because \sec^2\theta = \dfrac{4xy}{(x+y)^2}\right]$

or, $(x+y)^2 \leq 4xy$ or, $(x+y)^2 - 4xy \leq 0$ or, $(x-y)^2 \leq 0$

But, for real x and y, $(x-y)^2 \nless 0$

$\therefore$ $(x-y)^2 = 0$ or, $x - y = 0$

or, $x = y$ Hence **proved.**

**9. (ii) Solution :** Solution : Let $\cos^2\alpha = a$, $\cos^2\beta = b$ $\therefore$ $\sin^2\alpha = 1 - a$, $\sin^2\beta = 1 - b$.

Now $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$ or, $\dfrac{a^2}{b} + \dfrac{(1-a)^2}{1-b} = 1$

or, $a^2(1-b) + b(1-2a+a^2) = b(1-b)$

or, $a^2 - a^2b + b - 2ab + a^2b = b - b^2$

or, $a^2 - 2ab + b^2 = 0$ or, $(a-b)^2 = 0$ or, $a = b$

$\therefore$ $\cos^2\alpha = \cos^2\beta \Rightarrow 1 - \sin^2\alpha = 1 - \sin^2\beta \Rightarrow \sin^2\alpha = \sin^2\beta$

$\therefore$ $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = \dfrac{\cos^4\alpha}{\cos^2\alpha} + \dfrac{\sin^4\alpha}{\sin^2\alpha} = \cos^2\alpha + \sin^2\alpha = 1$ **(Proved)**

**9. (iv) Solution :** Given $\sin^4x + \sin^2x = 1$ or $\sin^4x = 1 - \sin^2x = \cos^2x$

$$\therefore \cot^4x + \cot^2x = \frac{\cos^4 x}{\sin^4 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{\cos^4 x}{\cos^2 x} + \frac{\sin^4 x}{\sin^2 x} = \cos^2x + \sin^2x = 1 \text{ (Proved)}$$

**10. (ii) Solution :** Let, $(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma) = (1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma) = k$

$\therefore k = (1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma)$, $k = (1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma)$

$\therefore$ multiplying we get,

$k^2 = (1 + \cos\alpha)(1 - \cos\alpha)(1 + \cos\beta)(1 - \cos\beta)(1 + \cos\gamma)(1 - \cos\gamma)$

$= (1 - \cos^2\alpha)(1 - \cos^2\beta)(1 - \cos^2\gamma) = \sin^2\alpha \,.\, \sin^2\beta \,.\, \sin^2\gamma$

$\therefore k = \pm \sin\alpha \sin\beta \sin\gamma$

Hence, each expression $= \pm \sin\alpha \sin\beta \sin\gamma$ **(Proved)**

**11. Solution :** Given, $\cos^2\theta - \sin^2\theta = \tan^2\alpha$ or, $\dfrac{\cos^2\theta - \sin^2\theta}{1} = \dfrac{\sin^2\alpha}{\cos^2\alpha}$

or, $\dfrac{1 - \cos^2\theta + \sin^2\theta}{1 + \cos^2\theta - \sin^2\theta} = \dfrac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha + \sin^2\alpha}$ [by componendo and dividendo]

or, $\dfrac{\sin^2\theta + \sin^2\theta}{\cos^2\theta + (1 - \sin^2\theta)} = \dfrac{\cos^2\alpha - \sin^2\alpha}{1}$ [$\because \sin^2x + \cos^2x = 1$]

or, $\dfrac{2\sin^2\theta}{2\cos^2\theta} = \cos^2\alpha - \sin^2\alpha$ or, $\cos^2\alpha - \sin^2\alpha = \tan^2\theta$ **(Proved)**

**12. Solution :** Given, $\tan k\theta = k\tan\theta$ or, $\dfrac{\sin k\theta}{\cos k\theta} = \dfrac{k\sin\theta}{\cos\theta}$ or, $\dfrac{\cos k\theta}{\sin k\theta} = \dfrac{\cos\theta}{k\sin\theta}$ [taking reciprocals of both sides]

or, $\dfrac{\cos^2 k\theta}{\sin^2 k\theta} = \dfrac{\cos^2\theta}{k^2\sin^2\theta}$ [squaring both sides] or, $\dfrac{\cos^2 k\theta}{\sin^2 k\theta} + 1 = \dfrac{\cos^2\theta}{k^2\sin^2\theta} + 1$ [adding 1 on both sides]

or, $\dfrac{\cos^2 k\theta + \sin^2 k\theta}{\sin^2 k\theta} = \dfrac{\cos^2\theta + k^2\sin^2\theta}{k^2\sin^2\theta}$ or, $\dfrac{1}{\sin^2 k\theta} = \dfrac{1 - \sin^2\theta + k^2\sin^2\theta}{k^2\sin^2\theta}$

or, $\dfrac{\sin^2\theta}{\sin^2 k\theta} = \dfrac{1 + (k^2 - 1)\sin^2\theta}{k^2}$

or, $\dfrac{\sin^2 k\theta}{\sin^2\theta} = \dfrac{k^2}{1 + (k^2 - 1)\sin^2\theta}$ **(Proved)**

**13. Solution :** Given, $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a + b}$

or, $\left(\dfrac{a+b}{a}\right).\sin^4\alpha + \left(\dfrac{a+b}{b}\right).\cos^4\alpha = 1$ or, $(\sin^4\alpha + \cos^4\alpha) + \left(\dfrac{b}{a}\sin^4\alpha + \dfrac{a}{b}\cos^4\alpha\right) = 1$

or, $(\sin^2\alpha + \cos^2\alpha)^2 - 2\sin^2\alpha\cos^2\alpha + \left(\dfrac{\sqrt{b}}{\sqrt{a}}.\sin^2\alpha - \dfrac{\sqrt{a}}{\sqrt{b}}\cos^2\alpha\right)^2$

$+2\sin^2\alpha\cos^2\alpha = 1$ [$\because a^2 + b^2 = (a+b)^2 - 2ab = (a-b)^2 + 2ab$]

or, $1 + \left(\dfrac{\sqrt{b}}{\sqrt{a}}.\sin^2\alpha - \dfrac{\sqrt{a}}{\sqrt{b}}\cos^2\alpha\right) = 1$ or, $\dfrac{\sqrt{b}}{\sqrt{a}}.\sin^2\alpha - \dfrac{\sqrt{a}}{\sqrt{b}}.\cos^2\alpha = 0$

or, $\dfrac{\sqrt{b}}{\sqrt{a}}.\sin^2\alpha = \dfrac{\sqrt{a}}{\sqrt{b}}.\cos^2\alpha$ or, $\dfrac{\sin^2\alpha}{a} = \dfrac{\cos^2\alpha}{b} = \dfrac{\sin^2\alpha + \cos^2\alpha}{a + b} = \dfrac{1}{a + b}$

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$\therefore \sin^2\alpha = \frac{a}{a+b}, \cos^2\alpha = \frac{b}{a+b}$

$\therefore$ L.H.S. $= \frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3} = \frac{1}{a^3} \cdot \frac{a^4}{(a+b)^4} + \frac{1}{b^3} \cdot \frac{b^4}{(a+b)^4}$

$= \frac{a}{(a+b)^4} + \frac{b}{(a+b)^4} = \frac{a+b}{(a+b)^4} = \frac{1}{(a+b)^3} =$ **R.H.S. (Proved)**

**14.** (i) **Solution :** Given expression $= \cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta.\cos(\alpha+\theta)$

$= \cos^2\theta + \cos(\alpha+\theta)[\cos(\alpha+\theta) - 2\cos\alpha\cos\theta] = \cos^2\theta + \cos(\alpha+\theta)[\cos\alpha\cos\theta - \sin\alpha\sin\theta - 2\cos\alpha\cos\theta]$

$= \cos^2\theta - \cos(\alpha+\theta).\cos(\alpha-\theta) = \cos^2\theta - \cos^2\alpha + \sin^2\theta$

$= 1 - \cos^2\alpha = \sin^2\alpha$, which is independent of $\theta$. **(Proved)**

**14.** (ii) **Solution :** $\sin^2(x+\alpha) + \sin^2(x+\beta) - 2\cos(\alpha-\beta).\sin(x+\alpha)\sin(x+\beta)$

$= \sin^2\theta + \sin^2\phi - 2\cos(\theta-\phi).\sin\theta\sin\phi$ [let $\theta = x+\alpha, \phi = x+\beta \Rightarrow \theta - \phi = \alpha - \beta$ ----(1)]

$= \frac{1}{2}\left[2\sin^2\theta + 2\sin^2\varphi - 2\cos(\theta-\varphi).2\sin\theta\sin\varphi\right]$

$= \frac{1}{2}\left[1-\cos 2\theta + 1 - \cos 2\varphi - 2\cos(\theta-\varphi).\{\cos(\theta-\varphi) - \cos(\theta+\varphi)\}\right]$

$= \frac{1}{2}\left[2 - (\cos 2\theta + \cos 2\varphi) - 2\cos^2(\theta-\varphi) + 2\cos(\theta-\varphi).\cos(\theta+\varphi)\right]$

$= \frac{1}{2}[2 - 2\cos(\theta+\varphi).\cos(\theta-\varphi) - 2\cos^2(\theta-\varphi) + 2\cos(\theta-\varphi).\cos(\theta+\varphi)]$

$= \frac{2}{2}\left[1-\cos^2(\theta-\varphi)\right] = \sin^2(\theta-\varphi) = \sin^2(\alpha-\beta)$ [by (1)], which is independent x. **(Proved)**

**15.** (i) **Solution :** Given, $\sec\alpha = \sec\beta\sec\gamma + \tan\beta\tan\gamma$

or, $\sec\alpha - \sec\beta\sec\gamma = \tan\beta\tan\gamma$ or, $(\sec\alpha - \sec\beta\sec\gamma)^2 = \tan^2\beta\tan^2\gamma$

or, $(\sec\alpha - \sec\beta\sec\gamma)^2 = (\sec^2\beta - 1)\tan^2\gamma$ or, $\sec^2\alpha - 2\sec\alpha\sec\beta\sec\gamma + \sec^2\beta.\sec^2\gamma = \sec^2\beta\tan^2\gamma - \tan^2\gamma$

or, $\sec^2\beta(\tan^2\gamma - \sec^2\gamma) + 2\sec\alpha\sec\gamma\sec\beta - (\tan^2\gamma + \sec^2\alpha) = 0$

or, $\sec^2\beta - 2\sec\alpha.\sec\gamma\sec\beta + (\tan^2\gamma + \sec^2\alpha) = 0$ [$\because \sec^2\gamma - \tan^2\gamma = 1$]

This is a quadratic equation of $\sec\beta$

Therefore, $\sec\beta = \frac{2\sec\alpha\sec\gamma \pm \sqrt{4\sec^2\alpha.\sec^2\gamma - 4(\tan^2\gamma + \sec^2\alpha)}}{2}$

$= \frac{2\sec\alpha\sec\gamma \pm 2\sqrt{\sec^2\alpha.\sec^2\gamma - \tan^2\gamma - \sec^2\alpha}}{2} = \sec\alpha\sec\gamma \pm \sqrt{\sec^2\alpha(\sec^2\gamma - 1) - \tan^2\gamma}$

$= \sec\alpha\sec\gamma \pm \sqrt{\sec^2\alpha.\tan^2\gamma - \tan^2\gamma}$

$= \sec\alpha\sec\gamma \pm \sqrt{\tan^2\gamma(\sec^2\alpha - 1)} = \sec\alpha\sec\gamma \pm \sqrt{\tan^2\alpha\tan^2\gamma}$

$\therefore \sec\beta = \sec\alpha\sec\gamma \pm \tan\alpha\tan\gamma$ **(Proved)**

**15.** (iii) **Solution :** Given, secx secy + tanx tany = secz or, $(\sec x \sec y + \tan x \tan y)^2 - 1 = \sec^2 z - 1$

or, $\sec^2 x \sec^2 y + \tan^2 x \tan^2 y + 2\sec x \sec y \tan x \tan y - 1 = \tan^2 z$

or, $\sec^2 x (\tan^2 y + 1) + \tan^2 x (\sec^2 y - 1) + 2\sec x \sec y \tan x \tan y - 1 = \tan^2 z$

or, $\sec^2 x \tan^2 y + \tan^2 x \sec^2 y + (\sec^2 x - \tan^2 x) + 2\sec x \tan y \tan x \sec y - 1 = \tan^2 z.$

or, $(\sec x \tan y + \tan x \sec y)^2 + 1 - 1 = \tan^2 z$ or, $(\sec x \tan y + \tan x \sec y)^2 = \tan^2 z$

$\therefore$ $\sec x \tan y + \tan x \sec y = \pm \tan z$ **(Proved)**

**16.** (i) **Solution :** $\cos(-1170^0) = \cos 1170^0$ $[\because \cos(-\theta) = \cos\theta] = \cos(90^0 \times 13 + 0^0) = \sin 0^0 = 0$ **(Ans)**

(ii) **Solution :** $\sin 1755^o = \sin(90^o \times 19 + 45^o) = -\cos 45^o = -\frac{1}{\sqrt{2}}$ **(Ans)**

(iii) **Solution :** $\cos(-1575^o) = \cos 1575^o$ $[\because \cos(-\theta) = \cos\theta]$

$= \cos(90^o \times 17 + 45^o) = -\sin 45^o = -\frac{1}{\sqrt{2}}$ **(Ans)**

**17.** (i) **Solution :** Given, tanx.tan3x = 1or, $\tan x = \frac{1}{\tan 3x} = \cot 3x$ or, $\tan x = \tan\left(\frac{\pi}{2} - 3x\right)$

or, $x = \frac{\pi}{2} - 3x$ or, $4x = \frac{\pi}{2}$ $\therefore 2x = \frac{\pi}{4}$ $\therefore$ $\tan 2x = \tan\frac{\pi}{4} = 1$ **Ans.**

**17.** (ii) **Solution :** Given, tanx.tan3x = 1 or, $\tan 3x = \cot x = \tan\left(\frac{\pi}{2} - x\right)$ or, $3x = \frac{\pi}{2} - x \Rightarrow 4x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{8}$

$\therefore$ $\tan x = \tan\frac{\pi}{8} = \frac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}} = \frac{2\sin^2\frac{\pi}{8}}{2\sin\frac{\pi}{8}\cos\frac{\pi}{8}} = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2} - 1$ **(Ans)**

**18.** (ii) **Solution :** L.H.S. = $\sin 420^o \cos 390^o - \cos(-300^o) . \sin(-330^o)$

$= \sin 420^o . \cos 390^o + \cos 300^o \sin 330^o$ $[\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]$

Now, $\sin 420^o = \sin(90^o \times 4 + 60^0) = +\sin 60^0 = +\frac{\sqrt{3}}{2}$

$\cos 390^o = \cos(90^o \times 4 + 30^o) = \cos 30^o = \frac{\sqrt{3}}{2}$

$\cos 300^o = \cos(90^o \times 3 + 30^o) = +\sin 30^o = +\frac{1}{2}$

$\sin 330^o = \sin(90^o \times 3 + 60^o) = -\cos 60^o = -\frac{1}{2}$

$\therefore$ **L. H. S** $= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$ = **R. H. S. (Proved)**

**18.** (iv) **Solution : L.H.S.** $= \cos^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\frac{5\pi}{4} + \sin^2\frac{7\pi}{4}$

$= \cos^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) + \sin^2\left(2\pi - \frac{\pi}{4}\right)$

$= \cos^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \cos^2\frac{3\pi}{4} + \sin^2\frac{\pi}{4}$

$= \left(\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4}\right) + \left(\sin^2\frac{3\pi}{4} + \cos^2\frac{3\pi}{4}\right)$

$= 1 + 1$ $[\because \sin^2\theta + \cos^2\theta = 1] = 2$ = **R.H.S. (Proved)**

**18.** (vi) **Solution : L.H.S.** $= \tan\frac{3\pi}{20} \cdot \tan\frac{5\pi}{20} \cdot \tan\frac{7\pi}{20} = \tan\frac{3\pi}{20}\cdot\tan\frac{\pi}{4}\cdot\tan\left(\frac{\pi}{2}-\frac{3\pi}{20}\right)$

$= \tan\frac{3\pi}{20}\cdot 1\cdot\cot\frac{3\pi}{20} = \tan\frac{3\pi}{20}\cdot\cot\frac{3\pi}{20} = 1 =$ **R.H.S. (Proved)**

**18.** (viii) **Solution** : $\cos 1^0 \cos 2^0 \cos 3^0 \ldots\ldots\ldots\ldots \cos 179^0$

$= \cos 1^0 \cos 2^0 \cos 3^0 \ldots\ldots \cos 89^0 \cos 90^0 \cos 91^0 \ldots\ldots\ldots \cos 179^0 = 0$ **(Proved)**

**18.** (ix) **Solution** : Given, $10\alpha = \frac{\pi}{2}$

Therefore, $\tan 3\alpha \tan 5\alpha \tan 7\alpha$

$= \tan\frac{3\pi}{20}\cdot\tan\frac{5\pi}{20}\cdot\tan\frac{7\pi}{20} = \tan\frac{3\pi}{20}\cdot\tan\frac{\pi}{4}\cdot\tan\left(\frac{\pi}{2}-\frac{3\pi}{20}\right) = \tan\frac{3\pi}{20}\cdot 1\cdot\cot\frac{3\pi}{20} = \tan\frac{3\pi}{20}\cdot\cot\frac{3\pi}{20} = 1$ **(Proved)**

**18.** (x) **Solution** : Given expression, $\tan 1^0 \tan 2^0 \tan 3^0 \ldots\ldots\ldots\ldots \tan 89^0$

$= (\tan 1^0 \tan 89^0)(\tan 2^0 \tan 88^0)(\tan 3^0 \tan 87^0) \ldots\ldots\ldots (\tan 44^0 \tan 46^0) \tan 45^0$

$= (\tan 1^0 \cot 1^0)(\tan 2^0 \cot 2^0)(\tan 3^0 \cot 3^0) \ldots\ldots\ldots (\tan 44^0 \cot 44^0) \tan 45^0$

$= 1$ [since $\tan 89^0 = \tan(90^0 - 1^0) = \cot 1^0$ etc.] **(Proved)**

**19.** (i) **Solution** : Given, $\tan 25^o = a$

$$\text{L.H.S.} = \frac{\tan 155^o - \tan 115^o}{1+\tan 155^o \cdot \tan 115^o} = \frac{\tan(180^o - 25^o) - \tan(90^o + 25^o)}{1+\tan(180^o - 25^o)\cdot\tan(90^o+25^o)}$$

$$= \frac{-\tan 25^o + \cot 25^o}{1+(-\tan 25^o)(-\cot 25^o)} = \frac{-\tan 25^o + \frac{1}{\tan 25^o}}{1+1} \quad [\because \tan\theta \cdot \cot\theta = 1]$$

$$= \frac{-a+\frac{1}{a}}{2} \quad [\because \tan 25^o = a] = \frac{1-a^2}{2a} = \text{R.H.S} \ \textbf{(Proved)}$$

**19.** (ii) **Solution** : Given expression,

$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$

$$= 3\left[\cos^4\alpha+\sin^4\alpha\right]-2\left[\cos^6\alpha+\sin^6\alpha\right]$$

$$= 3\left\{(\cos^2\alpha+\sin^2\alpha)^2 - 2\sin^2\alpha.\cos^2\alpha\right\} - 2\left\{(\cos^2\alpha+\sin^2\alpha)^3 - 3\cos^2\alpha.\sin^2\alpha\,(\cos^2\alpha+\sin^2\alpha)\right\}$$

$= 3(1-2\sin^2\alpha\cos^2\alpha) - 2(1-3\sin^2\alpha\cos^2\alpha)$ $[\because \sin^2\alpha + \cos^2\alpha = 1]$

$= 3 - 6\sin^2\alpha\cos^2\alpha - 2 + 6\sin^2\alpha\cos^2\alpha = 3 - 2 = 1$ **(Ans)**

**19.** (iv) **Solution** : Given, $\cot\theta = \frac{3}{4}$ or, $\frac{\cos\theta}{3} = \frac{\sin\theta}{4} = \pm\frac{\sqrt{\sin^2\theta+\cos^2\theta}}{\sqrt{3^2+4^2}} = \pm\frac{1}{5}$

$\therefore \cos\theta = \pm\frac{3}{5}, \sin\theta = \pm\frac{4}{5}$

$\because$ $\sin\theta$ is negative, $\therefore \sin\theta = -\frac{4}{5}$. Again, $\cot\theta$ is positive, $\sin\theta$ is negative.

$\therefore$ $\cos\theta$ also negative. $\therefore \cos\theta = -\frac{3}{5}$

$$\therefore \frac{\cos(-\theta)+\operatorname{cosec}\theta}{\cos\theta+\sin(-\theta)} = \frac{\cos\theta+\frac{1}{\sin\theta}}{\cos\theta-\sin\theta} = \frac{-\frac{3}{5}-\frac{5}{4}}{-\frac{3}{5}+\frac{4}{5}} = \frac{-37}{20}\times 5 = -\frac{37}{4} \quad \textbf{(Ans)}$$

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**20.** (ii) **Solution :** We know, $\sec^2\theta \geq 1$ or, $\frac{4xy}{(x+y)^2} \geq 1$ $\left[\because \sec^2\theta = \frac{4xy}{(x+y)^2}\right]$

or, $(x+y)^2 \leq 4xy$ or, $(x+y)^2 - 4xy \leq 0$ or, $(x-y)^2 \leq 0$

Since x, y are real, $(x-y)^2 < 0$ is not possible.

$\therefore (x-y)^2 = 0$ or, $x - y = 0$ or, $x = y$ **(Ans)**

**20.**(iv) Solution : Given, $\cos\theta = \frac{x^2+y^2+z^2}{xy+yz+zx}$

Now, we know, $-1 \leq \cos\theta \leq 1$ or, $-1 \leq \frac{x^2+y^2+z^2}{xy+yz+zx} \leq 1$

or, $-xy - yz - zx \leq x^2 + y^2 + z^2 \leq xy + yz + zx$

Now, $x^2 + y^2 + z^2 \leq xy + yz + zx$ gives,

$x^2 + y^2 + z^2 - xy - yz - zx \leq 0$

or, $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx \leq 0$

or, $(x-y)^2 + (y-z)^2 + (z-x)^2 \leq 0$ which is not true, since x, y, z are three unequal positive real numbers.

Therefore, $\cos\theta = \frac{x^2+y^2+z^2}{xy+yz+zx}$ is not possible for three unequal positive real numbers x, y and z. **(Ans)**

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Chapter 2

# Compound Angles,
# Transformation of Sums and Products,
# Multiple and Sub Multiple Angles

## 2.1 Compound Angles :

The algebraic sum of two or more angles is called a compound Angle. If A, B, C are three given angles, then A + B, A − B, A − B + C etc., are Compound Angles.

**Formulae :**

$\sin(A+B) = \sin A \cos B + \cos A \sin B$  $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$  $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

**Note :** The above formulae are true for any positive or negative values of A and B.

$\sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

$\cos(A+B).\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A.\tan B}$  $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A.\tan B}$

$\cot(A+B) = \dfrac{\cot A.\cot B - 1}{\cot B + \cot A}$  $\cot(A-B) = \dfrac{\cot A.\cot B + 1}{\cot B - \cot A}$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A.\tan B.\tan C}{1 - \tan A.\tan B - \tan B.\tan C - \tan C.\tan A}$$

$$\cot(A+B+C) = \frac{\cot A.\cot B.\cot C - \cot A - \cot B - \cot C}{\cot A.\cot B + \cot B.\cot C + \cot C.\cot A - 1}$$

$\sin(A+B+C) = \cos A.\cos B.\cos C(\tan A + \tan B + \tan C - \tan A.\tan B.\tan C)$

$\cos(A+B+C) = \cos A.\cos B.\cos C(1 - \tan A.\tan B - \tan B.\tan C - \tan C\tan A)$

## 2.2 Transformation of sums and product : Formulae :

$2\sin A.\cos B = \sin(A+B) + \sin(A-B)$  $2\cos A.\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A.\cos B = \cos(A+B) + \cos(A-B)$  $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin C + \sin D = 2\sin\dfrac{C+D}{2}.\cos\dfrac{C-D}{2}$  $\sin C - \sin D = 2\cos\dfrac{C+D}{2}.\sin\dfrac{C-D}{2}$

$\cos C + \cos D = 2\cos\dfrac{C+D}{2}.\cos\dfrac{C-D}{2}$  $\cos C - \cos D = 2\sin\dfrac{C+D}{2}.\sin\dfrac{D-C}{2}$

**2.3 Multiple Angle :** If A be a given angle, then 2A, 3A, 4A, etc. are called Multiple Angles.

**Formulae :**

$\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$ $\quad\cos 2A = \cos^2 A - \sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A} = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ $\quad\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

$\sin 3A = 3\sin A - 4\sin^3 A$ $\quad\cos 3A = 4\cos^3 A - 3\cos A$

$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$ $\quad\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

**2.4 Submultiple Angle :**

If A be the given angle, then $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$ etc. are called submultiple angles of the angle A.

**Formulae :**

$\sin A = 2\sin\frac{A}{2}.\cos\frac{A}{2} = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$ $\quad\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

$\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$ $\quad\cot A = \dfrac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}}$

$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$ $\quad\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$

$\tan A = \dfrac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1 - 3\tan^2\frac{A}{3}}$ $\quad\cot A = \dfrac{\cot^3\frac{A}{3} - 3\cot\frac{A}{3}}{3\cot^2\frac{A}{3} - 1}$

$\sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$ $\quad\cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$\tan\frac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{-1 \pm \sqrt{1+\tan^2 A}}{\tan A}$

**Note :** When A is given, we can find the respective quadrant in which $\frac{A}{2}$ will be and hence using "all, sin, tan, cos" rule we can determine the appropriate sign.

## Some Important Results

$\sin 18^0 = \dfrac{\sqrt{5}-1}{4}$; $\quad\cos 18^0 = \frac{1}{4}\sqrt{10+2\sqrt{5}}$ $\quad\cos 36^0 = \dfrac{\sqrt{5}+1}{4}$; $\quad\sin 36^0 = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

$\sin 54^0 = \dfrac{\sqrt{5}+1}{4}$; $\quad\cos 54^0 = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

$\sin 72^0 = \cos 18^0 = \frac{1}{4}\sqrt{10+2\sqrt{5}}$; $\quad\cos 72^0 = \sin 18^0 = \dfrac{\sqrt{5}-1}{4}$

$\sin 3^0 = \frac{1}{16}\left[(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \sqrt{10+2\sqrt{5}}(\sqrt{6}-\sqrt{2})\right]$ $\quad\cos 3^0 = \frac{1}{16}\left[\sqrt{10+2\sqrt{5}}(\sqrt{6}+\sqrt{2}) + (\sqrt{5}-1)(\sqrt{6}-\sqrt{2})\right]$

$\cos 15^0 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ $\quad\sin 15^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

# PROBLEM SET – I

[Problems with '*' marks are solved at the end of the problem set]

1. Prove that–

(i) $\sin^4\theta + \cos^4\theta = 1 - \frac{1}{2}.\sin^2 2\theta$ (ii) $\sin^6\theta + \cos^6\theta = 1 - \frac{3}{4}\sin^2 2\theta$

*(iii) $\sin^6\frac{\theta}{2} + \cos^6\frac{\theta}{2} = \frac{1}{4}(1 + 3\cos^2\theta)$

2. Prove that–

(i) $\frac{\sec\theta - \tan\theta + 1}{\sec\theta + \tan\theta + 1} = \frac{1 - \sin\theta}{\cos\theta} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$ *(ii) $\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{1 + \sin\theta}{\cos\theta} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

*(iii) $\frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\frac{\theta}{2}$ (iv) $\frac{1 - \cos 2\alpha + \sin 2\alpha}{1 + \cos 2\alpha + \sin 2\alpha} = \tan\alpha$

3. Prove that–

(i) $\frac{1 + \sin\theta}{1 - \sin\theta} = (\sec\theta + \tan\theta)^2 = \tan^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ (ii) $\sqrt{\frac{1 + \cos\theta}{1 - \cos\theta}} = \operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$

*(iii) $\alpha + 2\beta = 180^\circ$, if $\frac{1 - \sin\frac{\alpha}{2}}{1 + \sin\frac{\alpha}{2}} = \tan^2\frac{\beta}{2}$ (iv) $\frac{1}{\operatorname{cosec}\alpha - \cot\alpha} - \frac{1}{\sin\alpha} = \frac{1}{\sin\alpha} - \frac{1}{\operatorname{cosec}\alpha + \cot\alpha}$

*(v) $\frac{1}{\sec\theta + \tan\theta} - \frac{1}{\cos\theta} = \frac{1}{\cos\theta} - \frac{1}{\sec\theta - \tan\theta}$

4. Prove that–

*(i) $\frac{1 - \cos 2A}{\sin 2A} = \frac{\sin 2A}{1 + \cos 2A} = \tan A$ (ii) $\frac{1 + \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 - \cos\alpha} = \cot\frac{\alpha}{2}$

(iii) $\frac{\cos\alpha}{1 - \sin\alpha} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = \frac{\cot\frac{\alpha}{2} + 1}{\cot\frac{\alpha}{2} - 1}$ *(iv) $\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = \cot^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

5. Prove that–

*(i) $\cot 2A + \tan A = \operatorname{cosec} 2A$ (ii) $1 + \tan\theta.\tan\frac{\theta}{2} = \sec\theta$

(iii) $2\operatorname{cosec}\theta = \tan\frac{\theta}{2} + \cot\frac{\theta}{2}$ *(iv) $\tan\frac{\theta}{2} - \cot\frac{\theta}{2} + 2\cot\theta = \theta$

6. Prove that

(i) $\sin\theta.\sin\left(60^\circ + \theta\right).\sin\left(60^\circ - \theta\right) = \frac{1}{4}\sin 3\theta$ (ii) $\cos\alpha.\cos\left(120^\circ + \alpha\right).\cos\left(240^\circ + \alpha\right) = \frac{1}{4}\cos 3\alpha$

*(iii) $\tan\theta.\tan\left(\frac{\pi}{3} + \theta\right).\tan\left(\frac{\pi}{3} - \theta\right) = \tan 3\theta$

7. Prove that

*(i) $\frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ - \sin 8^\circ} = \tan 53^\circ$ **[WBSC – 82]** (ii) $\frac{\cos 9^\circ - \sin 9^\circ}{\cos 9^\circ + \sin 9^\circ} = \tan 36^\circ$

(iii) $\tan 25^\circ = \frac{\cos 20^\circ - \sin 20^\circ}{\cos 20^\circ + \sin 20^\circ}$ (iv) $\frac{\cos 7^0 + \sin 7^0}{\cos 7^0 - \sin 7^0} = \tan 52^0$ **[WBSC – 07]**

8.(a) Prove that–

*(i) $\tan 10A . \tan 6A . \tan 4A = \tan 10A - \tan 6A - \tan 4A$

(ii) $\tan 3A . \tan 2A . \tan A = \tan 3A - \tan 2A - \tan A$

8.(b) Prove that–

(i) $\tan 35^o + \tan 10^o + \tan 35^o . \tan 10^o = 1$ *(ii) $\sqrt{3}(\tan 170^o - \tan 140^o) = 1 + \tan 170^o . \tan 140^o$.

(iii) $1 - \tan 15^o - \tan 30^o = \tan 15^o . \tan 30^o$

8.(c) Prove that–

*(i) $\tan 50^o = \tan 40^o + 2 \tan 10^o$ (ii) $\tan 70^o = 2 \tan 50^o + \tan 20^o$

9. Prove that–

(i) $\frac{1}{\sin 10^o} - \frac{\sqrt{3}}{\cos 10^o} = 4$ *(ii) $\frac{1}{2\sin 10^o} - 2\sin 70^o = 1$

*(iii) $\frac{1}{2\sin 10^o} - \frac{\sqrt{3}}{2\cos 10^o} = 2$ **[WBSC – 96, 00]**

10. Prove that–

(i) $\cos 95^o + \cos 25^o = \frac{1}{\sqrt{2}}(\cos 10^o + \sin 10^o)$ *(ii) $\sin 16^o + \cos 16^o = \frac{1}{\sqrt{2}}(\sin 1^o + \sqrt{3}.\cos 1^o)$

11.(a) Prove that–

(i) $\sin 15^o \sin 75^o = \frac{1}{4}$ **[WBSC – 03]** (ii) $\cos 40^o \cos 100^o \cos 160^o = \frac{1}{8}$

*(iii) $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{1}{8}$ **[WBSC – 05]** *(iv) $\cos 20^o \cos 40^o \cos 60^o \cos 80^o = \frac{1}{16}$ **[WBSC – 84]**

*(v) $\sin 20^o \sin 40^o \sin 60^o \sin 80^o = \frac{3}{16}$ (vi) $\tan 20^o . \tan 40^o . \tan 60^o \tan 80^o = 3$

*(vii) $\tan 20^o . \tan 40^o . \tan 80^o = \sqrt{3}$ **[WBSC – 93]**

11.(b) Prove that–

*(i) $\tan 6^o . \tan 42^o . \tan 66^o . \tan 78^o = 1$

*(ii) $\sec^2 20^o + \sec^2 40^o + \sec^2 80^o = 36$ (iii) $\cot^2 20^o + \cot^2 40^o + \cot^2 80^o = 9$

12. Prove that–

(i) $4(\cos^3 10^o + \sin^3 20^o) = 3(\cos 10^o + \sin 20^o)$ *(ii) $4(\cos^3 25^o + \cos^3 35^o) = 3(\cos 25^o + \cos 35^o)$

13. Prove that–

(i) $\cos\alpha + \cos(120^o + \alpha) + \cos(120^o - \alpha) = 0$ (ii) $\sin\alpha + \sin(120^o + \alpha) - \sin(120^o - \alpha) = 0$

*(iii) $\cos^2(A - 120^o) + \cos^2 A + \cos^2(A + 120^o) = \frac{3}{2}$ (iv) $\sin^2\alpha + \sin^2(120^o - \alpha) + \sin^2(120^o + \alpha) = \frac{3}{2}$

(v) $\cos^3\alpha + \cos^3(120^o + \alpha) + \cos^3(240^o + \alpha) = \frac{3}{4} . \cos 3\alpha$

*(vi) $\sin^3\alpha + \sin^3(120^o + \alpha) + \sin^3(240^o + \alpha) = -\frac{3}{4}\sin 3\alpha$

14. Prove that–

(i) $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$ *(ii) $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$

*(iii) $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)=\frac{1}{8}$

15. Prove that–

(i) $16\cos\frac{2\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{8\pi}{15}.\cos\frac{14\pi}{15}=1$ (ii) $16\cos\frac{\pi}{15}.\cos\frac{2\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{8\pi}{15}=-1$

(iii) $\cos\theta . \cos2\theta . \cos4\theta . \cos8\theta = -\frac{1}{16}$, if $15\theta=\pi$

*(iv) $\cos\alpha . \cos2\alpha . \cos4\alpha . \cos7\alpha = \frac{1}{16}$, if $\alpha=\frac{2\pi}{15}$ [WBSC – 89]

*(v) $2^4\cos\theta . \cos2\theta . \cos2^2\theta . \cos2^3\theta = 1$, if $17\theta=\pi$ [WBSC – 86]

(vi) $\cos\theta . \cos2\theta . \cos4\theta . \cos8\theta . \cos16\theta = \frac{1}{32}$, if $\theta=\frac{2\pi}{31}$

*(vii) $\cos\theta . \cos2\theta . \cos3\theta . \cos4\theta . \cos5\theta = \frac{1}{2^5}$, if $\theta=\frac{\pi}{11}$ [WBSC – 90]

(viii) $\cos\theta . \cos2\theta . \cos3\theta . \cos4\theta . \cos5\theta . \cos6\theta = \frac{1}{2^6}$, if $\theta=\frac{\pi}{13}$

*(ix) $\cos\frac{\pi}{15}.\cos\frac{2\pi}{15}.\cos\frac{3\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{5\pi}{15}.\cos\frac{6\pi}{15}.\cos\frac{7\pi}{15}=(128)^{-1}$ [WBSC – 88]

*(x) $\sin\alpha . \sin3\alpha . \sin5\alpha = \frac{1}{8}$, if $\alpha=\frac{\pi}{14}$ [WBSC – 88]

**16.** *(i) If $A + B = 45^o$, prove that, $(1 + \tan A)(1 + \tan B) = 2$. Hence show that $\tan 22\frac{1}{2}^o=\sqrt{2}-1$. [WBSC – 97]

(ii) If $A + B = 45^o$, show that, $(\cot A - 1)(\cot B - 1) = 2$. Hence show that $\cot 22\frac{1}{2}^o=\sqrt{2}+1$.

*(iii) If $A + B = 225^o$, show that, $\frac{\cot A}{1+\cot A}\cdot\frac{\cot B}{1+\cot B}=\frac{1}{2}$

**17.** (i) If $A + B + C = \pi$, and $\cos A = \cos B . \cos C$, show that,

(a) $\tan B \tan C = 2$ *(b) $2\cot B \cot C = 1$ [WBSC – 83, 84, 88]

*(c) $\tan A = \tan B + \tan C$ [WBSC – 82]

(ii) If $A + B + C = \pi$, and $\sin A = -\sin B . \cos C$, prove that, $2\tan B + \tan C = 0$ [WBSC – 91]

*(iii) If $A + B + C = \pi$, show that, $\cot B . \cot C + \cot C . \cot A + \cot A . \cot B = 1$ [WBSC – 89]

(iv) If $\alpha + \beta + \gamma = 90^o$ show that, $\tan\alpha=\frac{1-\tan\beta.\tan\gamma}{\tan\beta+\tan\gamma}$.

18. Prove that–

(i) $\frac{\cot\theta}{\cot\theta-\cot3\theta}+\frac{\tan\theta}{\tan\theta-\tan3\theta}$ is independent of $\theta$.

*(ii) $\frac{1}{x}-\frac{1}{y}=\cot(\theta+\varphi)$, if $x=\tan\theta+\tan\varphi$, $y=\cot\theta+\cot\varphi$

**19.** *(i) If $\cos(x-y)=-1$, show that, $\cos x + \cos y = 0 = \sin x + \sin y$.

*(ii) If $\cos(A-B)+\cos(B-C)+\cos(C-A)=-\frac{3}{2}$, show that, $\cos A + \cos B + \cos C = 0 = \sin A + \sin B + \sin C$.

[WBSC – 85]

(iii) If $\sin(A + B) + \sin(B + C) + \cos(C - A) = -\frac{3}{2}$.

show that, $\sin A + \cos B + \sin C = 0$ and $\cos A + \sin B + \cos C = 0$..

**20.** If $\tan^2\theta = 1 + 2\tan^2\varphi$, show that,

(i) $\cos 2\theta + \sin^2\varphi = 0$ (ii) $1 + \sin^2\varphi = 2\sin^2\theta$

*(iii) $2\cos^2\theta = \cos^2\varphi$ *(iv) $\cos 2\varphi = 1 + 2\cos 2\theta$ **[WBSC – 83, 85]**

**21.** Prove that–

(i) $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma) = 4.\sin\frac{\beta+\gamma}{2}.\sin\frac{\gamma+\alpha}{2}.\sin\frac{\alpha+\beta}{2}$

*(ii) $\cos\alpha - \cos\beta + \cos\gamma - \cos(\alpha + \beta + \gamma) = 4.\sin\frac{\alpha+\beta}{2}.\sin\frac{\beta+\gamma}{2}.\cos\frac{\gamma+\alpha}{2}$

(iii) $\cos 2A + \cos 4A + \cos 6A + \cos 8A = 4\cos A . \cos 2A . \cos 5A$.

**22.** (i) If $\sin x - \sin y = a$, $\cos x + \cos y = b$, show that $\tan\frac{x+y}{2} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

(ii) If $\sin\alpha + \sin\beta = \frac{1}{3}$, $\cos\alpha + \cos\beta = \frac{1}{2}$, show that, $\tan\frac{\alpha+\beta}{2} = \frac{2}{3}$

*(iii) If $\sin\theta + \sin\varphi = a$, $\cos\theta + \cos\varphi = b$, show that, $\tan\frac{\theta-\varphi}{2} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$ **[WBSC – 86, 03]**

**23.** (i) If $\sec A - \operatorname{cosec} A = \operatorname{cosec} B - \sec B$, show that, $\tan A . \tan B = \tan\frac{A+B}{2}$

*(ii) If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, show that, **[WBSC – 85]**

**24.** *(i) If $\cos(A + B) . \sin(C + D) = \cos(A - B) . \sin(C - D)$, show that, $\cot A . \cot B . \cot C = \cot D$.

(ii) If $\cos 2A . \sin 2B = \cos 2C . \sin 2D$, prove that, $\tan(C + A) . \tan(C - A) . \tan(B + D) = \tan(D - B)$

*(iii) If $\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)} = \frac{\tan\gamma}{\tan\beta}$, prove that, either, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$, or, $\sin(\beta - \gamma) = 0$

(iv) If $\sin 2\alpha = 4\sin 2\beta$, show that, $5\tan(\alpha - \beta) = 3\tan(\alpha + \beta)$

**25.** *(i) If $\alpha$ and $\beta$ are the two solutions of the equation $a\tan\theta + b\sec\theta = c$,

express the value of $\tan(\alpha + \beta)$ in terms of a, b, c. [**Ans:** $2ac/(a^2 - c^2)$]

(ii) If between 0 and $2\pi$, $\alpha$ and $\beta$ be the two solutions of the equation $a\sin x + b\cos x + c = 0$, then prove that,

(a) $a - b.\tan\frac{\alpha+\beta}{2} = 0$ (b) $\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$

**26.** *(i) Show that, if an angle $\alpha$ be divided into two parts so that the ratio of the tangents of the parts is $\lambda$, the difference x between the parts is given by the equation. $\sin x = \frac{\lambda-1}{\lambda+1}.\sin\alpha$

(ii) If an angle $\theta$ is divided into two parts $\alpha$ and $\beta$, such that $\tan\alpha : \tan\beta = x : y$, prove that, $\sin(\alpha-\beta) = \frac{x-y}{x+y}.\sin\theta$

(iii) If $m\tan(\theta - 30^o) = n\tan(\theta + 120^o)$, show that, $\cos 2\theta = \frac{m+n}{2(m-n)}$

*(iv) If $\sin\theta = n\sin(\theta + 2\alpha)$, prove that, $\tan(\theta+\alpha) = \frac{1+n}{1-n}.\tan\alpha$ **[WBSC – 90]**

(v) If $A + B + C = \pi$ and $\sin\left(A + \frac{C}{2}\right) = n\sin\frac{C}{2}$, show that, $\tan\frac{A}{2}.\tan\frac{B}{2} = \frac{n-1}{n+1}$

**27.** *(i) If $2\tan\alpha = 3\tan\beta$, show that, $\tan(\alpha - \beta) = \dfrac{\sin 2\beta}{5 - \cos 2\beta}$ **[WBSC – 03, 07]**

*(ii) If $\tan(\alpha - \beta) = \dfrac{\sin 2\beta}{5 - \cos 2\beta}$, show that, $\tan\alpha : \tan\beta = 3 : 2$ **[WBSC – 04]**

*(iii) If $\tan\beta = \dfrac{\sin\alpha\cos\alpha}{2 + \cos^2\alpha}$, prove that, $3\tan(\alpha - \beta) = 2\tan\alpha$.

*(iv) If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, show that, $\tan(\alpha - \beta) = (1 - n)\tan\alpha$

(v) If $\tan\dfrac{\beta}{2} = 4\tan\dfrac{\alpha}{2}$, prove that, $\tan\dfrac{\beta - \alpha}{2} = \dfrac{3\sin\alpha}{5 - 3\cos\alpha}$

*(vi) If $2\tan\beta + \cot\beta = \tan\alpha$, show that, $\cot\beta = 2\tan(\alpha - \beta)$

*(vii) If $\tan\theta = \cos 2\alpha \cdot \tan\varphi$, prove that, $\tan(\varphi - \theta) = \dfrac{\tan^2\alpha \cdot \sin 2\varphi}{1 + \tan^2\alpha \cdot \cos 2\varphi}$

*(viii) If $\cot\theta = \cos(x + y)$ and $\cot\varphi = \cos(x - y)$, show that, $\tan(\theta - \varphi) = \dfrac{2\sin x\sin y}{\cos^2 x + \cos^2 y}$

*(ix) If $\sin\theta = k\sin(\theta + \varphi)$, prove that, $\tan(\theta + \varphi) = \dfrac{\sin\varphi}{\cos\varphi - k}$.

**28.** *(i) If A and B are positive acute angles and $\cos 2A = \dfrac{3\cos 2B - 1}{3 - \cos 2B}$ show that, $\tan A = \sqrt{2}.\tan B$. **[WBSC – 85]**

(ii) If $\tan\dfrac{\alpha}{2} = \dfrac{1}{\sqrt{3}}.\tan\dfrac{\beta}{2}$, show that, $\cos\beta = \dfrac{2\cos\alpha - 1}{2 - \cos\alpha}$

*(iii) If $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1 - e}{1 + e}}.\tan\dfrac{\varphi}{2}$, show that, $\cos\varphi = \dfrac{\cos\theta - e}{1 - e.\cos\theta}$ **[WBSC – 84, 97]**

(iv) If $\cos\theta = \dfrac{a\cos\varphi + b}{a + b\cos\varphi}$, show that, $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{a - b}{a + b}}.\tan\dfrac{\varphi}{2}$

(v) If $\tan\theta = \sqrt{\dfrac{a - b}{a + b}}.\tan\dfrac{A}{2}$, $\cos\varphi = \dfrac{b + a\cos A}{a + b\cos A}$, prove that, $\varphi = 2\theta$.

**29.** (i) If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha.\cos\beta}$, show that, one value of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}.\cot\dfrac{\beta}{2}$

*(ii) If $\tan\theta = \dfrac{\sin\alpha\sin\beta}{\cos\alpha + \cos\beta}$, show that, one value of the $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}.\tan\dfrac{\beta}{2}$.

**30.** *(i) If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, prove that, $1 + \cot\alpha \cdot \tan\beta = 0$ **[WBSC – 88]**

*(ii) If $\sin\alpha + \sin\beta = \sqrt{3}(\cos\beta - \cos\alpha)$, prove that, $\sin 3\alpha + \sin 3\beta = 0$ **[WBSC – 90, 91]**

*(iii) If $\dfrac{\cos\theta}{\cos\varphi} = \dfrac{a}{b}$, prove that, $(a + b).\tan\dfrac{\theta + \varphi}{2} = a\tan\theta + b\tan\varphi$. **[WBSC – 91]**

*(iv) If $\tan\theta = \sec 2\alpha$, then prove that, $\sin 2\theta = \dfrac{1 - \tan^4\alpha}{1 + \tan^4\alpha}$ **[WBSC – 99]**

*(v) Prove that, $2\,\mathrm{cosec}\,4\theta - \sec 2\theta = \dfrac{1 - \tan\theta}{1 + \tan\theta}.\mathrm{cosec}\,2\theta$.

(vi) If $\alpha = \dfrac{2\pi}{7}$, prove that, $\tan\alpha \cdot \tan 2\alpha + \tan 2\alpha \cdot \tan 4\alpha + \tan 4\alpha \cdot \tan\alpha = -7$

*(vii) If $\sqrt{2}\cos A = \cos B + \cos^3 B$, $\sqrt{2}\sin A = \sin B - \sin^3 B$ prove that, $\sin(A - B) = \pm\dfrac{1}{3}$ **[WBSC – 86]**

*(viii) If $\cos(\alpha + \beta) = \dfrac{4}{5}$ and $\sin(\alpha - \beta) = \dfrac{5}{13}$, and $\alpha$, $\beta$ lies between 0 and $\dfrac{\pi}{4}$; find the value of $\tan 2\alpha$.

**[Ans: 56/33] [WBSC – 89]**

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(ix) If $\tan\theta = \dfrac{x\sin\varphi}{1 - x\cos\varphi}$, $\tan\varphi = \dfrac{y\sin\theta}{1 - y\cos\theta}$, show that, $\dfrac{\sin\theta}{\sin\varphi} = \dfrac{x}{y}$

*(x) If $\dfrac{\tan(\alpha-\beta)}{\tan\alpha} + \dfrac{\sin^2\gamma}{\sin^2\alpha} = 1$, show that, $\tan\alpha \cdot \tan\beta = \tan^2\gamma$ [WBSC – 84]

*(xi) If $\tan\theta = n(\sec\theta - 1)^2$, prove that, $\cot^3\frac{\theta}{2} - \cot\frac{\theta}{2} = 2n$ [WBSC – 98, 00]

*(xii) If $\tan\frac{\theta}{2} = \tan^3\frac{\varphi}{2}$, and $\tan\varphi = 2\tan\alpha$, show that, $\theta + \varphi = 2\alpha$. [WBSC – 96, 00, 06, 08]

(xiii) Show that, $\sin^2 16^\circ + \sin^2 23^\circ + \sin^2 37^\circ + \sin^2 44^\circ = 1 + \sin^2 7^\circ + \sin^2 14^\circ$

*(xiv) Show that, $\tan 7^\circ 30' = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$ [WBSC – 96]

(xv) If $\sin\alpha + \sin\beta = \frac{1}{2}$, and $\cos\alpha + \cos\beta = \frac{5}{4}$, find the value of $\tan\alpha + \tan\beta$. [Ans: 256/117]

*(xvi) Prove that, $\cot\theta \cdot \cot 2\theta + \cot 2\theta \cdot \cot 3\theta + 2 = \cot\theta(\cot\theta - \cot 3\theta)$

*(xvii) Prove that, $\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta = \cot\theta$.

*(xviii) With the help of the value of $\cos 30^\circ$, find the value of $\cos 7\frac{1}{2}^0$ [WBSC – 92]

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1. (iii) Solution :** L.H.S. $= \sin^6\frac{\theta}{2} + \cos^6\frac{\theta}{2} = \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)^3 - 3\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)$

$= 1 - 3\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}$ $[\because \sin^2 x + \cos^2 x = 1]$

$= 1 - \frac{3}{4}\left(2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}\right)^2 = 1 - \frac{3}{4}\sin^2\theta = 1 - \frac{3}{4}(1 - \cos^2\theta)$

$= \dfrac{4 - 3 + 3\cos^2\theta}{4} = \frac{1}{4}(1 + 3\cos^2\theta)$ = R.H.S **(Proved)**

**2. (ii) Solution :** L.H.S. $= \dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta}$ [dividing numerator and denominator by $\cos\theta$]

$= \dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$ $[\because \sec^2\theta - \tan^2\theta = 1]$

$= \dfrac{(\tan\theta + \sec\theta) - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1} = \dfrac{(\sec\theta + \tan\theta)(1 - \sec\theta + \tan\theta)}{\tan\theta - \sec\theta + 1} = \sec\theta + \tan\theta$

$= \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1 + \sin\theta}{\cos\theta}$ **(Proved)**

$= \dfrac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} = \dfrac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)} = \dfrac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} = \dfrac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} = \dfrac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}.\tan\frac{\theta}{2}}$

$= \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ = R.H.S. **(Proved).**

**2. (iii) Solution :** L.H.S $= \dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \dfrac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta} = \dfrac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}$

$= \dfrac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)} = \tan\frac{\theta}{2}$ **(Proved)**

**3. (iii) Solution :** Given, $\frac{1-\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}} = \tan^2\frac{\beta}{2}$ or, $\frac{\cos^2\frac{\alpha}{4}+\sin^2\frac{\alpha}{4}-2\sin\frac{\alpha}{4}.\cos\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}+\sin^2\frac{\alpha}{4}+2\sin\frac{\alpha}{4}.\cos\frac{\alpha}{4}} = \tan^2\frac{\beta}{2}$

or, $\frac{\left(\cos\frac{\alpha}{4}-\sin\frac{\alpha}{4}\right)^2}{\left(\cos\frac{\alpha}{4}+\sin\frac{\alpha}{4}\right)^2} = \tan^2\frac{\beta}{2}$ or, $\left(\frac{1-\tan\frac{\alpha}{4}}{1+\tan\frac{\alpha}{4}}\right)^2 = \tan^2\frac{\beta}{2}$

or, $\frac{1-\tan\frac{\alpha}{4}}{1+\tan\frac{\alpha}{4}} = \tan\frac{\beta}{2}$ [taking positive sign] or, $\frac{\tan\frac{\pi}{4}-\tan\frac{\alpha}{4}}{1+\tan\frac{\pi}{4}.\tan\frac{\alpha}{4}} = \tan\frac{\beta}{2}$

or, $\tan\left(\frac{\pi}{4}-\frac{\alpha}{4}\right) = \tan\frac{\beta}{2}$ or, $\frac{\pi}{4}-\frac{\alpha}{4} = \frac{\beta}{2}$ or, $\pi-\alpha = 2\beta$ or, $\alpha+2\beta = 180^\circ$ **(Proved)**

**3. (v) Solution :** L.H.S. $= \frac{1}{\sec\theta+\tan\theta} - \frac{1}{\cos\theta} = \frac{\sec\theta-\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)} - \sec\theta$

$= \frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta} - \sec\theta = \sec\theta - \tan\theta - \sec\theta$ $[\because \sec^2\theta - \tan^2\theta = 1] = -\tan\theta$

R.H.S. $= \frac{1}{\cos\theta} - \frac{1}{\sec\theta-\tan\theta} = \sec\theta - \frac{\sec\theta+\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}$

$= \sec\theta - \frac{\sec\theta+\tan\theta}{\sec^2\theta-\tan^2\theta} = \sec\theta - \sec\theta - \tan\theta$ $[\because \sec^2\theta - \tan^2\theta = 1]$

$= -\tan\theta$ $\therefore$ L.H.S. = R.H.S. **(Proved)**.

**4. (i) Solution :** L.H.S. $= \frac{1-\cos 2A}{\sin 2A} = \frac{(1-\cos 2A)(1+\cos 2A)}{\sin 2A\,(1+\cos 2A)} = \frac{1-\cos^2 2A}{\sin 2A\,(1+\cos 2A)}$

$= \frac{\sin^2 2A}{\sin 2A\,(1+\cos 2A)} = \frac{\sin 2A}{1+\cos 2A}$ **(Proved)**.

$= \frac{2\sin A.\cos A}{1+2\cos^2 A-1} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin A}{\cos A} = \tan A =$ R.H.S. **(Proved)**

**4. (iv) Solution :** L.H.S. $= \frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta} = \frac{\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}} = \frac{1-\sin\theta}{1+\sin\theta}$

$= \frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} = \frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2} = \left(\frac{\cot\frac{\theta}{2}-1}{\cot\frac{\theta}{2}+1}\right)^2$

$= \left(\frac{\cot\frac{\theta}{2}.\cot\frac{\pi}{4}-1}{\cot\frac{\pi}{4}+\cot\frac{\theta}{2}}\right)^2 = \cot^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) =$ **R.H.S. (Proved)**

**5. (i) Solution :** L.H.S. $= \cot 2A + \tan A = \frac{\cos 2A}{\sin 2A} + \frac{\sin A}{\cos A} = \frac{\cos 2A.\cos A + \sin 2A.\sin A}{\sin 2A.\cos A}$

$= \frac{\cos(2A-A)}{\sin 2A.\cos A} = \frac{\cos A}{\sin 2A.\cos A} = \frac{1}{\sin 2A} = \operatorname{cosec} 2A =$ **R.H.S. (Proved)**

**5. (iv) Solution :** L.H.S. $= \tan\frac{\theta}{2} - \cot\frac{\theta}{2} + 2\cot\theta = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} - \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} + 2\cot\theta = \frac{\sin^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} + 2\cot\theta$

$= \frac{-2\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} + 2\cot\theta = -\frac{2\cos\theta}{\sin\theta} + 2\cot\theta = -2\cot\theta + 2\cot\theta = 0 =$ **R.H.S. (Proved)**

**6. (iii) Solution :** $L.H.S. = \tan\theta . \tan\left(\frac{\pi}{3}+\theta\right) . \tan\left(\frac{\pi}{3}-\theta\right)$

$$= \frac{\sin\theta . \sin(60^\circ+\theta) . \sin(60^\circ-\theta)}{\cos\theta . \cos(60^\circ+\theta) . \cos(60^\circ-\theta)} = \frac{\sin\theta\left(\sin^2 60^\circ - \sin^2\theta\right)}{\cos\theta\left(\cos^2 60^\circ - \sin^2\theta\right)}$$

[$\because$ $\sin(A+B) . \sin(A-B) = \sin^2 A - \sin^2 B$, $\cos(A+B) . \cos(A-B) = \cos^2 A - \sin^2 B$]

$$= \frac{\sin\theta\left(\frac{3}{4}-\sin^2\theta\right)}{\cos\theta\left(\frac{1}{4}-\sin^2\theta\right)} = \frac{3\sin\theta - 4\sin^3\theta}{\cos\theta - 4\cos\theta\left(1-\cos^2\theta\right)} = \frac{\sin 3\theta}{\cos\theta - 4\cos\theta + 4\cos^3\theta} = \frac{\sin 3\theta}{4\cos^3\theta - 3\cos\theta}$$

$= \frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta =$ R.H.S. **(Proved)**

**7. (i) Solution :** $L.H.S. = \frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ - \sin 8^\circ} = \frac{\frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}}{\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}}$ [dividing numerator and denominotor by $\cos 8^\circ$]

$= \frac{1+\tan 8^\circ}{1-\tan 8^\circ} = \frac{\tan 45^\circ + \tan 8^\circ}{1-\tan 45^\circ . \tan 8^\circ}$ $[\because \tan 45^\circ = 1]$

$= \tan(45^\circ + 8^\circ) = \tan 53^\circ =$ **R.H.S. (Proved)**

**8. a (i) Solution :** We know, $\tan 10A = \tan(6A + 4A)$ or, $\tan 10A = \frac{\tan 6A + \tan 4A}{1 - \tan 6A . \tan 4A}$

or, $\tan 6A + \tan 4A = \tan 10A - \tan 10A . \tan 6A \tan 4A$

or, $\tan 10A . \tan 6A . \tan 4A = \tan 10A - \tan 6A - \tan 4A$ **(Proved)**

**8. b (ii) Solution :** We know, $\tan 30^\circ = \tan(170^\circ - 140^\circ)$

or, $\tan 30^\circ = \frac{\tan 170^\circ - \tan 140^\circ}{1 + \tan 170^\circ . \tan 140^\circ}$ or, $\frac{1}{\sqrt{3}} = \frac{\tan 170^\circ - \tan 140^\circ}{1 + \tan 170^\circ . \tan 140^\circ}$

or, $\sqrt{3}(\tan 170^\circ - \tan 140^\circ) = 1 + \tan 170^\circ . \tan 140^\circ$ **(Proved)**

**8. c (i) Solution :** We know, $\tan 10^\circ = \tan(50^\circ - 40^\circ) = \frac{\tan 50^\circ - \tan 40^\circ}{1 + \tan 50^\circ . \tan 40^\circ}$

$= \frac{\tan 50^\circ - \tan 40^\circ}{1 + \tan 50^\circ . \cot 50^\circ}$ $[\because \tan 40^\circ = \tan(90^\circ - 50^\circ) = \cot 50^\circ]$

or, $\tan 10^\circ = \frac{\tan 50^\circ - \tan 40^\circ}{1+1}$ $[\because \tan\theta . \cot\theta = 1]$

or, $\tan 10^\circ = \frac{\tan 50^\circ - \tan 40^\circ}{2}$ or, $\tan 50^\circ - \tan 40^\circ = 2\tan 10^\circ$

or, $\tan 50^\circ = \tan 40^\circ + 2\tan 10^\circ$ **(Proved)**

**9.(ii) Solution :** $L.H.S. = \frac{1}{2\sin 10^\circ} - 2\sin 70^\circ = \frac{1 - 4\sin 70^\circ \sin 10^\circ}{2\sin 10^\circ} = \frac{1 - 2.(\cos 60^\circ - \cos 80^\circ)}{2\sin 10^\circ}$

$= \frac{1 - 2\left(\frac{1}{2} - \cos 80^\circ\right)}{2\sin 10^\circ} = \frac{1 - 1 + 2\cos 80^\circ}{2\sin(90^\circ - 80^\circ)} = \frac{2\cos 80^\circ}{2\cos 80^\circ} = 1 =$ **R.H.S. (Proved)**

**9.(iii) Solution :** L.H.S $= \dfrac{1}{2\sin 10^\circ} - \dfrac{\sqrt{3}}{2\cos 10^\circ} = \dfrac{\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$

$= \dfrac{2\left(\cos 60^\circ \cos 10^\circ - \sin 60^\circ \sin 10^\circ\right)}{2\sin 10^\circ \cos 10^\circ} = \dfrac{2\cos\left(60^\circ + 10^\circ\right)}{\sin 20^\circ}$

$= \dfrac{2\cos 70^\circ}{\sin\left(90^\circ - 70^\circ\right)} = \dfrac{2\cos 70^\circ}{\cos 70^\circ} = 2 =$ **R.H.S. proved.**

**10.(ii) Solution :** L.H.S. $= \sin 16^\circ + \cos 16^\circ = \sin(90^\circ - 74^\circ) + \cos 16^\circ = \cos 74^\circ + \cos 16^\circ$

$= 2\cos\left(\dfrac{74^\circ + 16^\circ}{2}\right).\cos\left(\dfrac{74^\circ - 16^\circ}{2}\right) = 2\cos 45^\circ . \cos 29^\circ$

$= 2 \times \dfrac{1}{\sqrt{2}}\cos\left(30^\circ - 1^\circ\right) = \dfrac{2}{\sqrt{2}}\left(\cos 30^\circ \cos 1^\circ + \sin 30^\circ \sin 1^\circ\right)$

$= \dfrac{2}{\sqrt{2}}\left(\dfrac{\sqrt{3}}{2}\cos 1^\circ + \dfrac{1}{2}\sin 1^\circ\right) = \dfrac{1}{\sqrt{2}}\left(\sin 1^\circ + \sqrt{3}.\cos 1^\circ\right) =$ **R.H.S. (Proved)**

**11. a (i) Solution :** L.H.S. $= \sin 15^0 \sin 75^0 = \frac{1}{2}(2\sin 75^0 \sin 15^0)$

$= \frac{1}{2}[\cos(75^0 - 15^0) - \cos(75^0 + 15^0)] = \frac{1}{2}\left(\cos 60^0 - \cos 90^0\right) = \frac{1}{2} \times \left(\frac{1}{2} - 0\right) = \frac{1}{4}$ = **R.H.S. (Proved)**

**11. a (iii) Solution :** L.H.S. $= \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

$= \dfrac{1}{2\sin\frac{\pi}{7}}\left(2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = \dfrac{1}{4\sin\frac{\pi}{7}}\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\cos\frac{4\pi}{7}$

$= \dfrac{1}{8\sin\frac{\pi}{7}}\left(2\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}\right) = \dfrac{1}{8\sin\frac{\pi}{7}} \times \sin\frac{8\pi}{7} = \dfrac{1}{8\sin\frac{\pi}{7}} \times \sin\left(\pi + \frac{\pi}{7}\right) = \dfrac{1}{8\sin\frac{\pi}{7}} \times \left(-\sin\frac{\pi}{7}\right) = -\frac{1}{8}$ **(Proved)**

**11. a (iv) Solution :** L.H.S. $= \cos 20^\circ \cos 40^\circ . \cos 60^\circ \cos 80^\circ = \frac{1}{2}\left(2\cos 40^\circ.\cos 20^\circ\right).\cos 60^\circ \cos 80^\circ$

$= \frac{1}{2}\left(\cos 60^\circ + \cos 20^\circ\right).\frac{1}{2}.\cos 80^\circ$ [$\because$ $2\cos A.\cos B = \cos(A+B) + \cos(A-B)$]

$= \frac{1}{4}\left(\frac{1}{2} + \cos 20^\circ\right).\cos 80^\circ = \frac{1}{4}\left(\frac{1}{2}.\cos 80^\circ + \cos 80^\circ.\cos 20^\circ\right)$

$= \frac{1}{8}\left(\cos 80^\circ + 2\cos 80^\circ.\cos 20^\circ\right) = \frac{1}{8}\left(\cos 80^\circ + \cos 100^\circ + \cos 60^\circ\right)$

$= \frac{1}{8}\left[\cos(90^\circ - 10^\circ) + \cos(90^\circ + 10^\circ) + \frac{1}{2}\right] = \frac{1}{8}\left[\sin 10^\circ - \sin 10^\circ + \frac{1}{2}\right] = \frac{1}{8}.\frac{1}{2} = \frac{1}{16}$ **(proved).**

**11. a.(v) Solution :** L.H.S. $= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$= \frac{1}{2}\left(2\sin 40^\circ.\sin 20^\circ\right).\frac{\sqrt{3}}{2}.\sin 80^\circ$

$= \frac{1}{2}\left(\cos 20^\circ - \cos 60^\circ\right).\frac{\sqrt{3}}{2}.\sin 80^\circ$ [$\because$ $2\sin A.\sin B = \cos(A-B) - \cos(A+B)$]

$= \frac{1}{2}.\frac{\sqrt{3}}{2}\left(\cos 20^\circ - \frac{1}{2}\right).\sin 80^\circ = \frac{\sqrt{3}}{4}\left(\sin 80^\circ.\cos 20^\circ - \frac{1}{2}\sin 80^\circ\right)$

$= \frac{\sqrt{3}}{8}\left[2\sin 80^\circ.\cos 20^\circ - \sin 80^\circ\right]$

$= \frac{\sqrt{3}}{8}\left[\sin 100^\circ + \sin 60^\circ - \sin 80^\circ\right]$ [$\because$ $2\sin A.\cos B = \sin(A+B) + \sin(A-B)$]

$= \frac{\sqrt{3}}{8}\left[\sin 60^\circ + \sin 100^\circ - \sin 80^\circ\right] = \frac{\sqrt{3}}{8}\left[\frac{\sqrt{3}}{2} + \sin(90^\circ + 10^\circ) - \sin(90^\circ - 10^\circ)\right]$

$= \frac{\sqrt{3}}{8}\left[\frac{\sqrt{3}}{2} + \cos 10^\circ - \cos 10^\circ\right] = \frac{\sqrt{3}}{8}.\frac{\sqrt{3}}{2}$ [$\because$ $\sin(90^\circ + \theta) = \cos\theta$, $\sin(90^\circ - \theta) = \cos\theta$] $= \frac{3}{16}$ **(proved).**

**11. a (vii) Solution :** L.H.S. = $\tan 20^\circ . \tan 40^\circ . \tan 80^\circ$

$$= \frac{\sin 20^\circ \sin 40^\circ \sin 80^\circ}{\cos 20^\circ.\cos 40^\circ.\cos 80^\circ} = \frac{(2\sin 40^\circ \sin 20^\circ)\sin 80^\circ}{(2\cos 40^\circ \cos 20^\circ).\cos 80^\circ} = \frac{(\cos 20^\circ - \cos 60^\circ)\sin 80^\circ}{(\cos 60^\circ + \cos 20^\circ)\cos 80^\circ}$$

$$= \frac{\left(\cos 20^\circ - \frac{1}{2}\right)\sin 80^\circ}{\left(\frac{1}{2} + \cos 20^\circ\right)\cos 80^\circ} = \frac{2\sin 80^\circ \cos 20^\circ - \sin 80^\circ}{\cos 80^\circ + 2\cos 80^\circ.\cos 20^\circ} = \frac{\sin 100^\circ + \sin 60^\circ - \sin 80^\circ}{\cos 80^\circ + \cos 100^\circ + \cos 60^\circ}$$

$$= \frac{\sin(90^\circ + 10^\circ) + \frac{\sqrt{3}}{2} - \sin(90^\circ - 10^\circ)}{\cos(90^\circ - 10^\circ) + \cos(90^\circ + 10^\circ) + \frac{1}{2}} = \frac{\cos 10^\circ + \frac{\sqrt{3}}{2} - \cos 10^\circ}{\sin 10^\circ - \sin 10^\circ + \frac{1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3} = \textbf{R.H.S. (Proved)}$$

**11. b (i) Solution :** L.H.S. = $\tan 6^\circ \tan 42^\circ \tan 66^\circ . \tan 78^\circ$

$$= \frac{(2\sin 6^\circ \sin 66^\circ)(2\sin 42^\circ \sin 78^\circ)}{(2\cos 6^\circ \cos 66^\circ)(2\cos 42^\circ.\cos 78^\circ)} = \frac{(\cos 60^\circ - \cos 72^\circ)(\cos 36^\circ - \cos 120^\circ)}{(\cos 72^\circ + \cos 60^\circ)(\cos 120^\circ + \cos 36^\circ)}$$

$$= \frac{\left(\frac{1}{2} - \sin 18^\circ\right)\left(\cos 36^\circ + \frac{1}{2}\right)}{\left(\sin 18^\circ + \frac{1}{2}\right)\left(-\frac{1}{2} + \cos 36^\circ\right)} = \frac{\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)}$$

$$= \frac{\left(\frac{2-\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}+1+2}{4}\right)}{\left(\frac{2+\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1-2}{4}\right)} = \frac{(3-\sqrt{5})(3+\sqrt{5})}{(1+\sqrt{5})(\sqrt{5}-1)} = \frac{9-5}{5-1} = \frac{4}{4} = 1 = \textbf{R.H.S. (Proved)}$$

**11. b (ii) Solution :** L.H.S. = $\sec^2 20^\circ + \sec^2 40^\circ + \sec^2 80^\circ$

$$= \frac{1}{\cos^2 20^\circ} + \frac{1}{\cos^2 40^\circ} + \frac{1}{\cos^2 80^\circ} = \frac{\cos^2 40^\circ \cos^2 80^\circ + \cos^2 80^\circ \cos^2 20^\circ + \cos^2 40^\circ \cos^2 20^\circ}{\cos^2 20^\circ \cos^2 40^\circ \cos^2 80^\circ}$$

Now, $\cos 20^\circ \cos 40^\circ \cos 80^\circ$

$$= \frac{1}{2\sin 20^\circ}(\sin 40^\circ \cos 40^\circ)\cos 80^\circ = \frac{1}{4\sin 20^\circ}(\sin 80^\circ \cos 80^\circ)$$

$$= \frac{1}{8\sin 20^\circ}(\sin 160^\circ) = \frac{\sin(180^\circ - 20^\circ)}{8\sin 20^\circ} = \frac{\sin 20^\circ}{8\sin 20^\circ} = \frac{1}{8} = \textbf{R.H.S. (Proved)}$$

Again, $\cos^2 40^\circ \cos^2 80^\circ + \cos^2 80^\circ \cos^2 20^\circ + \cos^2 40^\circ \cos^2 20^\circ$

$$= \frac{1}{4}\left[(2\cos 40^\circ \cos 80)^2 + (2\cos 80^\circ \cos 20^\circ)^2 + (2\cos 40^\circ \cos 20^\circ)^2\right]$$

$$= \frac{1}{4}\left[(\cos 120^\circ + \cos 40^\circ)^2 + (\cos 100^\circ + \cos 60^\circ)^2 + (\cos 60^\circ + \cos 20^\circ)^2\right]$$

$$= \frac{1}{4}\left[\left(-\frac{1}{2} + \cos 40^\circ\right)^2 + \left(-\cos 80^\circ + \frac{1}{2}\right)^2 + \left(\frac{1}{2} + \cos 20^\circ\right)^2\right]$$

$$= \frac{1}{4}\left[\frac{3}{4} - (\cos 40^\circ + \cos 80^\circ) + \cos 20^\circ + \cos^2 40^\circ + \cos^2 80^\circ + \cos^2 20^\circ\right]$$

$$= \frac{1}{4}\left[\frac{3}{4} - 2.\cos 60^\circ.\cos 20^\circ + \cos 20^\circ + \frac{1}{2}(1 + \cos 80^\circ)\right] + \frac{1}{2}(1 + \cos 160^\circ) + \frac{1}{2}(1 + \cos 40^\circ)]$$

$$= \frac{1}{4}\left\{\frac{3}{4} + \frac{3}{2} - 2.\frac{1}{2}.\cos 20^\circ + \cos 20^\circ + \frac{1}{2}[\cos 80^\circ + \cos 40^\circ - \cos 20^\circ]\right\} \quad [\because \cos 160^\circ = -\cos 20^\circ]$$

$$= \frac{1}{4}\left\{\frac{9}{4} - \cos 20^\circ + \cos 20^\circ + \frac{1}{2}\left[2\cos 60^\circ.\cos 20^\circ - \cos 20^\circ\right]\right\}$$

$$= \frac{1}{4}\left[\frac{9}{4} + \frac{1}{2}(\cos 20^\circ - \cos 20^\circ)\right] = \frac{1}{4}\left[\frac{9}{4} + 0\right] = \frac{9}{16} \quad \therefore \text{ L.H.S.} = \frac{9}{16}\times 64^{4} = 36 = \textbf{R.H.S. (Proved)}$$

**12.** (ii) **Solution :** L.H.S. $4(\cos^3 25^\circ + \cos^3 35^\circ) = 4\cos^3 25^\circ + 4\cos^3 35^\circ$

$= (\cos 75^\circ + 3\cos 25^\circ) + (\cos 105^\circ + 3\cos 35^\circ)$ $[\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta \therefore 4\cos^3\theta = \cos 3\theta + 3\cos\theta]$

$= 3(\cos 25^\circ + \cos 35^\circ) + \cos 75^\circ + \cos 105^\circ = 3(\cos 25^\circ + \cos 35^\circ) + \cos 75^\circ + \cos(180^\circ - 75^\circ)$

$= 3(\cos 25^\circ + \cos 35^\circ) + \cos 75^\circ - \cos 75^\circ = 3(\cos 25^\circ + \cos 35^\circ) =$ **R.H.S. (Proved)**

**13.** (iii) **Solution :** L.H.S. $= \cos^2(A - 120^\circ) + \cos^2 A + \cos^2(A + 120^\circ)$

$= (\cos A.\cos 120^\circ + \sin A.\sin 120^\circ)^2 + \cos^2 A + (\cos A.\cos 120^\circ - \sin A.\sin 120^\circ)^2$

$= 2\{\cos^2 A.\cos^2 120^\circ + \sin^2 A.\sin^2 120^\circ\} + \cos^2 A$ $[\because (a + b)^2 + (a - b)^2 = 2(a^2 + b^2)]$

$$= 2\left\{\cos^2 A\left(-\frac{1}{2}\right)^2 + \sin^2 A.\left(\frac{\sqrt{3}}{2}\right)^2\right\} + \cos^2 A = 2\left(\frac{1}{4}\cos^2 A + \frac{3}{4}\sin^2 A\right) + \cos^2 A$$

$$= \frac{1}{2}\cos^2 A + \frac{3}{2}\sin^2 A + \cos^2 A = \left(\frac{1}{2} + 1\right)\cos^2 A + \frac{3}{2}\sin^2 A = \frac{3}{2}\cos^2 A + \frac{3}{2}\sin^2 A$$

$= \frac{3}{2}(\cos^2 A + \sin^2 A) = \frac{3}{2} =$ **R.H.S. proved.**

**13.** (vi) **Solution :** L.H.S $= \sin^3\alpha + \sin^3(120^\circ + \alpha) + \sin^3(240^\circ + \alpha)$

$= \sin^3\alpha + (\sin 120^\circ \cos\alpha + \cos 120^\circ \sin\alpha)^3 + (\sin 240^\circ \cos\alpha + \cos 240^\circ \sin\alpha)^3$

$$= \sin^3\alpha + \left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)^3 + \left(-\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)^3$$

$$\left[\because \sin 120^\circ = \frac{\sqrt{3}}{2}, \cos 120^\circ = -\frac{1}{2}, \sin 240^\circ = -\frac{\sqrt{3}}{2}, \cos 240^\circ = -\frac{1}{2}\right]$$

$$= \sin^3\alpha + \frac{1}{8}\left\{(\sqrt{3}\cos\alpha - \sin\alpha)^3 - (\sqrt{3}\cos\alpha + \sin\alpha)^3\right\}$$

$$= \sin^3\alpha + \frac{1}{8}\left[-2\left\{3(\sqrt{3}\cos\alpha)^2.\sin\alpha + \sin^3\alpha\right\}\right] \left[\because (a - b)^3 - (a + b)^3 = -2(3a^2b + b^3)\right]$$

$$= \sin^3\alpha - \frac{1}{4}\left\{9\sin\alpha(1 - \sin^2\alpha) + \sin^3\alpha\right\} = \sin^3\alpha - \frac{1}{4}(9\sin\alpha - 9\sin^3\alpha + \sin^3\alpha)$$

$$= \frac{4\sin^3\alpha - 9\sin\alpha + 8\sin^3\alpha}{4} = \frac{1}{4}(12\sin^3\alpha - 9\sin\alpha)$$

$= -\frac{3}{4}(3\sin\alpha - 4\sin^3\alpha) = -\frac{3}{4}\sin 3\alpha =$ **R.H.S. proved.**

**14.** (ii) **Solution :** L.H.S. $= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\left(\pi - \frac{3\pi}{8}\right) + \cos^4\left(\pi - \frac{\pi}{8}\right)$$

$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{\pi}{8} = 2\left\{\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right\}$$

$$= 2\left[\left(\frac{1 + \cos\frac{\pi}{4}}{2}\right)^2 + \left(\frac{1 + \cos\frac{3\pi}{4}}{2}\right)^2\right] = \frac{1}{2}\left[\left(1 + \frac{1}{\sqrt{2}}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2\right]$$

$$\left[\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\right]$$

$= \frac{1}{2}\left[2\left(1 + \frac{1}{2}\right)\right]$ $[\because (a + b)^2 + (a - b)^2 = 2(a^2 + b^2)]$ $= \frac{3}{2} =$ **R. H. S. (Proved)**

**14. (iii) Solution :** L.H.S. $= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

$$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left\{1+\cos\left(\pi-\frac{3\pi}{8}\right)\right\}\left\{1+\cos\left(\pi-\frac{\pi}{8}\right)\right\}$$

$$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right) = \left(1+\cos\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) = \sin^2\frac{\pi}{8}.\sin^2\frac{3\pi}{8}$$

$$= \left(\frac{1-\cos\frac{\pi}{4}}{2}\right)\left(\frac{1-\cos\frac{3\pi}{4}}{2}\right) = \frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{4}\times\frac{1}{2} = \frac{1}{8} = \textbf{R.H.S. proved.}$$

**15. (iv) Solution :** L.H.S.$= \cos\alpha\ \cos2\alpha\ \cos4\alpha\ \cos7\alpha$

$$= \frac{1}{2\sin\alpha}(2\sin\alpha.\cos\alpha).\cos2\alpha.\cos4\alpha.\cos7\alpha$$

$$= \frac{1}{4\sin\alpha}(2\sin2\alpha\cos2\alpha)\cos4\alpha\cos(2\pi-8\alpha)\left[\because\ \alpha=\frac{2\pi}{15},\ \therefore\ 15\alpha=2\pi,\ \text{ or, }\ 7\alpha=2\pi-8\alpha\right]$$

$$= \frac{1}{8\sin\alpha}(2\sin4\alpha\ .\cos4\alpha)\cos8\alpha = \frac{1}{16\sin\alpha}(2\sin8\alpha\cos8\alpha)$$

$$= \frac{1}{16\sin\alpha}.\sin16\alpha = \frac{1}{16\sin\alpha}.\sin(15\alpha+\alpha) = \frac{1}{16\sin\alpha}.\sin(2\pi+\alpha)\left[\because\ \alpha=\frac{2\pi}{15},\ \therefore\ 15\pi=2\pi\right]$$

$$= \frac{1}{16\sin\alpha}.\sin\alpha = \frac{1}{16} = \textbf{R.H.S. (Proved)}$$

**15. (v) Solution :** $2^4\cos\theta.\cos2\theta\ \cos2^2\theta\ \cos2^3\theta$

$$= \frac{2^3}{\sin\theta}.(2\sin\theta.\cos\theta).\cos2\theta.\cos2^2\theta.\cos2^3\theta$$

$$= \frac{2^2}{\sin\theta}.(2\sin2\theta.\cos2\theta)\cos4\theta.\cos8\theta = \frac{2}{\sin\theta}.(2\sin4\theta.\cos4\theta).\cos8\theta$$

$$= \frac{1}{\sin\theta}.(2\sin8\theta.\cos8\theta) = \frac{1}{\sin\theta}.\sin16\theta = \frac{1}{\sin\theta}\sin(17\theta-\theta)$$

$$= \frac{1}{\sin\theta}.\sin(\pi-\theta)\ (\because\ 17\theta=\pi) = \frac{1}{\sin\theta}.\sin\theta = 1\ \textbf{(proved)}.$$

**15. (vii) Solution :** L.H.S. $\cos\theta\ .\ \cos2\theta\ .\ \cos3\theta\ .\ \cos4\theta\ .\ \cos5\theta$

$$= \frac{1}{2\sin\theta}(2\sin\theta.\cos\theta)\cos2\theta.\cos3\theta\ \cos4\theta.\cos5\theta$$

$$= \frac{1}{2.2\sin\theta}(2\sin2\theta.\cos2\theta).\cos4\theta.\cos3\theta.\cos5\theta$$

$$= \frac{1}{2.4\sin\theta}(2\sin4\theta.\cos4\theta).\cos(\pi-8\theta).\cos5\theta\ \left[\because\ \theta=\frac{\pi}{11}\ \therefore\ 11\theta=\pi\Rightarrow3\theta=\pi8\theta\right]$$

$$= \frac{-1}{2.8\sin\theta}.(2\sin8\theta.\cos8\theta)\cos5\theta = \frac{-1}{16\sin\theta}.\sin16\theta\ .\cos5\theta = \frac{-1}{16\sin\theta}\sin(11\theta+5\theta).\cos5\theta$$

$$= \frac{-1}{16\sin\theta}.\sin(\pi+5\theta).\cos5\theta = \frac{1}{2.16\sin\theta}.(2\sin5\theta.\cos5\theta) = \frac{1}{32\sin\theta}.\sin10\theta$$

$$= \frac{1}{32\sin\theta}\sin(11\theta-\theta) = \frac{1}{32\sin\theta}.\sin(\pi-\theta) = \frac{1}{32\sin\theta}(\sin\theta) = \frac{1}{32} = \frac{1}{32}\ \textbf{(proved)}$$

**15. (ix) Solution :** L.H.S. $= \cos\frac{\pi}{15}.\cos\frac{2\pi}{15}.\cos\frac{3\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{5\pi}{15}.\cos\frac{6\pi}{15}.\cos\frac{7\pi}{15}$

$$= \cos\theta\ .\ \cos2\theta\ .\ \cos3\theta\ .\cos4\theta\ .\ \cos5\theta\ .\ \cos6\theta\ .\ \cos7\theta\ \left[\text{where }\ \theta=\frac{\pi}{15}\right]$$

$$= \frac{1}{2\sin\theta}(2\sin\theta.\cos\theta).\cos2\theta\frac{1}{2\sin3\theta}(2\sin3\theta.\cos3\theta).\ \cos4\theta\ \cos5\theta\ .\ \cos6\theta\ .\ \cos7\theta$$

$= \frac{1}{2.2\sin\theta}(2\sin 2\theta\cos 2\theta).\frac{1}{2.2\sin 3\theta}(2\sin 6\theta.\cos 6\theta).\cos 4\theta.\cos 5\theta.\ \cos 7\theta$

$= \frac{1}{2.4\sin\theta}(2\sin 4\theta.\cos 4\theta)\frac{1}{4.\sin 3\theta}.\sin 12\theta.\cos 5\theta.\cos 7\theta$

$= \frac{1}{8\sin\theta}\sin 8\theta.\cos 7\theta.\cos 5\theta.\frac{1}{4\sin 3\theta}.\sin(\pi - 3\theta) = \frac{1}{8\sin\theta}.\sin 8\theta\cos(\pi - 8\theta).\cos 5\theta \times \frac{1}{4\sin 3\theta}.\sin 3\theta$

$= \frac{-1}{2.8.4\sin\theta}.(2\sin 8\theta.\cos 8\theta).\cos 5\theta = \frac{-1}{64\sin\theta}\sin 16\theta.\cos 5\theta = -\frac{1}{64\sin\theta}\sin(\pi + \theta).\cos 5\theta\ \ [\because\ 15\theta = \pi]$

$= \frac{-1}{64\sin\theta}(-\sin\theta)\cos 60^\circ\left[\because\ \theta = \frac{\pi}{15}, 5\theta = \frac{\pi}{3} = 60^\circ\right] = \frac{1}{64}.\frac{1}{2} = (128)^{-1} =$ **R.H.S. (Proved)**

**15.** (x) **Solution :** L.H.S. $= \sin\alpha\ \sin 3\alpha\ \sin 5\alpha = \sin\left(\frac{\pi}{2} - 6\alpha\right).\sin\left(\frac{\pi}{2} - 4\alpha\right)\sin\left(\frac{\pi}{2} - 2\alpha\right) = \cos 6\alpha\ .\ \cos 4\alpha\ \cos 2\alpha$

$\left[\because \frac{\pi}{14} = \alpha,\ \therefore \frac{\pi}{2} = 7\alpha \Rightarrow \frac{\pi}{2} - 6\alpha = \alpha, \frac{\pi}{2} - 4\alpha = 3\alpha, \frac{\pi}{2} - 2\alpha = 5\alpha\right]$

$= \frac{1}{2\sin 2\alpha}(2\sin 2\alpha\ .\cos 2\alpha).\cos 4\alpha\ .\ \cos 6\alpha = \frac{1}{4\sin 2\alpha}(2\sin 4\alpha\cos 4\alpha).\cos 6\alpha$

$= \frac{1}{4\sin 2\alpha}.\sin 8\alpha\ .\cos(\pi - 8\alpha) = -\frac{1}{8\sin 2\alpha}(2\sin 8\alpha\cos 8\alpha) = -\frac{1}{8\sin 2\alpha}.(\sin 16\alpha)$

$= -\frac{1}{8\sin 2\alpha}\sin(14\alpha + 2\alpha) = -\frac{1}{8\sin 2\alpha}\{\sin(\pi + 2\alpha)\} = -\frac{1}{8\sin 2\alpha}.(-\sin 2\alpha) = \frac{1}{8}$ **Ans.**

**16. (i) Solution :** Given, $A + B = 45^o$ or, $\tan(A + B) = \tan 45^o$ or, $\frac{\tan A + \tan B}{1 - \tan A.\tan B} = 1$

or, $\tan A + \tan B = 1 - \tan A\ .\ \tan B$ or, $\tan A + \tan B + \tan A\ \tan B = 1$

or, $(1 + \tan A) + \tan B\ (1 + \tan A) = 2$ or, $(1 + \tan A)(1 + \tan B) = 2$ **(Proved)** ——(1)

Now, putting $B = A$, we get, $A + A = 45^o$ or, $2A = 45^o \Rightarrow A = 22\frac{1}{2}^\circ$

And from (1) we get, $(1 + \tan A)(1 + \tan A) = 2$ or, $(1 + \tan A)^2 = 2$

or, $1 + \tan A = \sqrt{2}$ or, $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$ **(Proved)**

**16. (iii) Solution :** $A + B = 225^o$ or, $\cot(A + B) = \cot 225^o$ or, $\cot(A + B) = \cot(90^o \times 2 + 45^o) = \cot 45^o$

or, $\frac{\cot A\ .\cot B - 1}{\cot B + \cot A} = 1$ or, $\cot B + \cot A = \cot A\ .\ \cot B - 1$ or, $\cot B + \cot A + 1 = \cot A\ .\ \cot B$

or, $1 + \cot A + \cot B + \cot A\ .\ \cot B = 2\cot A\ .\ \cot B$ or, $(1 + \cot A) + \cot B(1 + \cot A) = 2\cot A\ .\ \cot B$

or, $(1 + \cot A)(1 + \cot B) = 2\cot A\ .\ \cot B$

$\therefore \frac{\cot A\ .\cot B}{(1 + \cot A)(1 + \cot B)} = \frac{1}{2}$ or, $\frac{\cot A}{1 + \cot A}.\frac{\cot B}{1 + \cot B} = \frac{1}{2}$ **proved.**

**17. (i) b. Solution :** Given, $A + B + C = \pi\ \therefore\ B + C = \pi - A$

or, $\cos(B + C) = \cos(\pi - A)$ or, $\cos B\cos C - \sin B\sin C = -\cos A$

or, $\cos B\cos C = \sin B\sin C - \cos B\cos C\ \ [\because\ \cos A = \cos B\cos C]$

or, $2\cos B\cos C = \sin B\sin C$ or, $\frac{\cos B\cos C}{\sin B\sin C} = \frac{1}{2}$ or, $\cot B\ .\cot C = \frac{1}{2}$ **proved.**

**17. (i) c. Solution :** $\tan A = \frac{\sin A}{\cos A} = \frac{\sin\{\pi - (B + C)\}}{\cos A}[\because\ A + B + C = \pi\ \therefore\ A = \pi - (B + C)]$

$= \frac{\sin(B + C)}{\cos A} = \frac{\sin B\cos C + \cos B\sin C}{\cos B\cos C} = \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C}$

$\therefore \tan A = \tan B + \tan C$ **(Proved)**

**17. (iii) Solution :** Given, $A + B + C = \pi$ or, $A + B = \pi - C$ or, $\cot(A + B) = \cot(\pi - C)$

or, $\frac{\cot A \cdot \cot B - 1}{\cot B + \cot A} = -\cot C$ or, $\cot A \cot B - 1 = -\cot B \cot C - \cot C \cdot \cot A$

or, $\cot B \cdot \cot C + \cot C \cdot \cot A + \cot A \cdot \cot B = 1$ **(Proved)**

**18. (ii) Solution :** Given, $x = \tan\theta + \tan\varphi$, $y = \cot\theta + \cot\varphi$

$\therefore \frac{1}{x} - \frac{1}{y} = \frac{1}{\tan\theta + \tan\varphi} - \frac{1}{\cot\theta + \cot\varphi} = \frac{1}{\frac{1}{\cot\theta} + \frac{1}{\cot\varphi}} - \frac{1}{\cot\theta + \cot\varphi}$

$= \frac{\cot\theta \cot\varphi}{\cot\theta + \cot\varphi} - \frac{1}{\cot\theta + \cot\varphi} = \frac{\cot\theta \cot\varphi - 1}{\cot\varphi + \cot\theta} = \cot(\theta + \varphi) =$ **R.H.S. (Proved)**

**19. (i) Solution :** $\cos(x - y) = -1$

or, $\cos x \cdot \cos y + \sin x \sin y = -1$ or, $2\cos x \cdot \cos y + 2\sin x \sin y + 2 = 0$

or, $\cos^2 x + \sin^2 x + \cos^2 y + \sin^2 y + 2\cos x \cos y + 2\sin x \sin y = 0$

or, $\cos^2 x + \cos^2 y + 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y = 0$

or, $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 0$ $\therefore \cos x + \cos y = 0, \sin x + \sin y = 0$

$\therefore \cos x + \cos y = \sin x + \sin y = 0$ **(Proved)**

**19. (ii) Solution :** $\cos(A - B) + \cos(B - C) + \cos(C - A) = -\frac{3}{2}$

or, $\cos A \cos B + \sin A \sin B + \cos B \cos C + \sin B \sin C + \cos C \cos A + \sin C \sin A = -\frac{3}{2}$

or, $2\cos A \cos B + 2\cos B \operatorname{coc} C + 2\cos C \cos A + 2\sin A \sin B + 2\sin B \sin C + 2\sin C \cdot \sin A + 3 = 0$

or, $\cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B + \cos^2 C + \sin^2 C + 2(\cos A \cdot \cos B$

$+ \cos B \cos C + \cos C \cdot \cos A) + 2(\sin A \cdot \sin B + \sin B \cdot \sin C + \sin C \sin A) = 0$

or, $\cos^2 A + \cos^2 B + \cos^2 C + 2(\cos A \cos B + \cos B \cdot \cos C + \cos C \cdot \cos A)$

$+ \sin^2 A + \sin^2 B + \sin^2 C + 2(\sin A \sin B + \sin B \cdot \sin C + \sin C \cdot \sin A) = 0$

or, $(\cos A + \cos B + \cos C)^2 + (\sin A + \sin B + \sin C)^2 = 0$

$\therefore \cos A + \cos B + \cos C = 0 = \sin A + \sin B + \sin C$ **(Proved)**

**20. (iv) Solution :** Given, $\tan^2\theta = 1 + 2\tan^2\varphi$ or, $1 + \tan^2\theta = 2 + 2\tan^2\varphi$

or, $\sec^2\theta = 2(1 + \tan^2\varphi) = 2\sec^2\varphi$ or, $\frac{1}{\cos^2\theta} = \frac{2}{\cos^2\varphi}$ or, $\cos^2\varphi = 2\cos^2\theta$ **(Proved) (iii)**

or, $2\cos^2\varphi = 2(2\cos^2\theta)$ or, $1 + \cos 2\varphi = 2(1 + \cos 2\theta)$

or, $1 + \cos 2\varphi = 2 + 2\cos 2\theta$ or, $\cos 2\varphi = 1 + 2\cos 2\theta$ **(Proved)**

**21. (ii) Solution :** L. H. S. $= \cos\alpha - \cos\beta + \cos\gamma - \cos(\alpha + \beta + \gamma)$

$= 2\sin\frac{\alpha + \beta}{2} \cdot \sin\frac{\beta - \alpha}{2} + 2\sin\frac{\alpha + \beta + 2\gamma}{2} \cdot \sin\frac{\alpha + \beta + \gamma - \gamma}{2}$ $\left[\because \cos C - \cos D = 2\sin\frac{C + D}{2} \cdot \sin\frac{D - C}{2}\right]$

$= 2\sin\frac{\alpha + \beta}{2}\sin\frac{\beta - \alpha}{2} + 2\sin\frac{\alpha + \beta + 2\gamma}{2} \cdot \sin\frac{\alpha + \beta}{2} = 2\sin\frac{\alpha + \beta}{2}\left\{\sin\frac{\alpha + \beta + 2\gamma}{2} + \sin\frac{\beta - \alpha}{2}\right\}$

$$= 2\sin\frac{\alpha+\beta}{2}\,.\,2\sin\left(\frac{\frac{\alpha+\beta+2\gamma}{2}+\frac{\beta-\alpha}{2}}{2}\right).\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\beta-\alpha}{2}}{2}\right)$$

$$= 2\sin\frac{\alpha+\beta}{2}\,.\,2\,.\sin\frac{\beta+\gamma}{2}\,.\cos\frac{\alpha+\gamma}{2} = 4\sin\frac{\alpha+\beta}{2}\,.\sin\frac{\beta+\gamma}{2}\,.\cos\frac{\gamma+\alpha}{2}$$ = **R. H. S. (Proved)**

**22.** (iii) **Solution :** $\sin\theta + \sin\varphi = a$ or, $2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right) = a$ or, $4\sin^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right) = a^2$ -----(1)

$\cos\theta + \cos\varphi = b$ or, $2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right) = b$ or, $4\cos^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right) = b^2$ -------(2)

Adding (1) and (2) we get, $4\sin^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right) + 4\cos^2\left(\frac{\theta+\varphi}{2}\right)\cos^2\left(\frac{\theta-\varphi}{2}\right) = a^2 + b^2$

or, $4\cos^2\left(\frac{\theta-\varphi}{2}\right)\left[\sin^2\left(\frac{\theta+\varphi}{2}\right) + \cos^2\left(\frac{\theta+\varphi}{2}\right)\right] = a^2 + b^2$ or, $4\cos^2\left(\frac{\theta-\varphi}{2}\right) = a^2 + b^2$ [$\because \sin^2x + \cos^2x = 1$]

or, $\cos^2\left(\frac{\theta-\varphi}{2}\right) = \frac{a^2+b^2}{4}$ or, $\sec^2\left(\frac{\theta-\varphi}{2}\right) = \frac{4}{a^2+b^2}$ $\therefore$ $\tan^2\left(\frac{\theta-\varphi}{2}\right) = \frac{4}{a^2+b^2} - 1$ [$\because \tan^2\theta = \sec^2\theta - 1$]

or, $\tan^2\left(\frac{\theta-\varphi}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$ $\therefore$ $\tan\left(\frac{\theta-\varphi}{2}\right) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$ **(Proved).**

**23.** (ii) **Solution :** cosecA + secA = coseeB + secB or, cosecA – cosecB = secB – secA

or, $\frac{1}{\sin A} - \frac{1}{\sin B} = \frac{1}{\cos B} - \frac{1}{\cos A}$ or, $\frac{\sin B - \sin A}{\sin A \sin B} = \frac{\cos A - \cos B}{\cos A \cos B}$

or, $\frac{\sin A\,.\sin B}{\cos A\,.\cos B} = \frac{\sin B - \sin A}{\cos A - \cos B}$ or, $\tan A.\tan B = \frac{\cancel{-}(\sin A - \sin B)}{\cancel{-}(\cos B - \cos A)} = \frac{2\sin\left(\frac{A-B}{2}\right).\cos\frac{A+B}{2}}{2\sin\frac{B+A}{2}\,.\sin\left(\frac{A-B}{2}\right)}$

or, $\tan A\,.\tan B = \cot\frac{1}{2}(A+B)$ **(proved).**

**24.** (i) **Solution :** Given, cos(A + B) . sin(C + D) = cos(A – B) . sin(C – D)

or, $\frac{\sin(C+D)}{\sin(C-D)} = \frac{\cos(A-B)}{\cos(A+B)}$ or, $\frac{\sin(C+D)+\sin(C-D)}{\sin(C+D)-\sin(C-D)} = \frac{\cos(A-B)+\cos(A+B)}{\cos(A-B)-\cos(A+B)}$

[by componendo and dividendo.]

or, $\frac{2\sin\frac{C+D+C-D}{2}\,.\cos\frac{C+D-C+D}{2}}{2\sin\frac{C+D-C+D}{2}\,.\cos\frac{C+D+C-D}{2}} = \frac{2\cos\frac{A-B+A+B}{2}\,.\cos\frac{A-B-A-B}{2}}{2\sin\frac{A-B+A+B}{2}\,.\sin\frac{A+B-A+B}{2}}$

or, $\frac{\sin C\,.\cos D}{\sin D\,.\cos C} = \frac{\cos A\,.\cos B}{\sin A\,.\sin B}$ or, $\frac{\cos A}{\sin A}\,.\,\frac{\cos B}{\sin B}\,.\,\frac{\cos C}{\sin C} = \frac{\cos D}{\sin D}$

or, cotA . cotB . cotC = cotD **(Proved)**

**24.** (iii) **Solution :** Given, $\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)} = \frac{\tan\gamma}{\tan\beta}$ or, $\frac{\frac{\sin(\alpha+\beta-\gamma)}{\cos(\alpha+\beta-\gamma)}}{\frac{\sin(\alpha-\beta+\gamma)}{\cos(\alpha-\beta+\gamma)}} = \frac{\frac{\sin\gamma}{\cos\gamma}}{\frac{\sin\beta}{\cos\beta}}$

or, $\frac{\sin(\alpha+\beta-\gamma)\,.\cos(\alpha-\beta+\gamma)}{\cos(\alpha+\beta-\gamma)\,.\sin(\alpha-\beta+\gamma)} = \frac{\sin\gamma.\cos\beta}{\cos\gamma.\sin\beta}$

By comp. and div. we get,

$$\frac{\sin(\alpha+\beta-\gamma).\cos(\alpha-\beta+\gamma)+\cos(\alpha+\beta-\gamma).\sin(\alpha-\beta+\gamma)}{\sin(\alpha+\beta-\gamma).\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma).\sin(\alpha-\beta+\gamma)}=\frac{\sin\gamma.\cos\beta+\cos\gamma.\sin\beta}{\sin\gamma.\cos\beta-\cos\gamma\sin\beta}$$

or, $\dfrac{\sin(\alpha+\beta-\gamma+\alpha-\beta+\gamma)}{\sin(\alpha+\beta-\gamma-\alpha+\beta-\gamma)}=\dfrac{\sin(\beta+\gamma)}{-\sin(\beta-\gamma)}$ or, $\dfrac{\sin 2\alpha}{\sin 2(\beta-\gamma)}=\dfrac{\sin(\beta+\gamma)}{-\sin(\beta-\gamma)}$

or, $\sin2(\beta-\gamma) . \sin(\beta+\gamma) = -\sin2\alpha . \sin(\beta-\gamma)$

or, $2\sin(\beta-\gamma) . \cos(\beta-\gamma) . \sin(\beta+\gamma) + \sin2\alpha . \sin(\beta-\gamma) = 0$

or, $\sin(\beta-\gamma)\{2\sin(\beta+\gamma) . \cos(\beta-\gamma) + \sin2\alpha\} = 0$

or, $\sin(\beta-\gamma)\{\sin(\beta+\gamma+\beta-\gamma) + \sin(\beta+\gamma-\beta+\gamma) + \sin2\alpha\} = 0$

or, $\sin(\beta-\gamma)(\sin2\beta + \sin2\gamma + \sin2\alpha) = 0$ or, $\sin(\beta-\gamma)(\sin2\alpha + \sin2\beta + \sin2\gamma) = 0$

$\therefore$ either, $\sin(\beta-\gamma) = 0$ or, $\sin2\alpha + \sin2\beta + \sin2\gamma = 0$ **(Proved)**

**25. (i) Solution :** $a\tan\theta + b\sec\theta = c$ or, $a\tan\theta - c = -b\sec\theta$

$(a\tan\theta - c)^2 = (-b\sec\theta)^2$ or, $a^2\tan^2\theta - 2ac\tan\theta + c^2 = b^2\sec^2\theta$

or, $a^2\tan^2\theta - 2ac\tan\theta + c^2 = b^2(1+\tan^2\theta) = b^2 + b^2\tan^2\theta$ or, $a^2\tan^2\theta - b^2\tan^2\theta - 2ac\tan\theta + c^2 - b^2 = 0$

or, $(a^2-b^2)\tan^2\theta - 2ac\tan\theta + (c^2-b^2) = 0$. This is a quadratic equation of $\tan\theta$.

So, by question $\tan\alpha$ and $\tan\beta$ are the roots of this equation.

$\therefore\ \tan\alpha + \tan\beta = \dfrac{2ac}{a^2-b^2}$, $\tan\alpha . \tan\beta = \dfrac{c^2-b^2}{a^2-b^2}$

$$\therefore\ \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha . \tan\beta} = \frac{\frac{2ac}{a^2-b^2}}{1-\frac{c^2-b^2}{a^2-b^2}} = \frac{2ac}{a^2-c^2}$$

$\therefore\ \tan(\alpha+\beta) = \dfrac{2ac}{a^2-c^2}$ **Ans.**

**26. (i) Solution :** Let A and B be two parts of the angle $\alpha$.

Then from the given condition, $\dfrac{\tan A}{\tan B} = \lambda$ ----------(1) and $\left.\begin{matrix}A+B=\alpha\\A-B=x\end{matrix}\right\}$ ----------(2)

Now from (1) we get, $\dfrac{\frac{\sin A}{\cos A}}{\frac{\sin B}{\cos B}} = \lambda$ or, $\dfrac{\sin A . \cos B}{\cos A . \sin B} = \dfrac{\lambda}{1}$

By componendo and dividendo we get, $\dfrac{\sin A . \cos B + \cos A\sin B}{\sin A\cos B - \cos A\sin B} = \dfrac{\lambda+1}{\lambda-1}$

or, $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{\lambda+1}{\lambda-1}$ or, $\dfrac{\sin\alpha}{\sin x} = \dfrac{\lambda+1}{\lambda-1}$ $\therefore\ \sin x = \dfrac{\lambda-1}{\lambda+1} . \sin\alpha$ **(proved).**

**26. (iv) Solution :** $\sin\theta = n\sin(\theta+2\alpha)$

or, $\dfrac{\sin\theta}{\sin(\theta+2\alpha)} = \dfrac{n}{1}$ or, $\dfrac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta} = \dfrac{1+n}{1-n}$ or, $\dfrac{2\sin\left(\frac{\theta+2\alpha+\theta}{2}\right).\cos\left(\frac{\theta+2\alpha-\theta}{2}\right)}{2\sin\left(\frac{\theta+2\alpha-\theta}{2}\right)\cos\left(\frac{\theta+2\alpha+\theta}{2}\right)} = \dfrac{1+n}{1-n}$

or, $\dfrac{\sin(\theta+\alpha) . \cos\alpha}{\cos(\theta+\alpha) . \sin\alpha} = \dfrac{1+n}{1-n}$ or, $\dfrac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)} = \dfrac{1+n}{1-n} . \dfrac{\sin\alpha}{\cos\alpha}$

or, $\tan(\theta+\alpha) = \dfrac{1+n}{1-n} . \tan\alpha$ **(proved).**

**27.** (i) **Solution :** 2 tanα = 3 tanβ or, $\tan\alpha = \frac{3}{2}\tan\beta$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\cdot\tan\beta} = \frac{\frac{3}{2}\tan\beta - \tan\beta}{1+\frac{3}{2}\tan\beta\cdot\tan\beta} = \frac{3\tan\beta - 2\tan\beta}{2+3\tan^2\beta} = \frac{\tan\beta}{2+3\tan^2\beta} = \frac{\frac{\sin\beta}{\cos\beta}}{2+\frac{3\sin^2\beta}{\cos^2\beta}} = \frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta}$$

$$= \frac{2\sin\beta\cos\beta}{2\cdot(2\cos^2\beta)+3\cdot(2\sin^2\beta)} = \frac{\sin 2\beta}{2\cdot(1+\cos 2\beta)+3\cdot(1-\cos 2\beta)} = \frac{\sin 2\beta}{2+2\cos 2\beta+3-3\cos 2\beta} = \frac{\sin 2\beta}{5-\cos 2\beta}$$ **(Proved).**

**27.** (ii) **Solution :** Here, $\frac{\sin 2\beta}{5-\cos 2\beta} = \frac{\frac{2\tan\beta}{1+\tan^2\beta}}{5-\frac{1-\tan^2\beta}{1+\tan^2\beta}} = \frac{2\tan\beta}{5+5\tan^2\beta-1+\tan^2\beta} = \frac{2\tan\beta}{4+6\tan^2\beta}$ or, $\frac{\sin 2\beta}{5-\cos 2\beta} = \frac{\tan\beta}{2+3\tan^2\beta}$

$\therefore \tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}$ gives, $\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta} = \frac{\tan\beta}{2+3\tan^2\beta}$

or, $2\tan\alpha - 2\tan\beta + 3\tan\alpha\tan^2\beta - 3\tan^3\beta = \tan\beta + \tan\alpha\tan^2\beta$

or, $2\tan\alpha - 3\tan\beta + 2\tan\alpha\tan^2\beta - 3\tan^3\beta = 0$ or, $(2\tan\alpha - 3\tan\beta) + \tan^2\beta(2\tan\alpha - 3\tan\beta) = 0$

or, $(2\tan\alpha - 3\tan\beta)(1+\tan^2\beta) = 0$

$\Rightarrow 2\tan\alpha - 3\tan\beta = 0$ [$\because$ $1+\tan^2\beta = 0$ gives, $\tan\beta = \pm\sqrt{-1}$ which is not real]

$\Rightarrow \frac{\tan\alpha}{\tan\beta} = \frac{3}{2}$ or, tanα : tanβ = 3 : 2 **(Proved)**

**27.** (iii) **Solution :** $3\tan(\alpha-\beta) = 3\left(\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\ .\tan\beta}\right)$

$$= 3\left(\frac{\tan\alpha - \frac{\sin\alpha\ .\cos\alpha}{2+\cos^2\alpha}}{1+\tan\alpha\ .\ \frac{\sin\alpha\ .\ \cos\alpha}{2+\cos^2\alpha}}\right) = 3\ .\ \frac{\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\alpha\ .\cos\alpha}{2+\cos^2\alpha}}{1+\frac{\sin\alpha}{\cos\alpha}\ .\ \frac{\sin\alpha\ \cos\alpha}{2+\cos^2\alpha}} = 3.\frac{2\sin\alpha+\sin\alpha.\cos^2\alpha-\sin\alpha.\cos^2\alpha}{\cos\alpha(2+\cos^2\alpha)+\sin^2\alpha.\cos\alpha}$$

$$= 3\ .\ \frac{2\sin\alpha}{2\cos\alpha+\cos^3\alpha+(1-\cos^2\alpha).\cos\alpha} = \frac{6\sin\alpha}{2\cos\alpha+\cos^3\alpha+\cos\alpha-\cos^3\alpha} = \frac{6\sin\alpha}{2\cos\alpha+\cos\alpha} = \frac{{}^{2}6\sin\alpha}{3\cos\alpha}$$

$\therefore$ 3 tan (α − β ) = 2 tanα **(Proved)**

**27.** (iv) **Solution :** Given, $\tan\beta = \frac{n\sin\alpha\ .\ \cos\alpha}{1-n\sin^2\alpha}$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\ .\tan\beta} = \frac{\tan\alpha - \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\tan\alpha.\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}$$

$$= \frac{\tan\alpha(1-n\sin^2\alpha)-n\sin\alpha.\cos\alpha}{1-n\sin^2\alpha+n.\tan\alpha.\sin\alpha.\cos\alpha} = \frac{\tan\alpha - n\tan\alpha\ .\sin^2\alpha - n\frac{\sin\alpha}{\cos\alpha}\ .\cos^2\alpha}{1-n\sin^2\alpha+n\frac{\sin\alpha}{\cos\alpha}\ .\sin\alpha\ .\cos\alpha}$$

$$= \frac{\tan\alpha - n\tan\alpha\sin^2\alpha - n\tan\alpha\ .\cos^2\alpha}{1-n\sin^2\alpha+n\sin^2\alpha} = \tan\alpha - n\tan\alpha(\sin^2\alpha+\cos^2\alpha)$$

or, tan (α − β) = tanα − n tanα or, tan (α − β) = (1 − n) tanα **(Proved)**

**27. (vi) Solution :** Given, $2\tan\beta + \cot\beta = \tan\alpha \Rightarrow \tan\alpha = 2\tan\beta + \cot\beta$

$$\therefore\ 2\tan(\alpha-\beta) = 2\cdot\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\ \tan\beta} = 2\cdot\frac{2\tan\beta+\cot\beta-\tan\beta}{1+(2\tan\beta+\cot\beta)\tan\beta}$$

$$= 2\cdot\frac{\tan\beta+\cot\beta}{1+1+2\tan^2\beta} = 2\,\frac{\dfrac{\sin^2\beta+\cos^2\beta}{\sin\beta.\cos\beta}}{2(1+\tan^2\beta)} = \frac{2}{\sin\beta\cos\beta}\times\frac{\cos^2\beta}{2} = \cot\beta \quad \therefore \cot\beta = 2\tan(\alpha-\beta)\ \textbf{(Proved)}$$

**27. (vii) Solution :** $\tan\theta = \cos2\alpha\ .\ \tan\varphi$.

$$\therefore\ \tan(\varphi-\theta) = \frac{\tan\varphi - \tan\theta}{1+\tan\varphi\ .\ \tan\theta} = \frac{\tan\varphi - \cos2\alpha\ .\tan\varphi}{1+\tan^2\varphi\ .\cos2\alpha} = \frac{\tan\varphi(1-\cos2\alpha)}{1+\tan^2\varphi\ .\cos2\alpha} = \frac{\dfrac{\sin\varphi}{\cos\varphi}\cdot\left(1-\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)}{1+\dfrac{\sin^2\varphi}{\cos^2\varphi}\cdot\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}}$$

$$= \frac{2\tan^2\alpha\ .\sin\varphi\cos\varphi}{(1+\tan^2\alpha)\ .\cos^2\varphi+(1-\tan^2\alpha)\sin^2\varphi} = \frac{\tan^2\alpha\ .\sin2\varphi}{\cos^2\varphi+\sin^2\varphi+\tan^2\alpha(\cos^2\varphi-\sin^2\varphi)}$$

$$\therefore\ \tan(\varphi-\theta) = \frac{\tan^2\alpha\ .\sin2\varphi}{1+\tan^2\alpha\ .\cos2\varphi}\ \textbf{ proved.}$$

**27. (viii) Solution :** $\cot\theta = \cos(x+y)$, $\cot\varphi = \cos(x-y)$,

$$\therefore\ \tan(\theta-\varphi) = \frac{\tan\theta-\tan\varphi}{1+\tan\theta\ .\tan\varphi} = \frac{\cot\varphi-\cot\theta}{\cot\theta\ .\cot\varphi+1}$$

$$= \frac{\cos(x-y)-\cos(x+y)}{\cos(x+y)\ .\cos(x-y)+1} = \frac{2\sin x\sin y}{\cos^2x-\sin^2y+1} = \frac{2\sin x\ .\sin y}{\cos^2x+\cos^2y}$$

**27. (ix) Solution :** Given, $\sin\theta = k\sin(\theta+\varphi)$ or, $\sin\{(\theta+\varphi)-\varphi\} = k\sin(\theta+\varphi)$

or, $\sin(\theta+\varphi)\cos\varphi - \cos(\theta+\varphi)\sin\varphi = k\sin(\theta+\varphi)$ or, $\sin(\theta+\varphi)\cos\varphi - k\sin(\theta+\varphi) = \cos(\theta+\varphi)\sin\varphi$

or, $\sin(\theta+\varphi)\{\cos\varphi - k\} = \cos(\theta+\varphi)\ .\ \sin\varphi$

or, $\dfrac{\sin(\theta+\varphi)}{\cos(\theta+\varphi)} = \dfrac{\sin\varphi}{\cos\varphi-k}$ or, $\tan(\theta+\varphi) = \dfrac{\sin\varphi}{\cos\varphi-k}$ **proved.**

**28. (i) Solution :** $\dfrac{\cos2A}{1} = \dfrac{3\cos2B-1}{3-\cos2B}$

By comp. and div. we get, $\dfrac{1-\cos2A}{1+\cos2A} = \dfrac{3-\cos2B-3\cos2B+1}{3-\cos2B+3\cos2B-1}$ or, $\dfrac{1-\cos2A}{1+\cos2A} = \dfrac{4-4\cos2B}{2+2\cos2B}$

or, $\dfrac{1-\cos2A}{1+\cos2A} = \dfrac{4(1-\cos2B)}{2(1+\cos2B)}$ or, $\dfrac{2\sin^2A}{2\cos^2A} = \dfrac{2\ .\ 2\sin^2B}{2\cos^2B}$ or, $\tan^2A = 2\tan^2B$ $\therefore \tan A = \sqrt{2}\tan B$ **(proved).**

**28. (iii) Solution :** $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1-e}{1-e}}\ .\ \tan\dfrac{\varphi}{2} \Rightarrow \tan^2\dfrac{\theta}{2} = \dfrac{1-e}{1+e}\ .\ \tan^2\dfrac{\varphi}{2}$

or, $\tan^2\dfrac{\varphi}{2} = \dfrac{1+e}{1-e}\ .\ \tan^2\dfrac{\theta}{2} = \dfrac{1+e}{1-e}\cdot\dfrac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \dfrac{(1+e)\ .\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2}}$

$$\text{Now, } \cos\varphi = \frac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}} = \frac{1-\dfrac{(1+e)\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2}}}{1+\dfrac{(1+e)\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2}}} = \frac{\dfrac{(1-e)\cos^2\frac{\theta}{2}-(1+e)\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2}}}{\dfrac{(1-e)\cos^2\frac{\theta}{2}+(1+e)\ .\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2}}}$$

$$= \frac{\cos^2\frac{\theta}{2} - e\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} - e.\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - e\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + e\sin^2\frac{\theta}{2}}$$

$$= \frac{\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) - e\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) - e\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)} = \frac{\cos\theta - e.1}{1 - e.\cos\theta} \quad \left[\because \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \text{ and } \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta\right]$$

$\therefore \cos\varphi = \dfrac{\cos\theta - e}{1 - e\cos\theta}$ **(proved).**

**29. (ii) Solution :** Given, $\tan\theta = \dfrac{\sin\alpha\sin\beta}{\cos\alpha + \cos\beta}$

or, $\dfrac{\sin\theta}{\sin\alpha\sin\beta} = \dfrac{\cos\theta}{\cos\alpha + \cos B} = \dfrac{\sqrt{\sin^2\theta + \cos^2\theta}}{\sqrt{\sin^2\alpha\sin^2\beta + (\cos\alpha + \cos\beta)^2}}$

$$= \frac{1}{\sqrt{(1-\cos^2\alpha)(1-\cos^2\beta) + (\cos\alpha + \cos\beta)^2}}$$

$$= \frac{1}{\sqrt{1 - \cos^2\alpha - \cos^2\beta + \cos^2\alpha\cos^2\beta + \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta}}$$

$$= \frac{1}{\sqrt{1 + 2\cos\alpha\cos\beta + \cos^2\alpha\cos^2\beta}} = \frac{1}{\sqrt{(1+\cos\alpha\cos\beta)^2}} = \frac{1}{1+\cos\alpha\cos\beta}$$

$\therefore \cos\theta = \dfrac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta} \quad \therefore \dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1 + \cos\alpha\cos\beta - \cos\alpha - \cos\beta}{1 + \cos\alpha\cos\beta + \cos\alpha + \cos\beta}$ [by componendo and dividendo]

or, $\dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)} = \dfrac{1-\cos\alpha}{1+\cos\alpha} \cdot \dfrac{1-\cos\beta}{1+\cos\beta}$

$\therefore \tan^2\frac{\theta}{2} = \tan^2\frac{\alpha}{2} \cdot \tan^2\frac{\beta}{2}$ or, $\quad \left[\because \dfrac{1-\cos x}{1+\cos x} = \tan^2\frac{x}{2}\right]$

$\therefore \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2}$

$\therefore$ one value of $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2}$ **proved.**

**30. (i) Solution :** $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$

or, $-(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = -1$ or, $\cos(\alpha+\beta) = 1$

or, $\cos^2(\alpha+\beta) = 1^2$ or, $1 - \sin^2(\alpha+\beta) = 1$ or, $-\sin^2(\alpha+\beta) = 0$

or, $\sin^2(\alpha+\beta) = 0$ or, $\sin(\alpha+\beta) = 0$

Now, $1 + \cot\alpha \cdot \tan\beta = 1 + \dfrac{\cos\alpha}{\sin\alpha} \cdot \dfrac{\sin\beta}{\cos\beta} = 1 + \dfrac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$

$= \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta} = \dfrac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta} = \dfrac{0}{\sin\alpha.\cos\beta} = 0$ **(proved).**

**30. (ii) Solution :** Given, $\sin\alpha + \sin\beta = \sqrt{3}(\cos\beta - \cos\alpha)$

or, $\dfrac{\sin\alpha + \sin\beta}{\cos\beta - \cos\alpha} = \sqrt{3}$ or, $\dfrac{2\sin\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2}} = \sqrt{3}$

or, $\cot\left(\dfrac{\alpha-\beta}{2}\right) = \sqrt{3}$ or, $\cot\dfrac{\alpha-\beta}{2} = \cot\dfrac{\pi}{6}$ or, $\dfrac{\alpha-\beta}{2} = \dfrac{\pi}{6}$ or, $\alpha - \beta = \dfrac{\pi}{3}$ or, $3\alpha - 3\beta = \pi$ or, $3\alpha = \pi + 3\beta$

$\therefore \sin3\alpha = \sin(\pi + 3\beta) = -\sin3\beta \; \therefore \sin3\alpha + \sin3\beta = 0$ **(Proved)**

**30. (iii) Solution :** $\dfrac{\cos\theta}{\cos\varphi} = \dfrac{a}{b}$ or, $a = b\,.\,\dfrac{\cos\theta}{\cos\varphi}$

Now, $(a+b)\tan\dfrac{\theta+\varphi}{2} = \left(b\,.\,\dfrac{\cos\theta}{\cos\varphi} + b\right).\dfrac{\sin\dfrac{\theta+\varphi}{2}}{\cos\dfrac{\theta+\varphi}{2}} = b\left(\dfrac{\cos\theta}{\cos\varphi}+1\right).\dfrac{\sin\dfrac{\theta+\varphi}{2}}{\cos\dfrac{\theta+\varphi}{2}}$

$= b\,.\left(\dfrac{\cos\theta+\cos\varphi}{\cos\varphi}\right).\dfrac{\sin\dfrac{\theta+\varphi}{2}}{\cos\dfrac{\theta+\varphi}{2}} = b\dfrac{2\cos\dfrac{\theta+\varphi}{2}\,.\cos\dfrac{\theta-\varphi}{2}}{\cos\varphi}.\dfrac{\sin\dfrac{\theta+\varphi}{2}}{\cos\dfrac{\theta+\varphi}{2}}$

$= b\,.\dfrac{2\sin\dfrac{\theta+\varphi}{2}\,.\cos\dfrac{\theta-\varphi}{2}}{\cos\varphi} = b\,.\dfrac{\sin\left(\dfrac{\theta+\varphi}{2}+\dfrac{\theta-\varphi}{2}\right)+\sin\left(\dfrac{\theta+\varphi}{2}-\dfrac{\theta-\varphi}{2}\right)}{\cos\varphi}$

$= b\,.\dfrac{\sin\theta+\sin\varphi}{\cos\varphi} = b\dfrac{\sin\theta}{\cos\varphi}+b\dfrac{\sin\varphi}{\cos\varphi} = a\,.\dfrac{\cos\varphi}{\cos\theta}.\dfrac{\sin\theta}{\cos\varphi}+b\tan\varphi = a\tan\theta + b\tan\varphi =$ **R.H.S. (Proved)**

**30. (iv) Solution :** $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{2\,.\sec 2\alpha}{1+\sec^2 2\alpha} = \dfrac{2\cos 2\alpha}{\cos^2 2\alpha+1} = \dfrac{2\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}}{\dfrac{(1-\tan^2\alpha)^2}{(1+\tan^2\alpha)^2}+1}$

$= \dfrac{2(1+\tan^2\alpha)(1-\tan^2\alpha)}{(1+\tan^2\alpha)^2+(1-\tan^2\alpha)^2} = \dfrac{2(1-\tan^4\alpha)}{2(1+\tan^4\alpha)}$

$\therefore\ \sin 2\theta = \dfrac{1-\tan^4\alpha}{1+\tan^4\alpha}$ **proved.**

**30. (v) Solution :** L.H.S. $= 2\,\text{cosec}4\theta - \sec 2\theta = \dfrac{2}{\sin 4\theta}-\dfrac{1}{\cos 2\theta} = \dfrac{2}{2\sin 2\theta.\cos 2\theta}-\dfrac{1}{\cos 2\theta} = \dfrac{1-\sin 2\theta}{\sin 2\theta.\cos 2\theta}$

$= \dfrac{\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta}{\cos 2\theta}.\dfrac{1}{\sin 2\theta} = \dfrac{(\cos\theta-\sin\theta)^2}{\cos^2\theta-\sin^2\theta}.\text{cosec}2\theta$

$= \dfrac{(\cos\theta-\sin\theta)^2}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}.\text{cosec}2\theta = \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}.\text{cosec}2\theta$

$= \dfrac{\dfrac{\cos\theta-\sin\theta}{\cos\theta}}{\dfrac{\cos\theta+\sin\theta}{\cos\theta}}.\text{cosec}2\theta = \dfrac{1-\tan\theta}{1+\tan\theta}.\text{cosec}2\theta =$ **R.H.S. (Proved)**

**30. (vii) Solution :** Given, $\sqrt{2}\cos A = \cos B + \cos^3 B$ ------------ (1)

$\sqrt{2}\sin A = \sin B - \sin^3 B$ ------------ (2)

$\therefore\ \sqrt{2}.\sin(A-B) = \sqrt{2}(\sin A\cos B - \cos A\sin B) = \cos B(\sin B-\sin^3 B) - \sin B(\cos B+\cos^3 B)$

$= \cos B\sin B - \cos B.\sin^3 B - \sin B\cos B - \sin B.\cos^3 B = -\cos B\sin B(\sin^2 B+\cos^2 B)$

$\sqrt{2}\sin(A-B) = -\cos B\sin B \Rightarrow \sin^2(A-B) = \frac{1}{2}.\cos^2 B.\sin^2 B$ ---------- (3)

Squaring (1) and (2) and adding we get,

$2(\cos^2 A+\sin^2 A) = (\cos B+\cos^3 B)^2 + (\sin B-\sin^3 B)^2$

or, $2 = \cos^2 B + \sin^2 B + \cos^6 B + \sin^6 B + 2\cos^4 B - 2\sin^4 B$

or, $2 = 1 + (\cos^2 B+\sin^2 B)^3 - 3\cos^2 B\sin^2 B.(\cos^2 B+\sin^2 B) + 2(\cos^2 B+\sin^2 B).(\cos^2 B-\sin^2 B)$

or, $1 = 1 - 3\cos^2 B \sin^2 B + 2(\cos^2 B - \sin^2 B)$ or, $2(2\cos^2 B - 1) - 3\cos^2 B(1 - \cos^2 B) = 0$

or, $4\cos^2 B - 2 - 3\cos^2 B + 3\cos^4 B = 0$ or, $3\cos^4 B + \cos^2 B - 2 = 0$

or, $3\cos^4 B + 3\cos^2 B - 2\cos^2 B - 2 = 0$ or, $3\cos^2 B(\cos^2 B + 1) - 2(\cos^2 B + 1) = 0$

or, $(\cos^2 B + 1)(3\cos^2 B - 2) = 0 \Rightarrow 3\cos^2 B - 2 = 0$ [ $\because \cos^2 B = -1 \Rightarrow \cos B = \pm\sqrt{-1}$, which is not real. ]

or, $\cos^2 B = \frac{2}{3}$ $\therefore \sin^2 B = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ from (3) we get, $\sin^2(A - B) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{9}$ $\therefore \sin(A - B) = \pm\frac{1}{3}$ **proved.**

**30. (viii) Solution :** Given, $\cos(\alpha + \beta) = \frac{4}{5}$

We know, $\sin^2(\alpha + \beta) = 1 - \cos^2(\alpha + \beta) = 1 - \left(\frac{4}{5}\right)^2 = \frac{25 - 16}{25} = \frac{9}{25} = \left(\frac{3}{5}\right)^2$

$\therefore \sin(\alpha + \beta) = \frac{3}{5}$

$$\therefore \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\frac{3}{5}}{\frac{4}{5}} \quad \therefore \tan(\alpha + \beta) = \frac{3}{4}$$

Again, $\sin(\alpha - \beta) = \frac{5}{13}$

But, $\cos^2(\alpha - \beta) = 1 - \sin^2(\alpha - \beta) = 1 - \frac{25}{169} = \frac{169 - 25}{169} = \frac{144}{169}$

$\therefore \cos(\alpha - \beta) = \frac{12}{13}$

$$\therefore \tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\frac{5}{13}}{\frac{12}{13}} \quad \therefore \tan(\alpha - \beta) = \frac{5}{12}$$

Now, $2\alpha = (\alpha + \beta) + (\alpha - \beta)$ or, $\tan 2\alpha = \tan\{(\alpha + \beta) + (\alpha - \beta)\}$

$$\text{or, } \tan 2\alpha = \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta) \cdot \tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{\frac{36 + 20}{48}}{\frac{48 - 15}{48}} = \frac{56}{33}$$

$\therefore \tan 2\alpha = \frac{56}{33}$ **Ans.**

**30. (x) Solution :** $\frac{\tan(\alpha - \beta)}{\tan\alpha} + \frac{\sin^2\gamma}{\sin^2\alpha} = 1$ or, $\frac{\sin^2\gamma}{\sin^2\alpha} = 1 - \frac{\tan(\alpha - \beta)}{\tan\alpha} = 1 - \frac{\sin(\alpha - \beta) \cdot \cos\alpha}{\cos(\alpha - \beta) \cdot \sin\alpha}$

$$= \frac{\sin\alpha \cdot \cos(\alpha - \beta) - \cos\alpha \cdot \sin(\alpha - \beta)}{\sin\alpha \cdot \cos(\alpha - \beta)} = \frac{\sin(\alpha - \alpha + \beta)}{\sin\alpha \cdot \cos(\alpha - \beta)}$$

$$\text{or, } \frac{\sin^2\gamma}{\sin^2\alpha} = \frac{\sin\beta}{\sin\alpha \cdot \cos(\alpha - \beta)} \quad \therefore \sin^2\gamma = \frac{\sin\beta \cdot \sin\alpha}{\cos(\alpha - \beta)} \quad \text{--------- (1)}$$

$$\therefore \cos^2\gamma = 1 - \sin^2\gamma = 1 - \frac{\sin\beta \sin\alpha}{\cos(\alpha - \beta)} = \frac{\cos(\alpha - \beta) - \sin\alpha \sin\beta}{\cos(\alpha - \beta)}$$

$$= \frac{\cos\alpha \cdot \cos\beta + \sin\alpha \sin\beta - \sin\alpha \sin\beta}{\cos(\alpha - \beta)} \quad \therefore \cos^2\gamma = \frac{\cos\alpha \cos\beta}{\cos(\alpha - \beta)} \quad \text{-------- (2)}$$

From (1) and (2) we get,

$$\frac{\sin^2\gamma}{\cos^2\gamma} = \frac{\sin\alpha \cdot \sin\beta}{\cos(\alpha - \beta)} \times \frac{\cos(\alpha - \beta)}{\cos\alpha \cdot \cos\beta} = \frac{\sin\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta}$$

$\therefore \tan^2\gamma = \tan\alpha \cdot \tan\beta$ $\quad\therefore \tan\alpha \cdot \tan\beta = \tan^2\gamma$ **(Proved)**

**30. (xi) Solution :** Given, $\tan\theta = n(\sec\theta - 1)^2$ or, $\frac{\sin\theta}{\cos\theta} = n\left(\frac{1}{\cos\theta} - 1\right)^2$

or, $\frac{\sin\theta}{\cos\theta} = n\frac{(1-\cos\theta)^2}{\cos^2\theta}$ or, $\sin\theta\,.\cos\theta = n\left(2\sin^2\frac{\theta}{2}\right)^2$ or, $2\sin\frac{\theta}{2}\,.\cos\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = 4n\sin^4\frac{\theta}{2}$

or, $\cos^3\frac{\theta}{2} - \cos\frac{\theta}{2}\sin^2\frac{\theta}{2} = 2n\sin^3\frac{\theta}{2}$ or, $\frac{\cos^3\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} - \frac{\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} = 2n$ or, $\cot^3\frac{\theta}{2} - \cot\frac{\theta}{2} = 2n$ **proved.**

**30. (xii) Solution :** $\tan\left(\frac{\theta}{2} + \frac{\varphi}{2}\right) = \frac{\tan\frac{\theta}{2} + \tan\frac{\varphi}{2}}{1 - \tan\frac{\theta}{2}\,.\tan\frac{\varphi}{2}} = \frac{\tan^3\frac{\varphi}{2} + \tan\frac{\varphi}{2}}{1 - \tan^3\frac{\varphi}{2}\,.\tan\frac{\varphi}{2}}$ $\left[\because \tan\frac{\theta}{2} = \tan^3\frac{\varphi}{2}\right]$

$$= \frac{\tan\frac{\varphi}{2}\left(\tan^2\frac{\varphi}{2} + 1\right)}{1 - \tan^4\frac{\varphi}{2}} = \frac{\tan\frac{\varphi}{2}\left(1 + \tan^2\frac{\varphi}{2}\right)}{\left(1 + \tan^2\frac{\varphi}{2}\right)\left(1 - \tan^2\frac{\varphi}{2}\right)} = \frac{\tan\frac{\varphi}{2}}{1 - \tan^2\frac{\varphi}{2}}$$

$$= \frac{\frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}}{1 - \frac{\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}} = \frac{\frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}}{\frac{\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}} = \frac{2\sin\frac{\varphi}{2}\,.\cos\frac{\varphi}{2}}{2\cos\varphi} = \frac{\sin\varphi}{2\cos\varphi} = \frac{\tan\varphi}{2} = \frac{2\tan\alpha}{2} \quad [\because \tan\varphi = 2\tan\alpha]$$

$\therefore \tan\left(\frac{\theta}{2} + \frac{\varphi}{2}\right) = \tan\alpha \quad \therefore \frac{\theta}{2} + \frac{\varphi}{2} = \alpha \Rightarrow \theta + \varphi = 2\alpha$ **(Proved)**

**30. (xiv) Solution :** $\tan 7\frac{1}{2}^\circ = \frac{\sin\frac{15^\circ}{2}}{\cos\frac{15^\circ}{2}} = \frac{2\sin^2\frac{15^\circ}{2}}{2\sin\frac{15^\circ}{2}\,.\cos\frac{15^\circ}{2}} = \frac{1-\cos 15^\circ}{\sin 15^\circ}$

$$= \frac{1 - \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} \quad \left[\begin{array}{l}\because \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} \\ \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}\end{array}\right] = \frac{\frac{2\sqrt{2} - \sqrt{3} - 1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{(\sqrt{3}+1)(2\sqrt{2} - \sqrt{3} - 1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{2\sqrt{6} + 2\sqrt{2} - 3 - \sqrt{3} - \sqrt{3} - 1}{3 - 1}$$

$= \frac{2\sqrt{6} + 2\sqrt{2} - 2\sqrt{3} - 4}{2} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$ **(Proved)**

**30. (xvi) Solution :** We have, $\cot 3\theta = \cot(2\theta + \theta) = \frac{\cot 2\theta\cot\theta - 1}{\cot\theta + \cot 2\theta}$

or, $\cot 3\theta\,.\cot\theta + \cot 3\theta\,.\cot 2\theta = \cot 2\theta\cot\theta - 1$

or, $\cot\theta\cot 2\theta + \cot 2\theta\cot 3\theta + 1 = 2\cot\theta\cot 2\theta - \cot\theta\cot 3\theta$ [adding $\cot\theta\,.\cot 2\theta$ on both sides.]

or, $\cot\theta\cot 2\theta + \cot 2\theta\cot 3\theta + 2 = 1 + 2\cot\theta\cot 2\theta - \cot\theta\cot 3\theta$ ------- (1)

Again, $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$ or, $2\cot\theta\,.\cot 2\theta = \cot^2\theta - 1$ ------- (2)

$\therefore$ (1) $\Rightarrow \cot\theta\,.\cot 2\theta + \cot 2\theta\,.\cot 3\theta + 2 = 1 + \cot^2\theta - 1 - \cot\theta.\cot 3\theta = \cot\theta(\cot\theta - \cot 3\theta)$

[putting the value of $2\cot\theta\,.\cot 2\theta$ from (2)] **(Proved)**

**30.** (xvii) **Solution :** L.H.S. = $\tan\theta + 2\tan2\theta + 4\tan4\theta + 8\cot8\theta$

$= \tan\theta + 2\tan2\theta + 4\left(\tan4\theta + 2 \cdot \frac{\cot^2 4\theta - 1}{2\cot4\theta}\right) = \tan\theta + 2\tan2\theta + 4\{\tan4\theta + (\cot^2 4\theta - 1)\tan4\theta\}$

$= \tan\theta + 2\tan2\theta + 4\tan4\theta\,(1 + \cot^2 4\theta - 1) = \tan\theta + 2\tan2\theta + 4\cot4\theta$

$= \tan\theta + 2\left(\tan2\theta + 2 \cdot \frac{\cot^2 2\theta - 1}{2\cot2\theta}\right) = \tan\theta + 2\{\tan2\theta + (\cot^2 2\theta - 1)\tan2\theta\}$

$= \tan\theta + 2\tan2\theta\,(1 + \cot^2 2\theta - 1) = \tan\theta + 2\cot2\theta = \tan\theta + 2 \cdot \frac{\cot^2\theta - 1}{2\cot\theta} = \tan\theta + (\cot^2\theta - 1)\tan\theta$

$= \tan\theta\,(1 + \cot^2\theta - 1) = \tan\theta \cdot \cot^2\theta = \cot\theta$ = **R.H.S. (Proved)**

**30.** (xviii) **Solution :** Let $\theta = 7\frac{1}{2}^\circ = \frac{15^\circ}{2}$ $\therefore 2\theta = 15^\circ$ or, $4\theta = 30^\circ$

$\therefore \cos4\theta = \cos30^\circ$ or, $2\cos^2 2\theta - 1 = \frac{\sqrt{3}}{2}$ or, $2\cos^2 2\theta = 1 + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$

$\therefore \cos^2 2\theta = \frac{2+\sqrt{3}}{4} = \frac{4+2\sqrt{3}}{8} = \frac{(\sqrt{3})^2 + 2 \cdot \sqrt{3} \cdot 1 + 1^2}{8} = \frac{(\sqrt{3}+1)^2}{8}$

$\therefore \cos2\theta = \frac{\sqrt{3}+1}{2\sqrt{2}}$ or, $2\cos^2\theta - 1 = \frac{\sqrt{3}+1}{2\sqrt{2}}$ or, $2\cos^2\theta = \frac{2\sqrt{2}+\sqrt{3}+1}{2\sqrt{2}}$

$\therefore \cos^2\theta = \frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}} = \frac{4+\sqrt{6}+\sqrt{2}}{8}$

$\therefore \cos\theta = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$ $\therefore \cos7\frac{1}{2}^\circ = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$ **(Ans)**

## PROBLEM SET – II

[Problems with '*' marks are solved at the end of the problem set]

**31.** (i) If $x = r\cos\theta \cdot \cos\varphi$, $y = r\cos\theta \cdot \sin\varphi$, $z = r\sin\theta$ proved that, $x^2 + y^2 + z^2 = r^2$

*(ii) If $x = a\sec\theta \cdot \cos\varphi$, $y = b\sec\theta\sin\varphi$, $z = c\tan\theta$, show that, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(iii) If $\sin A + \cos A = m$, $\sec A + \operatorname{cosec} A = n$, show that, $n(m^2 - 1) = 2m$

(iv) If $\sin\theta + \cos\theta = a$ and $\tan\theta + \cot\theta = b$, proved that, $b(a^2 - 1) = 2$

*(v) If $\tan A + \sin A = m$, $\tan A - \sin A = n$, proved that, $m^2 - n^2 = 4\sqrt{mn}$ **[WBSC – 84]**

(vi) If $2y\cos\theta = x\sin\theta$, $2x\sec\theta - y\operatorname{cosec}\theta = 3$, show that, $\frac{x^2}{4} + \frac{y^2}{1} = 1$

*(vii) If $x\sin\theta - y\cos\theta = \sqrt{x^2+y^2}$ and $\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{x^2+y^2}$ show that $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

(viii) If $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha \cdot \cos\alpha$ and $x\sin\alpha - y\cos\alpha = 0$, prove that, $x^2 + y^2 = 1$

(ix) If $\tan\theta - \cot\theta = a$ and $\cos\theta - \sin\theta = b$ show that, $(a^2 + 4)(b^2 - 1)^2 = 4$

*(x) If $x = \operatorname{cosec}\theta - \sin\theta$, $y = \sec\theta - \cos\theta$, show that, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (xy)^{-\frac{2}{3}}$

*(xi) If $a\sin x = b\cos x = \frac{2c\tan x}{1-\tan^2 x}$ prove that, $(a^2 - b^2)^2 = 4c^2(a^2 + b^2)$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**31.** (ii) **Solution :** Given, $x = a\sec\theta\cos\varphi$ , $y = b\sec\theta\sin\varphi$, $z = c\tan\theta$

$$\therefore \frac{x^2}{a^2} = \sec^2\theta\,.\cos^2\varphi,\ \frac{y^2}{b^2} = \sec^2\theta\,.\sin^2\varphi,\ \frac{z^2}{c^2} = \tan^2\theta$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = \sec^2\theta\cos^2\varphi+\sec^2\theta\sin^2\varphi-\tan^2\theta = \sec^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)-\tan^2\theta = \sec^2\theta-\tan^2\theta = 1$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$$ **(Proved)**

**31.** (v) **Solution :** Given, $\tan A + \sin A = m$, $\tan A - \sin A = n$ $\therefore m^2 - n^2 = (\tan A + \sin A)^2 - (\tan A - \sin A)^2 = 4\tan A$ $\sin A$

$$\text{Again, } 4\sqrt{mn} = 4\sqrt{(\tan A+\sin A)(\tan A-\sin A)} = 4\sqrt{\tan^2 A-\sin^2 A} = 4\sqrt{\frac{\sin^2 A}{\cos^2 A}-\sin^2 A}$$

$$= 4\sqrt{\left(\sec^2 A-1\right)\sin^2 A} = 4\sqrt{\tan^2 A\sin^2 A} = 4\tan A\,.\sin A \quad \therefore m^2-n^2 = 4\sqrt{mn}$$ proved.

**31.** (vii) **Solution :** Given, $x\sin\theta - y\cos\theta = \sqrt{x^2+y^2}$ or, $x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta = x^2 + y^2$ [squaring both sides]

or, $x^2(1-\sin^2\theta) + y^2(1-\cos^2\theta) + 2xy\sin\theta\cos\theta = 0$ or, $x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta = 0$

or, $(x\cos\theta + y\sin\theta)^2 = 0$ or, $x\cos\theta + y\sin\theta = 0$ or, $x\cos\theta = -y\sin\theta$

$$\text{or, } \frac{\cos\theta}{-y} = \frac{\sin\theta}{x} = \frac{\sqrt{\cos^2\theta+\sin^2\theta}}{\sqrt{y^2+x^2}} = \frac{1}{\sqrt{x^2+y^2}} \quad \therefore \cos\theta = -\frac{y}{\sqrt{x^2+y^2}},\ \sin\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2} = \frac{1}{x^2+y^2} \text{ gives, } \frac{y^2}{a^2\left(x^2+y^2\right)}+\frac{x^2}{b^2\left(x^2+y^2\right)} = \frac{1}{x^2+y^2} \quad \therefore \frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$$ **(Proved)**

**31.** (x) **Solution :** Given, $x = \operatorname{cosec}\theta - \sin\theta$, $y = \sec\theta - \cos\theta$

$$\therefore x = \frac{1}{\sin\theta}-\sin\theta = \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} \text{ and } y = \frac{1}{\cos\theta}-\cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$$\therefore xy = \frac{\cos^2\theta}{\sin\theta}\times\frac{\sin^2\theta}{\cos\theta} = \sin\theta\cos\theta \qquad \text{--------- (1)}$$

$$\text{Again, } x^{\frac{2}{3}}+y^{\frac{2}{3}} = \frac{\cos^{\frac{4}{3}}\theta}{\sin^{\frac{2}{3}}\theta}+\frac{\sin^{\frac{4}{3}}\theta}{\cos^{\frac{2}{3}}\theta} = \frac{\cos^2\theta+\sin^2\theta}{\sin^{\frac{2}{3}}\theta\,.\cos^{\frac{2}{3}}\theta} = \frac{1}{(\sin\theta\cos\theta)^{\frac{2}{3}}} = (xy)^{-\frac{2}{3}} \quad [\text{from} - (1)]$$

$$\therefore x^{\frac{2}{3}}+y^{\frac{2}{3}} = (xy)^{-\frac{2}{3}}$$ **(Proved)**

**31.** (xi) **Solution :** Given, $a\sin x = b\cos x = \dfrac{2c\tan x}{1-\tan^2 x}$ or, $a\sin x = b\cos x$

$$\text{or, } \frac{\sin x}{b} = \frac{\cos x}{a} = \frac{\sqrt{\sin^2 x+\cos^2 x}}{\sqrt{b^2+a^2}} = \frac{1}{\sqrt{a^2+b^2}} \quad \therefore \sin x = \frac{b}{\sqrt{a^2+b^2}},\ \cos x = \frac{a}{\sqrt{a^2+b^2}} \text{ and, } \tan x = \frac{b}{a}$$

$\therefore\ b\cos x = \dfrac{2c\tan x}{1-\tan^2 x} \Rightarrow \dfrac{ba}{\sqrt{a^2+b^2}} = \dfrac{2c\cdot\frac{b}{a}}{1-\frac{b^2}{a^2}}$ or, $\dfrac{ab}{\sqrt{a^2+b^2}} = \dfrac{2abc}{a^2-b^2}$ or, $a^2-b^2 = 2c\sqrt{a^2+b^2}$

or, $\left(a^2-b^2\right)^2 = 4c^2(a^2+b^2)$ **(Proved)**

## PROBLEM SET – III

[Problems with '*' marks are solved at the end of the problem set]

**32.** (a) Find the minimum value of

(i) $\sin^2\theta + \text{cosec}^2\theta$ *(ii) $16\cos^2\theta + 9\sec^2\theta$

(iii) $9\tan^2\theta + 4\cot^2\theta$ [Ans : (1) 2 (ii) 24 (iii) 12

(b) Find the maximum and minimum value of

(i) $\cos^6 x + \sin^6 x$ *(ii) $\sin^2 x + \cos^4 x$ $\left[\text{Ans.: (i) } 1, \frac{1}{4}\ \text{(ii) } 1, \frac{3}{4}\right]$

**33.** Show that the expression $(3+\cos\theta)\,\text{cosec}\theta$ can have no value between $-2\sqrt{2}$ and $2\sqrt{2}$. **[WBSC – 93]**

**34.** Find the range of the values of

(i) $A = 6\cos^2\theta + 8\sin\theta.\cos\theta$ *(ii) $a = (3\cos x + 4\sin x)\sin x$

$\left[\text{Ans: (i) } -2 \le A \le 8\quad \text{(ii) } -\frac{1}{2} \le A \le \frac{9}{2}\right]$

**35.** Prove that,

(i) $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$ (ii) $-\sqrt{2} \le \cos\theta + \sin\theta \le \sqrt{2}$ **[WBSC – 06]**

*(iii) $-4 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{6}\right) + 3 \le 10$ (iv) $-\sqrt{19} \le 3\cos x + 5\sin\left(x-\frac{\pi}{6}\right) \le \sqrt{19}$

(v) $0 \le 3\cos\theta + 4\sin\theta + 5 \le 10$ *(vi) $-1 \le 5\cos\theta + 12\sin\theta + 12 \le 25$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**32. a (ii) Solution :** Let, $A = 16\cos^2\theta + 9\sec^2\theta = (4\cos\theta - 3\sec\theta)^2 + 2\,.\,4\cos\theta\,.\,3\sec\theta = (4\cos\theta - 3\sec\theta)^2 + 24$

$\because (4\cos\theta - 3\sec\theta)^2$ is a perfect square, its minimum value is 0.

Hence minimum value of the given expression is 24. **(Ans)**

**32. b (ii) Solution :** Let $A = \sin^2 x + \cos^4 x = \frac{1}{2}\left(2\sin^2 x\right) + \frac{1}{4}\left(2\cos^2 x\right)^2 = \frac{1}{2}(1-\cos 2x) + \frac{1}{4}(1+\cos 2x)^2$

$= \frac{1}{2} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) = \frac{1}{2} - \frac{1}{2}\cos 2x + \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}\left(2\cos^2 2x\right)$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8}(1+\cos 4x) = \frac{7}{8} + \frac{1}{8}\cos 4x$

Now, $\because -1 \le \cos 4x \le 1 \therefore -\frac{1}{8} \le \frac{1}{8}\cos 4x \le \frac{1}{8}$ or, $\frac{7}{8} - \frac{1}{8} \le \frac{7}{8} + \frac{1}{8}\cos 4x \le \frac{7}{8} + \frac{1}{8} \Rightarrow \frac{3}{4} \le A \le 1$

Hence maximum and minimum value of the given expression are respectively 1 and $\frac{3}{4}$. **(Ans)**

**33. Solution :** Let, $(3+\cos\theta)\operatorname{cosec}\theta = k$ or, $3+\cos\theta = k\sin\theta$

or, $3+\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = k\cdot\dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ or, $3+\dfrac{1-t^2}{1+t^2} = \dfrac{2kt}{1+t^2}$ $\left[\text{Assuming } t=\tan\dfrac{\theta}{2}\right]$

or, $3+3t^2+1-t^2-2kt=0$ or, $2t^2-2kt+4=0$ or, $t^2-kt+2=0$

$\because$ t is real, discriminant $(B^2-4AC)\geq 0$

$\therefore (-k)^2-4.1.2\geq 0$ or, $k^2\geq 8=(2\sqrt{2})^2 \Rightarrow k\geq 2\sqrt{2},\ k\leq -2\sqrt{2}$

Thus the given expression cannot have any value between $-2\sqrt{2}$ and $2\sqrt{2}$. (proved).

**34.** (ii) **Solution :** Given $A=(3\cos x+4\sin x)\sin x$

$=\frac{3}{2}(2\sin x\cos x)+2(2\sin^2 x) = \frac{3}{2}\sin 2x + 2(1-\cos 2x)$

$=2+\frac{3}{2}\sin 2x - 2\cos 2x = 2+r(\sin 2x\cos\theta - \cos 2x\,.\sin\theta)$

$\left[\text{Let } r\cos\theta=\frac{3}{2},\ r\sin\theta=2 \quad \therefore r=\sqrt{\left(\frac{3}{2}\right)^2+2^2}=\sqrt{\frac{9}{4}+4}=\sqrt{\frac{16+9}{4}}=\frac{5}{2}\right]$

$=2+r\sin(2x-\theta) = 2+\frac{5}{2}\sin(2x-\theta)$

Now, $-1\leq\sin(2x-\theta)\leq 1$ or, $\frac{-5}{2}\leq\frac{5}{2}\sin(2x-\theta)\leq\frac{5}{2}$

or, $2-\frac{5}{2}\leq 2+\frac{5}{2}\sin(2x-\theta)\leq 2+\frac{5}{2}$ or, $-\frac{1}{2}\leq A\leq\frac{9}{2}$ proved.

**35.** (iii) **Solution :** Let $A=5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3 = 5\cos\theta+3\left(\cos\theta\,.\cos\frac{\pi}{3}-\sin\theta\,.\sin\frac{\pi}{3}\right)+3$

$=5\cos\theta+3\left(\cos\theta\,.\frac{1}{2}-\sin\theta\,.\frac{\sqrt{3}}{2}\right)+3 = 5\cos\theta+\frac{3}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$

$=\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3 = r\cos\theta\,.\cos\alpha - r\sin\theta\sin\alpha+3$ where, $r\cos\alpha=\frac{13}{2},\ r\sin\theta=\frac{3\sqrt{3}}{2}$

$\therefore r=\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}=\sqrt{\frac{169+27}{4}}=\sqrt{\frac{196}{4}}=7$

$\therefore A=7\cos(\theta+\alpha)+3$

But we know, $-1\leq\cos(\theta+\alpha)\leq 1$ or, $-7\leq 7\cos(\theta+\alpha)\leq 7$

or, $-7+3\leq 7\cos(\theta+\alpha)+3\leq 7+3$

or, $-4\leq A\leq 10$ **(Proved)**

**35.** (vi) **Solution :** Let $A=5\cos\theta+12\sin\theta+12 = r(\cos\theta\cos\alpha+\sin\theta\sin\alpha)+12$ [where, $r\cos\alpha=5$, $r\sin\alpha=12$

$\therefore A = 13\cos(\theta-\alpha)+12$

Now, $-1\leq\cos(\theta-\alpha)\leq 1$ or, $-13\leq 13\cos(\theta-\alpha)\leq 13$

or, $-13+12\leq 13\cos(\theta-\alpha)+12\leq 13+12$ or, $-1\leq A\leq 25$

or, $-1\leq 5\cos\theta+12\sin\theta+12\leq 25$ **(Proved)**

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# Trigonometrical Equations and General Values

**3.1** A relation involving one or more trigonometrical functions which is satisfied by a definite set of values (finite or infinite) of the associated angle(s) is called a **Trigonometrical Equation**.

## General values of some standard Trigonometrical Equations :

**1.** (i) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha$

(ii) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$ (iii) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$

**2.** (i) If $\sin\theta = 0$, then $\theta = n\pi$ (ii) If $\cos\theta = 0$, then $\theta = (2n + 1)\frac{\pi}{2}$

(iii) If $\tan\theta = 0$ then $\theta = n\pi$

**3.** (i) If $\sin\theta = 1$, then $\theta = (4n + 1)\frac{\pi}{2}$ (ii) If $\cos\theta = 1$, then $\theta = 2n\pi$

**4.** (i) If $\sin\theta = -1$, then $\theta = (4n - 1)\frac{\pi}{2}$ (ii) If $\cos\theta = -1$, then $\theta = (2n + 1)\pi$, where n = 0 or any integer.

## 3.2 To find the general solution of the equation of the form :

$a\cos\theta + b\sin\theta = c$ [a, b, c are constant and $|c| \le \sqrt{a^2 + b^2}$ ]

**Solution :** $a\cos\theta + b\sin\theta = c$ or, $r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = c$ ....... (1) where,

$a = r\cos\alpha, b = r\sin\alpha$ and $r > 0$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ $\therefore$ $r = \sqrt{a^2 + b^2}$ ; $\alpha = \tan^{-1}\frac{b}{a}$

From (2) we get, $r\cos(\theta - \alpha) = c$ or, $\cos(\theta - \alpha) = \frac{c}{r} = \cos\beta$ (say)

or, $\theta - \alpha = 2n\pi \pm \beta$ or, $\theta = 2n\pi \pm \beta + \alpha$, n = 0 or any integer and $\cos\beta = \frac{c}{r} = \frac{c}{\sqrt{a^2 + b^2}}$

Clearly equation (1) is solvable when, $|\cos\beta| \le 1$ i.e., if $\left|\frac{c}{\sqrt{a^2 + b^2}}\right| \le 1 \Rightarrow |c| \le \sqrt{a^2 + b^2}$

If $|c| > \sqrt{a^2 + b^2}$ then equation (1) cannot be solved.

# PROBLEM SET

[Problems with '*' marks are solved at the end of the problem set]

**1.** (i) Find the general values of x which satisfy the equation $\sin^2 x = \frac{3}{4}$

*(ii) Find the values of θ which satisfy $\tan^2\theta = \frac{1}{3}$, $-\pi \le \theta \le \pi$

**2.** Find the general solution :

(i) $\sin4\theta = \sin\theta$ *(ii) $\sin2\theta = \cos\theta$

(iii) $\tan p\theta = \cot q\theta$ (iv) $4\sin4\theta + 1 = \sqrt{5}$

*(v) $\cos2\theta - \cos4\theta = 0$ $0^\circ < \theta < 360^\circ$

**3.** Solve for θ :

*(i) $7\cos^2\theta + 3\sin^2\theta = 4$ $[0 \le \theta \le 2\pi]$ **[WBSC – 87]**

(ii) $2\sin^2\theta + 3\cos\theta = 0$

(iii) $3\sin^2\theta - 8\sin\theta + 5 = 0$, $0 \le \theta \le 2\pi$ *(iv) $2\cos^2\theta - \sin\theta + 1 = 0$, $0 \le \theta \le 2\pi$

**4.** Solve :

*(i) $\cos3x + \cos2x + \cos x = 0$ (ii) $\sin2x + \sin4x + \sin6x = 0$

(iii) $\sin5x - \sin3x - \sin x = 0$, $0^\circ < x < 360^\circ$ *(iv) $\cos6\theta + \cos4\theta + \cos2\theta + 1 = 0$

(v) $\sin\theta + \sin5\theta = \sin3\theta$, $0 \le \theta \le \pi$ **[WBSC – 08]**

**5.** Solve :

*(i) $\cot x - \tan x = 2$ (ii) $\tan x + \cot x = 2$

**6.** Solve :

*(i) $\tan x + \tan2x + \tan x \cdot \tan2x = 1$ (ii) $\tan\theta + \tan2\theta + \sqrt{3} \cdot \tan\theta \cdot \tan2\theta = \sqrt{3}$

*(iii) $\tan\theta + \tan2\theta + \tan3\theta = \tan\theta\tan2\theta\tan3\theta$ **[WBSC – 91]**

**7.** Solve :

*(i) $\sin3\theta \cdot \cos^3\theta + \cos3\theta \cdot \sin^3\theta = \frac{3}{4}$ (ii) $\cos3x \cdot \cos^3x + \sin3x \cdot \sin^3x = 0$

**8.** Solve for θ :

(i) $\sqrt{3}\cos\theta + \sin\theta = 1$, $-2\pi < x < 2\pi$ *(ii) $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}}$, $-\pi < \theta < \pi$ **[WBSC – 88]**

*(iii) $\cos\theta + \sqrt{3}\sin\theta = \sqrt{2}$, $0^\circ < \theta < 360^\circ$ *(iv) $\cos\theta + \sqrt{3}\sin\theta = 1$, $0^\circ < \theta < 360^\circ$ **[WBSC – 82]**

*(v) $\sin\theta - \sqrt{3}\cos\theta = 1$, $0 \le \theta \le 2\pi$ **[WBSC – 04]** *(vi) $\sqrt{3}\cos x + \sin x = 1$, $0 < x < 2\pi$ **[WBSC – 05]**

*(vii) $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ where θ lies between 0 to 2π. **[WBSC – 07]**

**9.** Sove for x :

*(i) $\frac{\sin\alpha}{\sin2x} + \frac{\cos\alpha}{\cos2x} = 2$ (ii) $\frac{\cos\alpha}{\cos x} + \frac{\sin\alpha}{\sin x} = 2$

**10.** Solve for θ :

*(i) $3\tan(\theta - 15^\circ) = \tan(\theta + 15^\circ)$ *(ii) $\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=4$

**11.** Solve :

(i) $\operatorname{cosec}^2\theta + \cot^2\theta = 3\cot\theta$ *(ii) $\tan^2\theta + \cot^2\theta = 2$

**12.** Solve :

(i) $4\sin\theta\cos\theta = 1 + 2\cos\theta - 2\sin\theta,\ 0 < \theta < \pi$ *(ii) $1 + 2\sin\theta\cos\theta - 2\sin\theta - \cos\theta = 0,\ 0^0 \le \theta \le 360^\circ$

(iii) $1 - 2\sin\theta - 2\cos\theta + \cot\theta = 0,\ 0 < \theta < 2\pi$ (iv) $\sin2\theta \,.\, \tan\theta + 1 = \sin2\theta + \tan\theta$

*(v) $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$ *(vi) $2(\sin x - \cos2x) - \sin2x(1 + 2\sin x) + 2\cos x = 0$

**[WBSC – 98]**

**13.** Solve :

*(i) $2^{\sin x + \cos y} = 1,\ 16^{\sin^2 x + \cos^2 y} = 4$ **[WBSC – 90]**

(ii) Solve: $16^{\sin^2 x} + 16^{\cos^2 x} = 10\ (0 \le x \le \pi)$ **[JEE – 99]**

## ANSWERS

**1.** (i) $n\pi \pm (-1)^n\frac{\pi}{3}$ (ii) $\frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{6}, -\frac{5\pi}{6}$

**2.** (i) $\frac{2n\pi}{3}, (2n+1)\frac{\pi}{5}$ (ii) $(2n+1)\frac{\pi}{2},\ n\pi+(-1)^n.\frac{\pi}{6}$ (iii) $\frac{(2n+1)\pi}{2(p+q)}$ (iv) $\frac{n\pi}{4}+(-1)^n.\frac{\pi}{40}$ (v) $\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

**3.** (i) 60°, 120°, 240°, 300° (ii) $2n\pi \pm \frac{2\pi}{3}$ (iii) 90°, 450°, 810° (iv) $\frac{\pi}{2}$

**4.** (i) $n\pi \pm \frac{\pi}{4},\ 2n\pi \pm \frac{2\pi}{3}$ (ii) $\frac{n\pi}{4},\ n\pi \pm \frac{\pi}{3}$ (iii) 15°, 75°, 105°, 165°, 180°, 195°, 255°, 285°, 345°

(iv) $(2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{4}, (2n+1)\frac{\pi}{6}$ (v) $0, \frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

**5.** (i) $\frac{1}{2}\left(n\pi+\frac{\pi}{4}\right)$ (ii) $n\pi+\frac{\pi}{4}$ **6.** (i) $\frac{1}{3}\left(n\pi+\frac{\pi}{4}\right)$ (ii) $(3n+1)\frac{\pi}{9}$ (iii) $\frac{n\pi}{3}$

**7.** (i) $(4n+1)\frac{\pi}{8}$ (ii) $(2n+1)\frac{\pi}{4}$

**8.** (i) $\frac{\pi}{2}, \frac{\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$, (ii) $\frac{\pi}{12}, -\frac{7\pi}{12}$ (iii) 15°, 105° (iv) 120° (v) $\frac{\pi}{2}, \frac{7\pi}{6}$ (vi) $\frac{\pi}{2}, \frac{11\pi}{6}$ (vii) $\frac{5\pi}{12}, \frac{23\pi}{12}$

**9.** (i) $\frac{n\pi-\alpha}{2-(-1)^n.4}$. (ii) $2n\pi+\alpha,\ \frac{(2n+1)\pi-\alpha}{3}$

**10.** (i) $(4n+1)\frac{\pi}{4}$ (ii) $n\pi \pm (-1)^n\frac{\pi}{6}$

**11.** (i) $n\pi+\frac{\pi}{4},\ n\pi+\cot^{-1}\frac{1}{2}$ (ii) $n\pi \pm \frac{\pi}{4}$

**12.** (i) $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$ (ii) $0, \frac{\pi}{6}, \frac{5\pi}{6}, 2\pi$ (iii) $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$, (iv) $n\pi+\frac{\pi}{4},\ n\pi+(-1)^n\frac{\pi}{3}$

(v) $n\pi+\frac{\pi}{4},\ n\pi+\frac{\pi}{3}$ (vi) $2n\pi,\ \frac{2n\pi+\frac{\pi}{2}}{2\pm1}$

**13.** (i) $x = n\pi \pm (-1)^n.\frac{\pi}{6},\ y = 2n\pi \pm \frac{\pi}{3},\ 2n\pi \pm \frac{2\pi}{3}$ (ii) $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1.(ii) Solution :** $\tan^2\theta = \frac{1}{3}$ or, $\tan\theta = \pm\frac{1}{\sqrt{3}} = \tan\left(\pm\frac{\pi}{6}\right)$

$\therefore\ \theta = n\pi \pm \frac{\pi}{6}$, where, $n = 0, \pm 1, \pm 2, \ldots$

When, $n = 0$, $\theta = \pm\frac{\pi}{6}$,

When, $n = -1$, $\theta = -\pi \pm \frac{\pi}{6} = \frac{-5\pi}{6}, -\frac{7\pi}{6}$

When, $n = 1$, $\theta = \pi \pm \frac{\pi}{6} = \frac{7\pi}{6}, \frac{5\pi}{6}$

Therefore the required solutions are, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{6}, -\frac{5\pi}{6}$ **(Ans)**

**2. (ii) Solution :** Given $\sin 2\theta = \cos\theta$

or, $2\sin\theta\cos\theta - \cos\theta = 0$ or, $\cos\theta(2\sin\theta - 1) = 0$

$\therefore$ either, $\cos\theta = 0$, or $2\sin\theta - 1 = 0$ or, $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

When, $\cos\theta = 0$, $\theta = (2n+1)\frac{\pi}{2}$ and when, $\sin\theta = \sin\frac{\pi}{6}$, $\theta = n\pi + (-1)^n\frac{\pi}{6}$, where $n = 0, \pm 1, \pm 2, \ldots\ldots$

$\therefore\ \theta = (2n+1)\frac{\pi}{2}$, $n\pi + (-1)^n\frac{\pi}{6}$, $n = 0, \pm 1, \pm 2, \ldots\ldots$ are the required solutions. **(Ans)**

**2. (v) Solution :** Given, $\cos 2\theta - \cos 4\theta = 0$ or, $2\sin\frac{2\theta + 4\theta}{2}.\sin\left(\frac{4\theta - 2\theta}{2}\right)$ or, $\sin 3\theta \,.\, \sin\theta = 0$

$\therefore$ either, $\sin 3\theta = 0$ or $\sin\theta = 0$

$\therefore\ 3\theta = n\pi$, or $\theta = n\pi \Rightarrow \theta = \frac{n\pi}{3}, \theta = n\pi, n = 0, \pm 1, \pm 2, \ldots\ldots.$

For, $n = 0$, $\theta = 0$

For, $n = 3$, $\theta = \pi, 3\pi$

For, $n = 6$, $\theta = 2\pi, 6\pi$

For, $n = 1$, $\theta = \frac{\pi}{3}, \pi$

For, $n = 4$, $\theta = \frac{4\pi}{3}, 4\pi$

For, $n = 2$, $\theta = \frac{2\pi}{3}, 2\pi$

For, $n = 5$, $\theta = \frac{5\pi}{3}, 5\pi$

$\therefore$ the required solutions are, $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ **(Ans)**

**3. (i) Solution :** $7\cos^2\theta + 3\sin^2\theta = 4$ or, $7\cos^2\theta + 3\sin^2\theta = 4(\sin^2\theta + \cos^2\theta)$

or, $7\cos^2\theta + 3\sin^2\theta = 4\sin^2\theta + 4\cos^2\theta$ or, $3\cos^2\theta = \sin^2\theta$ or, $\tan^2\theta = 3$ or, $\tan\theta = \pm\sqrt{3}$ or, $\tan\theta = \tan(\pm 60^o)$

$\therefore\ \theta = n\pi \pm 60^o$, where $n = 0, \pm 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

For $n = 0$, $\theta = \pm 60^o$

For $n = 1$, $\theta = \pi \pm 60^o = 180^o + 60^o, 180^o - 60^o = 240^o, 120^o$

For $n = 2$, $\theta = 2\pi \pm 60^o = 2\pi + 60^o, 2\pi - 60^o = 420^o, 300^o$

$\therefore$ the required solutions are $\theta = 60^o, 240^o, 120^o, 300^o$ **(Ans)**

**3. (iv) Solution :** $2\cos^2\theta - \sin\theta + 1 = 0$

or, $2(1 - \sin^2\theta) - \sin\theta + 1 = 0$ or, $2 - 2\sin^2\theta - \sin\theta + 1 = 0$ or, $2\sin^2\theta + \sin\theta - 3 = 0$

or, $2\sin^2\theta + 3\sin\theta - 2\sin\theta - 3 = 0$ or, $\sin\theta(2\sin\theta + 3) - 1(2\sin\theta + 3) = 0$ or, $(\sin\theta - 1)(2\sin\theta + 3) = 0$

$\therefore\ \sin\theta - 1 = 0$ $\left[\because\ \sin\theta \neq -\frac{3}{2},\ -1 \leq \sin\theta \leq 1\right]$

or, $\sin\theta = 1$ or, $\theta = (4n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots\ldots$

When, $n = 0$, $\theta = \frac{\pi}{2}$. When, $n = 1$, $\theta = \frac{5\pi}{2}$

$\therefore$ the required solution is $\theta = \frac{\pi}{2}$ $[\because 0 \le \theta \le 2\pi]$ **(Ans)**

**4. (i) Solution :** $\cos 3x + \cos 2x + \cos x = 0$ or, $\cos 3x + \cos x + \cos 2x = 0$

or, $2\cos\left(\frac{3x+x}{2}\right) . \cos\left(\frac{3x-x}{2}\right) + \cos 2x = 0$ or, $2\cos 2x . \cos x + \cos 2x = 0$ or, $\cos 2x (2\cos x + 1) = 0$

or, $\cos 2x = 0 = \cos 90^o$ | or, $2\cos x + 1 = 0$

$\therefore 2x = 2n\pi \pm 90^o$ | or, $\cos x = -\frac{1}{2} = \cos 120^o$

$\therefore x = n\pi \pm \frac{\pi}{4}$ | $\therefore x = 2n\pi \pm \frac{2\pi}{3}$

where, $n = 0, \pm 1, \pm 2,$ .......... **(Ans)**

**4. (iv) Solution :** $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$ or, $2\cos\left(\frac{6\theta+4\theta}{2}\right) . \cos\left(\frac{6\theta-4\theta}{2}\right) + (\cos 2\theta + 1) = 0$

or, $2\cos 5\theta . \cos\theta + 2\cos^2\theta = 0$ or, $\cos\theta (\cos 5\theta + \cos\theta) = 0$

or, $\cos\theta . 2\cos\left(\frac{5\theta+\theta}{2}\right) . \cos\left(\frac{5\theta-\theta}{2}\right) = 0$ or, $\cos\theta . \cos 3\theta . \cos 2\theta = 0$

$\therefore$ either, $\cos\theta = 0$, $\therefore \theta = (2n+1)\frac{\pi}{2}$

or, $\cos 3\theta = 0$, $\therefore 3\theta = (2n+1)\frac{\pi}{2}$

or, $\cos 2\theta = 0$, $2\theta = (2n+1)\frac{\pi}{2}$

$\therefore \theta = (2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{4}, (2n+1)\frac{\pi}{6}$, where, $n = 0, \pm 1, \pm 2,$ ...... **(Ans)**

**5. (i) Solution :** $\cot x - \tan x = 2$ or, $\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = 2$ or, $\frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = 2$

or, $\cos 2x = 2\sin x \cos x = \sin 2x$ or, $\tan 2x = 1 = \tan\frac{\pi}{4}$

$\therefore 2x = n\pi + \frac{\pi}{4}$ $\therefore x = \frac{1}{2}\left(n\pi + \frac{\pi}{4}\right)$, where, $n = 0, \pm 1, \pm 2$..... **(Ans)**

**6. (i) Solution :** $\tan x + \tan 2x + \tan x \tan 2x = 1$ or, $\tan x + \tan 2x = 1 - \tan x . \tan 2x$

or, $\frac{\tan x + \tan 2x}{1 - \tan x . \tan 2x} = 1$ or, $\tan(x + 2x) = 1$ or, $\tan 3x = \tan\frac{\pi}{4}$

$\therefore 3x = n\pi + \frac{\pi}{4}$ $\therefore x = \frac{1}{3}\left(n\pi + \frac{\pi}{4}\right)$, where $n = 0, \pm 1$,.....**Ans.**

**6. (iii) Solution :** $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta . \tan 2\theta . \tan 3\theta$

or, $\tan\theta + \tan 2\theta = -\tan 3\theta + \tan\theta . \tan 2\theta \tan 3\theta = -\tan 3\theta (1 - \tan\theta . \tan 2\theta)$

$\therefore \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta . \tan 2\theta} = -\tan 3\theta$ or, $\tan(\theta + 2\theta) = -\tan 3\theta$

or, $\tan 3\theta + \tan 3\theta = 0$ or, $2\tan 3\theta = 0$ or $\tan 3\theta = 0$ $\therefore 3\theta = n\pi$

$\therefore \theta = \frac{n\pi}{3}$, where, $n = 0, \pm 1, \pm 2$........... **(Ans)**

**7. (i) Solution :** $\sin 3\theta . \cos^3\theta + \cos 3\theta . \sin^3\theta = \frac{3}{4}$

or, $\sin 3\theta (4\cos^3\theta) + \cos 3\theta (4\sin^3\theta) = 3$ or, $\sin 3\theta (\cos 3\theta + 3\cos\theta) + \cos 3\theta (3\sin\theta - \sin 3\theta) = 3$

$[\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta, \sin 3\theta = 3\sin\theta - 4\sin^3\theta]$

or, $\sin 3\theta \cos 3\theta + 3\sin 3\theta \cos\theta + 3\cos 3\theta . \sin\theta - \cos 3\theta \sin 3\theta = 3$

or, 3(sin3θ cosθ + cos3θ sinθ) = 3 or, sin (3θ + θ) = 1 or, sin4θ = 1

$\therefore\ 4\theta = (4n+1)\frac{\pi}{2}\ \therefore\ \theta = (4n+1)\frac{\pi}{8}$, where n = 0, ± 1, ± 2..... **(Ans)**

**8. (ii) Solution :** $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}}$

or, $\cos\theta \cdot \frac{1}{\sqrt{2}} - \sin\theta \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}\cdot\sqrt{2}}$ or, $\cos\theta \cdot \cos 45^\circ - \sin\theta \cdot \sin 45^\circ = \frac{1}{2}$ or, $\cos(\theta + 45^\circ) = \cos 60^\circ$

$\therefore\ \theta + 45^\circ = 2n\pi \pm 60^\circ$, or, $\theta = 2n\pi \pm 60^\circ - 45^\circ$, n = 0, ±1, ±2,......

For, n = 0, $\theta = \pm 60^\circ - 45^\circ = 15^\circ, -105^\circ$

For, n = 1, $\theta = 2\pi \pm 60^\circ - 45^\circ = 2\pi + 15^\circ, 2\pi - 105^\circ$

$\therefore\ \theta = 15^\circ, -105^\circ = \frac{\pi}{12}, -\frac{7\pi}{12}$ **(Ans)**

**8. (iii) Solution :** $\cos\theta + \sqrt{3}\sin\theta = \sqrt{2}$ or, $\frac{1}{2} \cdot \cos\theta + \frac{\sqrt{3}}{2}\sin\theta = \frac{\sqrt{2}}{2}$ or, $\cos\theta \cdot \cos\frac{\pi}{3} + \sin\theta \cdot \sin\frac{\pi}{3} = \frac{1}{\sqrt{2}}$

or, $\cos\left(\theta - \frac{\pi}{3}\right) = \cos\frac{\pi}{4}\ \therefore\ \theta - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{4}$ or, $\theta = 2n\pi \pm 45^\circ + 60^\circ$, where, n = 0, ± 1, ± 2,.....

For n = 0, $\theta = \pm 45^\circ + 60^\circ = 105^\circ, 15^\circ$

For n = 1, $\theta = 2\pi \pm 45^\circ + 60^\circ = 360^\circ + 105^\circ, 360^\circ + 15^\circ = 465^\circ, 375^\circ$

$\therefore\ \theta = 15^\circ, 105^\circ$ **(Ans)**

**8. (iv) Solution :** $\cos\theta + \sqrt{3}\sin\theta = 1$ or, $\frac{1}{2} \cdot \cos\theta + \frac{\sqrt{3}}{2} \cdot \sin\theta = \frac{1}{2}$ or, $\cos\theta \cdot \cos 60^\circ + \sin\theta \cdot \sin 60^\circ = \cos 60^\circ$

or, $\cos(\theta - 60^\circ) = \cos 60^\circ\ \therefore\ \theta - 60^\circ = 2n\pi \pm 60^\circ\quad \therefore\ \theta = 2n\pi \pm 60^\circ + 60^\circ$, n = 0, ±1, ±2,........

For, n = 0, $\theta = \pm 60^\circ + 60^\circ = 120^\circ, 0^\circ$

For, n = 1, $\theta = 2\pi \pm 60^\circ + 60^\circ = 2\pi, 2\pi + 120^\circ\ \therefore\ \theta = 120^\circ$ **(Ans)**

**8. (v) Solution :** $\sin\theta - \sqrt{3}\cos\theta = 1$

or, $\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2}$ or, $\sin\theta\cos\frac{\pi}{3} - \cos\theta\sin\frac{\pi}{3} = \frac{1}{2}$ or, $\sin\left(\theta - \frac{\pi}{3}\right) = \sin\frac{\pi}{6}$ or, $\theta - \frac{\pi}{3} = n\pi + (-1)^n\frac{\pi}{6}$

or, $\theta = n\pi + (-1)^n\frac{\pi}{6} + \frac{\pi}{3}$ where, $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$

Putting n = 0 we get, $\theta = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

Putting n = 1 we get, $\theta = \pi - \frac{\pi}{6} + \frac{\pi}{3} = \frac{7\pi}{6}$

Putting n = 2 we get, $\theta = 2\pi + \frac{\pi}{6} + \frac{\pi}{3} > 2\pi$

Therefore the required solutions are $\frac{\pi}{2}, \frac{7\pi}{6}$ **(Ans)**

**8. (vi) Solution :** $\sqrt{3}\cos x + \sin x = 1$

or, $\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{2}$ or, $\cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \frac{1}{2}$ or, $\cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$ or, $x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

or, $x = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{6}$ where, $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$

Putting n = 0 we get, $x = \pm\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}, -\frac{\pi}{6}$ Putting n = 1 we get, $x = 2\pi \pm \frac{\pi}{3} + \frac{\pi}{6} = 2\pi + \frac{\pi}{2}, 2\pi - \frac{\pi}{6} = \frac{5\pi}{2}, \frac{11\pi}{6}$

Putting n = 2 we get, $x = 4\pi \pm \frac{\pi}{3} + \frac{\pi}{6} = 4\pi + \frac{\pi}{2}, 4\pi - \frac{\pi}{6}$

Therefore the required solutions are $\frac{\pi}{2}, \frac{11\pi}{6}$ **(Ans)**

**8. (vii) Solution :** $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$ or, $\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta=\frac{\sqrt{2}}{2}$ or, $\cos\theta\cos\frac{\pi}{6}+\sin\theta\sin\frac{\pi}{6}=\frac{1}{\sqrt{2}}$

or, $\cos\left(\theta-\frac{\pi}{6}\right)=\cos\frac{\pi}{4}$ $\therefore\ \theta-\frac{\pi}{6}=2n\pi\pm\frac{\pi}{4}$ or, $\theta=2n\pi\pm\frac{\pi}{4}+\frac{\pi}{6}$, $n=0,\pm1,\pm2,\pm3,\ldots\ldots$

Putting n = 0 we get, $\theta=\pm\frac{\pi}{4}+\frac{\pi}{6}=\frac{\pi}{4}+\frac{\pi}{6},-\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{12},-\frac{\pi}{12}$

Putting n = 1 we get, $\theta=2\pi\pm\frac{\pi}{4}+\frac{\pi}{6}=2\pi+\frac{5\pi}{12},2\pi-\frac{\pi}{12}=\frac{29\pi}{12},\frac{23\pi}{12}$

Putting n = 2 we get, $\theta=4\pi\pm\frac{\pi}{4}+\frac{\pi}{6}>2\pi$

$\therefore\ \theta=\frac{5\pi}{12},\frac{23\pi}{12}$ **(Ans)**

**9. (i) Solution :** $\frac{\sin\alpha}{\sin 2x}+\frac{\cos\alpha}{\cos 2x}=2$ or, $\frac{\sin\alpha\ .\ \cos 2x+\cos\alpha\ .\ \sin 2x}{\sin 2x\ .\ \cos 2x}=2$

or, $\sin(\alpha+2x)=2\sin 2x\ .\ \cos 2x$ or, $\sin(\alpha+2x)=\sin 4x$

$\therefore\ \alpha+2x=n\pi+(-1)^n\ .\ 4x$ or, $2x-(-1)^n\ .\ 4x=n\pi-\alpha$

or, $\left\{2-(-1)^n\ .\ 4\right\}x=n\pi-\alpha$ or, $x=\frac{n\pi-\alpha}{2-(-1)^n.4}$, where, n = 0, ± 1, .... **(Ans)**

**10. (i) Soluton :** $3\tan(\theta-15^\circ)=\tan(\theta+15^\circ)$

or, $\frac{3\sin(\theta-15^\circ)}{\cos(\theta-15^\circ)}=\frac{\sin(\theta+15^\circ)}{\cos(\theta+15^\circ)}$ or, $3\sin(\theta-15^\circ)\ .\ \cos(\theta+15^\circ)=\sin(\theta+15^\circ)\ .\ \cos(\theta-15^\circ)$

or, $2\sin(\theta-15^\circ)\ .\ \cos(\theta+15^\circ)=\sin(\theta+15^\circ)\ .\ \cos(\theta-15^\circ)-\cos(\theta+15^\circ)\ .\ \sin(\theta-15^\circ)$

or, $\sin(\theta-15^\circ+\theta+15^\circ)+\sin(\theta-15^\circ-\theta-15^\circ)=\sin(\theta+15^\circ-\theta+15^\circ)$

or, $\sin 2\theta+\sin(-30^\circ)=\sin 30^\circ$ or, $\sin 2\theta-\sin 30^\circ=\sin 30^\circ$

or, $\sin 2\theta=2\sin 30^\circ$ or, $\sin 2\theta=2\ .\ \frac{1}{2}=1$

or, $2\theta=(4n+1)\frac{\pi}{2}$ $\therefore\ \theta=(4n+1).\frac{\pi}{4}$, where, n = 0, ± 1, ± 2,...... **(Ans)**

**10. (ii) Solution :** $\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=4$ or, $\frac{\sin\left(\frac{\pi}{4}+\theta\right)}{\cos\left(\frac{\pi}{4}+\theta\right)}+\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)}=4$

or, $\frac{\sin\left(\frac{\pi}{4}+\theta\right).\cos\left(\frac{\pi}{4}-\theta\right)+\sin\left(\frac{\pi}{4}-\theta\right).\cos\left(\frac{\pi}{4}+\theta\right)}{\cos\left(\frac{\pi}{4}+\theta\right).\cos\left(\frac{\pi}{4}-\theta\right)}=4$ or, $\frac{\sin\left(\frac{\pi}{4}+\theta+\frac{\pi}{4}-\theta\right)}{\cos^2\frac{\pi}{4}-\sin^2\theta}=4$

or, $\sin\frac{\pi}{2}=4\left(\cos^2 45^\circ-\sin^2\theta\right)$ or, $1=4\left(\frac{1}{2}-\sin^2\theta\right)=2-4\sin^2\theta$ or, $4\sin^2\theta=2-1=1$

$\therefore\ \sin^2\theta=\frac{1}{4}$ $\therefore\ \sin\theta=\pm\frac{1}{2}=\sin(\pm 30^\circ)$ $\therefore\ \theta=n\pi+(-1)^n\left(\pm\frac{\pi}{6}\right)$, where n = 0, ± 1, ± 2, ......... **(Ans)**

**11. (ii) Solution :** $\tan^2\theta+\cot^2\theta=2$ or, $(\tan\theta-\cot\theta)^2+2\tan\theta\cot\theta=2$

or, $(\tan\theta-\cot\theta)^2+2=2$ or, $(\tan\theta-\cot\theta)^2=0$ or, $\tan\theta-\cot\theta=0$

or, $\tan\theta=\cot\theta$ or, $\tan\theta=\frac{1}{\tan\theta}$ or, $\tan^2\theta=1$

or, $\tan\theta=\pm1=\tan\left(\pm\frac{\pi}{4}\right)$ $\therefore\ \theta=n\pi\pm\frac{\pi}{4}$, where, n = 0, ± 1, ± 2, .. **(Ans)**

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**12. (ii) Solution :** $1 + 2\sin\theta\cos\theta - 2\sin\theta - \cos\theta = 0$ or, $(1 - 2\sin\theta) - \cos\theta\,(1 - 2\sin\theta) = 0$

or, $(1 - 2\sin\theta)(1 - \cos\theta) = 0$

$\therefore$ either, $1 - 2\sin\theta = 0$ or, $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$ $\therefore$ $\theta = n\pi + (-1)^n\frac{\pi}{6}$

or $1 - \cos\theta = 0$ or, $\cos\theta = 1$ $\therefore$ $\theta = 2n\pi$ where, $n = 0, \pm 1, \pm 2, \ldots\ldots$

For, $n = 0$, $\theta = \frac{\pi}{6}, 0$. For, $n = 1$, $\theta = \pi - \frac{\pi}{6}, 2\pi = \frac{5\pi}{6}, 2\pi$

For, $n = 2$, $\theta = 2\pi + \frac{\pi}{6}, 4\pi$

$\therefore$ the required solutions are $0, \frac{\pi}{6}, \frac{5\pi}{6}, 2\pi$ **(Ans)**

**12. (v) Solution :** $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$ or, $\tan^2\theta - \tan\theta - \sqrt{3}\tan\theta + \sqrt{3} = 0$

or, $\tan\theta(\tan\theta - 1) - \sqrt{3}(\tan\theta - 1) = 0$ or, $(\tan\theta - 1)(\tan\theta - \sqrt{3}) = 0$

$\therefore$ either,

$\tan\theta - 1 = 0$

or, $\tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore$ $\theta = n\pi + \frac{\pi}{4}$

or, $\tan\theta - \sqrt{3} = 0$

or, $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$

$\therefore$ $\theta = n\pi + \frac{\pi}{3}$ where, $n = 0, \pm 1, \pm 2, \ldots\ldots$ **(Ans)**

**12. (vi) Solution :** $2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$

or, $2\sin x - 2\cos 2x - \sin 2x - 2\sin 2x\sin x + 2\cos x = 0$

or, $2\sin x - 2\cos 2x - 2\sin x\,.\cos x - 4\sin^2 x\,.\cos x + 2\cos x = 0$

or, $2\sin x(1 - \cos x) - 2\cos 2x + 2\cos x(1 - 2\sin^2 x) = 0$ or, $2\sin x(1 - \cos x) - 2\cos 2x + 2\cos x\,.\cos 2x = 0$

or, $2\sin x(1 - \cos x) - 2\cos 2x(1 - \cos x) = 0$ or, $2(1 - \cos x)(\sin x - \cos 2x) = 0$

$\therefore$ either $1 - \cos x = 0$ or, $\sin x - \cos 2x = 0$

or, $\cos x = 1 = \cos 0$ $\therefore$ $x = 2n\pi$

or, $\cos 2x = \sin x$ or, $\cos 2x = \cos(90^\circ - x)$ $\therefore$ $2x = 2n\pi \pm (90^\circ - x)$

or, $2x \pm x = 2n\pi \pm 90^0$ or, $x(2 \pm 1) = 2n\pi \pm 90^\circ$ $\therefore$ $x = \frac{2n\pi \pm 90^\circ}{2 \pm 1}$

$\therefore$ $x = 2n\pi, \frac{2n\pi \pm 90^\circ}{2 \pm 1}$, where $n = 0, \pm 1, \pm 2, \ldots$ **(Ans)**

**13. (i) Solution :** $2^{\sin x + \cos y} = 1$ $\therefore$ $\sin x + \cos y = 0$ -----(1)

$16^{\sin^2 x + \cos^2 y} = 4$ or $4^{2(\sin^2 x + \cos^2 y)} = 4^1$ or, $2(\sin^2 x + \cos^2 y) = 1$ ......(2)

or, $\sin^2 x + \sin^2 x = \frac{1}{2}$ [ $\therefore$ from (1) $\sin x + \cos y = 0$ $\therefore$ $\cos y = -\sin x$ $\therefore$ $\cos^2 y = \sin^2 x$]

or, $2\sin^2 x = \frac{1}{2}$ or, $\sin^2 x = \frac{1}{4}$ or, $\sin x = \pm\frac{1}{2} = \sin\left(\pm\frac{\pi}{6}\right)$ $\therefore$ $x = n\pi + (-1)^n.\left(\pm\frac{\pi}{6}\right)$

Again from (2) we get,

$\cos^2 y + \cos^2 y = \frac{1}{2}$ or, $\cos^2 y = \frac{1}{4}$ or, $\cos y = \frac{1}{2}, -\frac{1}{2} = \cos\frac{\pi}{3}, \cos\frac{2\pi}{3}$ $\therefore y = 2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \frac{2\pi}{3}$

$\therefore$ $x = n\pi + (-1)^n.\left(\pm\frac{\pi}{6}\right)$, $y = 2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \frac{2\pi}{3}$ where $n = 0, \pm 1, \pm 2, \ldots$ **(Ans)**

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## Chapter 4

# Inverse Circular Functions

**4.1** If x $(-1 \le x \le 1)$ be a real number then we may have infinitely many values of $\theta$, which satisfy the equation $\sin\theta = x$. In this case the angle $\theta$ is represented as $\sin^{-1}x$ (read as sine inverse x).

Therefore, if $\sin\theta = x\ (-1 \le x \le 1)$, then $\theta = \sin^{-1} x$ Similarly, if $\cos\theta = x(-1 \le x \le 1)$, then $\theta = \cos^{-1}x$

if $\tan\theta = x\ (-\infty < x < \infty)$, then, $\theta = \tan^{-1}x$ if $\cot\theta = x\ (-\infty < x < \infty)$, then, $\theta = \cot^{-1}x$

if $\sec\theta = x\ (|x| \ge 1)$, then, $\theta = \sec^{-1}x$ and if $\operatorname{cosec}\theta = x\ (|x| \ge 1)$, then, $\theta = \operatorname{cosec}^{-1}x$

$\sin^{-1}x$ represents an angle while $\sin\theta$ represents a pure number.

For a given value of x $(-1 \le x \le 1)$ we may have infinitely many values of $\sin^{-1}x$, but a given value of $\theta$ gives definite finite value of $\sin\theta$.

i.e., $\sin^{-1}x$, is a multi-valued function but $\sin\theta$ is a single–valued function.

Conversely, if $\sin^{-1}x = \theta$, then $\sin\theta = x$, if $\cos^{-1}x = \theta$, then $\cos\theta = x$ etc.

The trigonometrical functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ are called Inverse Circular Function.

**Note :** $\sin^{-1}x$ is a circular function and it represents an angle. $(\sin x)^{-1} = \frac{1}{\sin x}$ = a pure number.

## 4.2 Formulae :

| | | |
|---|---|---|
| $\sin(\sin^{-1}x) = x$ | $\cos(\cos^{-1}x) = x$ | $\tan(\tan^{-1}x) = x$ |
| $\cot(\cot^{-1}x) = x$ | $\sec(\sec^{-1}x) = x$ | $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$ |

**Note :** Here x $(-1 \le x \le 1)$ is a pure number.

| | | |
|---|---|---|
| $\sin^{-1}(\sin\theta) = \theta$ | $\cos^{-1}(\cos\theta) = \theta$ | $\tan^{-1}(\tan\theta) = \theta$ |
| $\cot^{-1}(\cot\theta) = \theta$ | $\sec^{-1}(\sec\theta) = \theta$ | $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$ |

**Note :** Here $\theta$ represents an angle.

| | | |
|---|---|---|
| $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$ | $\cos^{-1}x = \sec^{-1}\frac{1}{x}$ | $\tan^{-1}x = \cot^{-1}\frac{1}{x}$ |
| $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ | $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ | $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ |
| $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$ | $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$ | |
| $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\frac{xy-1}{y+x}$ | $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\frac{xy+1}{y-x}$ | |

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-zy-zx}\right)$

$\cot^{-1}x+\cot^{-1}y+\cot^{1}z=\cot^{-1}\left(\dfrac{xyz-x-y-z}{xy+yz+zx-1}\right)$

$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$

$\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$

$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left\{xy-\sqrt{(1-x^2)(1-y^2)}\right\}$

$\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left\{xy+\sqrt{(1-x^2)(1-y^2)}\right\}$

$2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\cos^{-1}(1-2x^2)$

$2\cos^{-1}x=\cos^{-1}(2x^2-1)$ $\quad 2\cot^{-1}x=\cot^{-1}\left(\dfrac{x^2-1}{2x}\right)$

$2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}=\sin^{-1}\dfrac{2x}{1+x^2}=\cos^{-1}\dfrac{1-x^2}{1+x^2}$

$3\sin^{-1}x=\sin^{-1}(3x-4x^3)$

$3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

$3\tan^{-1}x=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

$3\cot^{-1}x=\cot^{-1}\dfrac{x^3-3x}{3x^2-1}$

## PROBLEM SET

[Problems with '*' marks are solved at the end of the problem set]

1. Prove that,

(i) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$ *(ii) $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$ **[WBSC – 82]**

*(iii) $3\tan^{-1}\dfrac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{3}$ (iv) $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{32}{43}$

*(v) $4\left(\cot^{-1}3+\operatorname{cosec}^{-1}\sqrt{5}\right)=\pi$

2. Prove that,

(i) $\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{7}-\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{1}{18}$ *(ii) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$ **[WBSC – 89]**

*(iii) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=0$ **[WBSC – 93]** (iv) $2\cot^{-1}5+\cot^{-1}7+2\cot^{-1}8=\frac{\pi}{4}$

3. Prove that,

*(i) $\sin^{-1}\frac{16}{65}+2\tan^{-1}\frac{1}{5}=\cos^{-1}\frac{4}{5}$ (ii) $\cos^{-1}\dfrac{2}{\sqrt{5}}+\sin^{-1}\dfrac{1}{\sqrt{10}}=\frac{\pi}{4}$

*(iii) $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{56}{33}$ (iv) $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{16}{65}$

4. Prove that,

*(i) $\sec^2(\cot^{-1}2)+\operatorname{cosec}^2(\tan^{-1}3)=2\frac{13}{36}$ (ii) $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=15$ **[WBSC – 83]**

5. Prove that,

*(i) $2\tan^{-1}\left(\dfrac{1+x}{1-x}\right)+\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)=0$ **[WBSC – 91]** *(ii) $\tan\left(\frac{1}{2}.\sin^{-1}\dfrac{2x}{1+x^2}+\frac{1}{2}.\cos^{-1}\dfrac{1-y^2}{1+y^2}\right)=\dfrac{x+y}{1-xy}$

(iii) $\tan(\sin^{-1}x+\sin^{-1}y)+\tan(\cos^{-1}x+\cos^{-1}y)=0$ (iv) $\cot^{-1}\dfrac{xy+1}{x-y}+\cot^{-1}\dfrac{yz+1}{y-z}+\cot^{-1}\dfrac{zx+1}{z-x}=0$

*(v) $\tan^{-1}\dfrac{a-b}{1+ab}+\tan^{-1}\dfrac{b-c}{1+bc}+\tan^{-1}\dfrac{c-a}{1+ca}=0$

**6.** Prove that,

(i) $\tan^{-1}\sqrt{x} = \frac{1}{2}.\cos^{-1}\frac{1-x}{1+x}$

*(ii) $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}} = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

(iii) $\sin^{-1}\sqrt{\frac{x-q}{p-q}} = \cos^{-1}\sqrt{\frac{p-x}{p-q}} = \cot^{-1}\sqrt{\frac{p-x}{x-q}}$

**7.** Prove that :

*(i) $\sin\left\{\cos^{-1}\left(-\frac{1}{2}\right)\right\} = \frac{\sqrt{3}}{2}$ **[WBSC – 03, 07]**

*(ii) $\cos\tan^{-1}\cot\sin^{-1}x = x$

(iii) $\sin\cot^{-1}\tan\cos^{-1}x = x$

(iv) $\sin\cos^{-1}\tan\sec^{-1}x = \sqrt{2-x^2}$

*(v) $\cos\tan^{-1}\sin\cot^{-1}x = \left(\frac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}$

**8.** (i) Show that, $\tan^{-1}\frac{1}{a+b} + \tan^{-1}\frac{b}{a^2+ab+1} = \tan^{-1}\frac{1}{a}$

*(ii) If $xy = 1 + a^2$, then prove that, $\tan^{-1}\frac{1}{a+x} + \tan^{-1}\frac{1}{a+y} = \tan^{-1}\frac{1}{a}$; $x + y + 2a \neq 0$. **[WBSC – 04]**

*(iii) In $\Delta ABC$, $\angle ACB = 90^\circ$. Prove that, $\tan^{-1}\frac{a}{b+c} + \tan^{-1}\frac{b}{c+a} = \frac{\pi}{4}$

*(iv) If $A + B + C = \pi$, and $A = \tan^{-1}2$, $B = \tan^{-1}3$, show that $\angle C = \frac{\pi}{4}$.

**9.** *(i) Prove that, $\tan^{-1}(\cot x) + \cot^{-1}(\tan x) = \pi - 2x$

(ii) Prove that, $\tan^{-1}\left(\frac{x\cos\varphi}{1-x\sin\varphi}\right) - \cot^{-1}\left(\frac{\cos\varphi}{x-\sin\varphi}\right)$ is independent of x and find its simplest value.

*(iii) If $\varphi = \cot^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$, show that, $\sin\varphi = \tan^2\theta$.

(iv) Show that, $\cot^{-1}(\tan 2x) + \cot^{-1}(-\tan 3x) = x$

*(v) Prove that, $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}.\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$. **[WBSC – 19]**

**10.** *(i) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, show tht, $x^2 + y^2 + z^2 + 2xyz = 1$ **[WBSC – 07]**

(ii) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, then prove that, $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$.

**11.** (i) If $\cos^{-1}x + \cos^{-1}y = \theta$, prove that, $x^2 - 2xy\cos\theta + y^2 = \sin^2\theta$.

*(ii) If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \theta$, prove that, $\frac{x^2}{a^2} - \frac{2xy}{ab}.\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$ **[WBSC – 03]**

**12.** *(i) Prove that, $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{zx}{yr} + \tan^{-1}\frac{xy}{zr} = \frac{\pi}{2}$, where, $r^2 = x^2 + y^2 + z^2$ **[WBSC – 89]**

*(ii) If $\tan^{-1}\frac{yz}{xp} + \tan^{-1}\frac{zx}{yp} + \tan^{-1}\frac{xy}{zp} = \frac{\pi}{2}$, prove that, $x^2 + y^2 + z^2 = p^2$.

(iii) Show that, $\tan^{-1}\sqrt{\frac{xr}{yz}} + \tan^{-1}\sqrt{\frac{yr}{zx}} + \tan^{-1}\sqrt{\frac{zr}{xy}} = \pi$, where $x + y + z = r$.

**13.** (i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, show that, $xy + yz + zx = 1$ **[WBSC – 07]**

*(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that, $x + y + z = xyz$.

**14.** (i) If If $\sec\theta - \operatorname{cosec}\theta = \frac{4}{3}$, show that $\theta = \frac{1}{2}.\sin^{-1}\frac{3}{4}$

*(ii) If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then show that, $\theta = \pm\frac{1}{2}.\sin^{-1}\frac{3}{4}$

**15.** Solve :

*(i) $\tan(\cos^{-1}x) = \sin(\tan^{-1}2)$ **[WBSC – 85]**

(ii) $2\sin^{-1}x = \cos^{-1}x$

(iii) $\tan^{-1}x = \cot^{-1}x$

(iv) $\tan^{-1}(\cot x) + \cot^{-1}(\tan x) = \frac{\pi}{4}$

*(v) $\tan^{-1}\left(\frac{1}{2}.\sec x\right) + \cot^{-1}(2\cos x) = \frac{\pi}{3}$ **[WBSC – 04]**

**16.** Solve :

*(i) $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$ **[WBSC – 86]**

(ii) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$

*(iii) $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{4x+1} = \tan^{-1}\frac{2}{x^2}$

(iv) $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{6}{17}$

**[Ans. 16** (i) $\left[\pm\frac{1}{\sqrt{2}}\right]$ (ii) $\left[\frac{4}{3}, -\frac{3}{8}\right]$ (iii) $\left[0, 3, -\frac{2}{3}\right]$ (iv) $\left[\frac{1}{3}\right]$ ]

**17.** Solve :

*(i) $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$ **[WBSC – 92]**

(ii) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

*(iii) $\sin^{-1}x + \sin^{-1}(2x) = \frac{\pi}{3}$

**[Ans. 17** (i) [± 13] (ii) $\left[0, \frac{1}{2}\right]$ (iii) $\left[\pm\frac{1}{2}\sqrt{\frac{3}{7}}\right]$ ]

**18.** Solve :

(i) $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$ $\left[\frac{1}{2}, 1\right]$

*(ii) $\sin^{-1}x - \sin^{-1}y = \frac{\pi}{3}$, $\cos^{-1}x + \cos^{-1}y = \frac{2\pi}{3}$ $\left[\frac{\sqrt{3}}{2}, 0\right]$

***19.** Solve $\sin^{-1}\frac{2a}{1+a^2} + \sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}x$ **Ans.** $\left[\frac{a+b}{1-ab}\right]$

***20.** Express $\tan^{-1}\alpha$ as the sum of two angles of which $\alpha$ is one. **[WBSC – 87]**

***21.** Prove that, $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) = \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$

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## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1. (ii) Solution :** $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{2.\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)+\tan^{-1}\frac{1}{7}$ $\left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$

$$=\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right)=\tan^{-1}\left(\frac{25}{25}\right)=\tan^{-1}(1)=\frac{\pi}{4}\ \textbf{(Ans)}$$

**1. (iii) Solution :** $\because \tan 15^\circ=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ.\tan 30^\circ}=\frac{1-\frac{1}{\sqrt{3}}}{1+1.\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$

$$=\frac{(\sqrt{3}-1)(\sqrt{3}+1)}{(\sqrt{3}+1)^2}=\frac{3-1}{3+1+2\sqrt{3}}=\frac{2}{4+2\sqrt{3}}=\frac{1}{2+\sqrt{3}}$$

$\therefore\ 15^\circ=\tan^{-1}\frac{1}{2+\sqrt{3}}$ or, $45^\circ=3\tan^{-1}\frac{1}{2+\sqrt{3}}$ or, $\frac{\pi}{4}=3\tan^{-1}\frac{1}{2+\sqrt{3}}$ --------- (1)

Now, $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)=\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$ -------(2)

From (1) and (2) we can write $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=3\tan^{-1}\frac{1}{2+\sqrt{3}}$

$\therefore\ 3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{3}$ **(Proved)**

**1. (v) Solution :** Let $\text{cosce}^{-1}\sqrt{5}=\theta$ $\therefore$ $\text{cosce}\theta=\sqrt{5}$

$\therefore\ \cot\theta=\sqrt{\text{cosec}^2\theta-1}=\sqrt{5-1}=\sqrt{4}=2\Rightarrow\theta=\cot^{-1}2$

$\therefore$ L.H.S. $=4\left(\cot^{-1}3+\text{cosec}^{-1}\sqrt{5}\right)=4\left(\cot^{-1}3+\cot^{-1}2\right)$

$=4\cot^{-1}\left(\frac{3\times 2-1}{3+2}\right)=4\,.\,\text{co5t}^{-1}\left(\frac{5}{5}\right)=4\,.\cot^{-1}(1)=4\,.\frac{\pi}{4}=\pi=$ R.H.S **(Proved)**

**2. (ii) Solution :** $4\tan^{-1}\frac{1}{5}=2\,.\,2\tan^{-1}\frac{1}{5}=2\,.\tan^{-1}\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)=2\,.\tan^{-1}\frac{5}{12}=\tan^{-1}\left(\frac{2\times\frac{5}{12}}{1-\frac{25}{144}}\right)=\tan^{-1}\frac{120}{119}$

Again, $\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}=\tan^{-1}\left(\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right)=\tan^{-1}\frac{29}{6931}=\tan^{-1}\frac{1}{239}$

$\therefore$ L.H.S. $=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=4\tan^{-1}\frac{1}{5}-\left(\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right)$

$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}\tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}-\frac{1}{239}}\right]=\tan^{-1}\left(\frac{28561}{28561}\right)=\tan^{-1}(1)=\frac{\pi}{4}=$ R.H.S. **(Proved)**

**2. (iii) Solution :** L.H.S. $= \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \tan^{-1}1 + \tan^{-1}\left(\frac{2+3}{1-2\times3}\right)$

$= \tan^{-1}(1) + \tan^{-1}\left(\frac{5}{-5}\right) = \tan^{-1}(1) - \tan^{-1}(1) = 0$ **proved.**

**3. (i) Solution :** L.H.S. $= \sin^{-1}\frac{16}{65} + 2\tan^{-1}\frac{1}{5} = \cos^{-1}\sqrt{1-\frac{(16)^2}{(65)^2}} + \cos^{-1}\left(\frac{1-\frac{1}{25}}{1+\frac{1}{25}}\right)$

$\left[\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} \text{ and } 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$

$= \cos^{-1}\frac{\sqrt{(65)^2-(16)^2}}{65} + \cos^{-1}\left(\frac{24}{26}\right) = \cos^{-1}\left(\frac{63}{65}\right) + \cos^{-1}\frac{12}{13}$

$= \cos^{-1}\left[\frac{63}{65}\times\frac{12}{13} - \sqrt{\left\{1-\left(\frac{63}{65}\right)^2\right\}\left\{1-\left(\frac{12}{13}\right)^2\right\}}\right] = \cos^{-1}\left[\frac{63\times12}{65\times13} - \sqrt{\left\{\frac{(65)^2-(63)^2}{(65)^2}\right\}\times\left\{\frac{(13)^2-(12)^2}{(13)^2}\right\}}\right]$

$= \cos^{-1}\left[\frac{63\times12}{65\times13} - \frac{16\times5}{65\times13}\right] = \cos^{-1}\left(\frac{63\times12-16\times5}{65\times13}\right) = \cos^{-1}\left(\frac{676}{65\times13}\right) = \cos^{-1}\frac{4}{5} =$ R.H.S. **(proved).**

**3. (iii) Solution :** L.H.S. $= \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{4}{5} = \cos^{-1}\sqrt{1-\frac{25}{169}} + \cos^{-1}\frac{4}{5}$ $\left[\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}\right]$

$= \cos^{-1}\sqrt{\frac{169-25}{169}} + \cos^{-1}\frac{4}{5} = \cos^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5}$

$= \cos^{-1}\left\{\frac{12}{13}\times\frac{4}{5} - \sqrt{\left(1-\frac{144}{169}\right)\left(1-\frac{16}{25}\right)}\right\} = \cos^{-1}\left[\frac{48}{65} - \sqrt{\left(\frac{169-144}{169}\right)\left(\frac{25-16}{25}\right)}\right]$

$= \cos^{-1}\left(\frac{48}{65} - \frac{5\times3}{13\times5}\right) = \cos^{-1}\left(\frac{48-15}{65}\right) = \cos^{-1}\left(\frac{33}{65}\right) = \theta$ (say).

$\therefore \cos\theta = \frac{33}{65}$ $\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{(33)^2}{(65)^2}} = \frac{\sqrt{(65)^2-(33)^2}}{65} = \frac{\sqrt{3136}}{65} = \frac{56}{65}$

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{56}{65}}{\frac{33}{65}} = \frac{56}{33}$ $\therefore \theta = \tan^{-1}\frac{56}{33}$ **Hence proved.**

**4. (i) Solution :** L.H.S. $= \sec^2(\cot^{-1}2) + \text{cosec}^2(\tan^{-1}3)$

$= \sec^2\left(\tan^{-1}\frac{1}{2}\right) + \text{cosec}^2\left(\cot^{-1}\frac{1}{3}\right) = 1 + \tan^2\left(\tan^{-1}\frac{1}{2}\right) + 1 + \cot^2\left(\cot^{-1}\frac{1}{3}\right)$

$= 1 + \left\{\tan\left(\tan^{-1}\frac{1}{2}\right)\right\}^2 + 1 + \left\{\cot\left(\cot^{-1}\frac{1}{3}\right)\right\}^2$

$= 1 + \left(\frac{1}{2}\right)^2 + 1 + \left(\frac{1}{3}\right)^2$ $[\because \tan(\tan^{-1}x) = x \text{ etc.}] = 2\frac{13}{36} =$ **R.H.S. proved.**

**5. (i) Solution :** $2\tan^{-1}\frac{1+x}{1-x} + \sin^{-1}\frac{1-x^2}{1+x^2}$

$= \tan^{-1}\left\{\frac{2\cdot\frac{1+x}{1-x}}{1-\left(\frac{1+x}{1-x}\right)^2}\right\} + \alpha$ (let)

$= \tan^{-1}\left\{\frac{2\cdot\frac{1+x}{1-x}}{\frac{(1-x)^2-(1+x)^2}{(1-x)^2}}\right\} + \alpha$

Let $\sin^{-1}\frac{1-x^2}{1+x^2} = \alpha$ or, $\sin\alpha = \frac{1-x^2}{1+x^2}$

$\therefore \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}$

$= \sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2}$

$\therefore \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1-x^2}{2x}$ $\therefore \alpha = \tan^{-1}\frac{1-x^2}{2x}$

$= \tan^{-1}\left\{\frac{2(1+x)(1-x)}{1+x^2-2x-1-x^2-2x}\right\}+\alpha$

$= \tan^{-1}\left\{\frac{2(1-x^2)}{-4x}\right\}+\alpha \quad = -\tan^{-1}\frac{1-x^2}{2x}+\tan^{-1}\frac{1-x^2}{2x} = 0$ **(Proved)**

**5. (ii) Solution :** $\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right)$

$= \tan\left(\frac{1}{2}\cdot 2\tan^{-1}x+\frac{1}{2}\cdot 2\tan^{-1}y\right) = \tan(\tan^{-1}x+\tan^{-1}y) = \tan\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{x+y}{1-xy}$ **proved.**

**5. (v) Solution :** $\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}\frac{c-a}{1+ca}$

$= \tan^{-1}a-\tan^{-1}b+\tan^{-1}b-\tan^{-1}c+\tan^{-1}c-\tan^{-1}a \left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y\right] = 0$ **(Proved)**

**6. (ii) Solution :** Let $\cos^{-1}x=\theta \therefore \cos\theta = x \Rightarrow x=\cos\theta$

$\therefore\ 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\sin^{-1}\sqrt{\frac{1-\cos\theta}{2}} = 2\sin^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2}} = 2\sin^{-1}\left(\sin\frac{\theta}{2}\right) = 2\cdot\frac{\theta}{2} = \theta = \cos^{-1}x$

$2\cos^{-1}\sqrt{\frac{1+x}{2}} = 2\cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = 2\cos^{-1}\sqrt{\frac{2\cos^2\frac{\theta}{2}}{2}} = 2\cos^{-1}\left(\cos\frac{\theta}{2}\right) = 2\cdot\frac{\theta}{2} = \theta = \cos^{-1}x$

$2\tan^{-1}\sqrt{\frac{1-x}{1+x}} = 2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = 2\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = 2\tan^{-1}\left(\tan\frac{\theta}{2}\right) = 2\cdot\frac{\theta}{2} = \theta = \cos^{-1}x$

$\therefore\ \cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}} = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ **(Proved)**

**7. (i) Solution :** Given: $\sin\left\{\cos^{-1}\left(-\frac{1}{2}\right)\right\}$ $\quad$ [Let $\cos^{-1}\left(-\frac{1}{2}\right)=\theta,\ \therefore \cos\theta = -\frac{1}{2} = \cos 120^0 \ \therefore\ \theta = 120^0$]

$= \sin 120^0 = \frac{\sqrt{3}}{2}$ **(Proved)**

**7. (ii) Solution :** L.H.S. $= \cos\tan^{-1}\cot\sin^{-1}x$

$= \cos\tan^{-1}\cot\theta$

$= \cos\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \cos\alpha = x$

**= R.H.S. (Proved)**

Let $\sin^{-1}x=\theta \therefore \sin\theta = x$

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2} \ \therefore\ \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-x^2}}{x}$

Let, $\tan^{-1}\frac{\sqrt{1-x^2}}{x} = \alpha \therefore \tan\alpha = \frac{\sqrt{1-x^2}}{x}$

or, $\sec^2\alpha = 1+\tan^2\alpha = 1+\frac{1-x^2}{x^2} = \frac{1}{x^2} \therefore \cos^2\alpha = x^2 \ \therefore \cos\alpha = x$

**7. (v) Solution :** L.H.S. $= \cos\tan^{-1}\sin\cot^{-1}x$

$= \cos\tan^{-1}\sin\theta$

$= \cos\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \cos\alpha = \left(\frac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}$

**= R.H.S. (Proved)**

Let $\cot^{-1}x=\theta \therefore \cot\theta = x$

$\operatorname{cosec}^2\theta = 1+\cot^2\theta = 1+x^2 \therefore \sin^2\theta = \frac{1}{1+x^2}$ or, $\sin\theta = \frac{1}{\sqrt{1+x^2}}$

Let $\tan^{-1}\frac{1}{\sqrt{1+x^2}} = \alpha$ or, $\tan\alpha = \frac{1}{\sqrt{1+x^2}}$

or, $\sec^2\alpha = 1+\tan^2\alpha = 1+\frac{1}{1+x^2} = \frac{2+x^2}{1+x^2}$

or, $\cos^2\alpha = \frac{x^2+1}{x^2+2} \ \therefore \cos\alpha = \left(\frac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}$

**8. (ii) Solution :** L. H. S. $= \tan^{-1}\left(\frac{1}{a+x}\right)+\tan^{-1}\left(\frac{1}{a+y}\right) = \tan^{-1}\left[\frac{\frac{1}{a+x}+\frac{1}{a+y}}{1-\left(\frac{1}{a+x}\right)\left(\frac{1}{a+y}\right)}\right]$

$$= \tan^{-1}\left[\frac{\frac{a+y+a+x}{(a+x)(a+y)}}{\frac{(a+x)(a+y)-1}{(a+x)(a+y)}}\right] = \tan^{-1}\left[\frac{x+y+2a}{a^2+ay+ax+xy-1}\right] = \tan^{-1}\left[\frac{x+y+2a}{a^2+ay+ax+a^2}\right] \text{ [ since, } xy = 1+a^2 \text{ ]}$$

$$= \tan^{-1}\left[\frac{x+y+2a}{a(x+y+2a)}\right] = \tan^{-1}\left(\frac{1}{a}\right) \quad [\because x+y+2a \neq 0]$$ **(Proved)**

**8. (iii) Solution :** $\tan^{-1}\frac{a}{b+c}+\tan^{-1}\frac{b}{c+a} = \tan^{-1}\left(\frac{\frac{a}{b+c}+\frac{b}{c+a}}{1-\frac{a}{b+c}\cdot\frac{b}{c+a}}\right) = \tan^{-1}\left[\frac{\frac{ac+a^2+b^2+bc}{(b+c)(c+a)}}{\frac{(b+c)(c+a)-ab}{(b+c)(c+a)}}\right]$

$$= \tan^{-1}\left(\frac{ac+bc+a^2+b^2}{bc+c^2+ab+ac-ab}\right) = \tan^{-1}\left(\frac{ac+bc+c^2}{bc+c^2+ac}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$ **(proved).**

**8. (iv) Solution :** Given, $A+B+C=\pi$ $\therefore$ $C=\pi-(A+B)$ or, $\tan C = \tan\{\pi-(A+B)\} = -\tan(A+B)$

or, $\tan C = -\frac{\tan A+\tan B}{1-\tan A\,.\,\tan B} = -\frac{2+3}{1-2\,.\,3}$ $[\because A=\tan^{-1}2,\ B=\tan^{-1}3\ \therefore \tan A=2,\ \tan B=3]$

$= -\frac{5}{-5} = 1$ $\therefore \tan C = \tan\frac{\pi}{4}$ $\therefore C=\frac{\pi}{4}$ $[\because \angle ACB = 9^{o}\ \therefore c^2=a^2+b^2]$ **(Proved)**

**9. (i) Solution :** L.H.S. $\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-x\right)\right\}+\cot^{-1}\left\{\cot\left(\frac{\pi}{2}-x\right)\right\} = \frac{\pi}{2}-x+\frac{\pi}{2}-x = \pi-2x = \text{R.H.S.}$$ **proved.**

**9. (iii) Solution :** Given, $\varphi = \cot^{-1}\sqrt{\cos 2\theta}-\tan^{-1}\sqrt{\cos 2\theta}$

$$= \tan^{-1}\frac{1}{\sqrt{\cos 2\theta}}-\tan^{-1}\sqrt{\cos 2\theta} = \tan^{-1}\left(\frac{\frac{1}{\sqrt{\cos 2\theta}}-\sqrt{\cos 2\theta}}{1+\frac{1}{\sqrt{\cos 2\theta}}\cdot\sqrt{\cos 2\theta}}\right) = \tan^{-1}\left(\frac{1-\cos 2\theta}{2\sqrt{\cos 2\theta}}\right)$$

or, $\tan\varphi = \frac{1-\cos 2\theta}{2\sqrt{\cos 2\theta}}$ or, $\tan^2\varphi = \frac{\left(\frac{1-\cos 2\varphi}{2}\right)^2}{\cos 2\theta} = \frac{\sin^4\theta}{\cos 2\theta}$

or, $\cot^2\varphi = \frac{\cos 2\theta}{\sin^4\theta}$ or, $1+\cot^2\varphi = 1+\frac{\cos 2\theta}{\sin^4\theta} = \frac{\sin^4\theta+1-2\sin^2\theta}{\sin^4\theta}$

or, $\operatorname{cosec}^2\varphi = \frac{(1-\sin^2\theta)^2}{\sin^4\theta} = \frac{\cos^4\theta}{\sin^4\theta} = \cot^4\theta$ $\therefore \sin^2\varphi = \tan^4\theta$ $\therefore \sin\varphi = \tan^2\theta$ **(Proved)** [taking positive sign]

**9. (v) Solution :** L.H.S. $= \tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)$

$= \tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right),\ \theta = \frac{1}{2}\cos^{-1}\frac{a}{b}$ (say)

$$= \frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\,.\,\tan\theta}+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\,.\,\tan\theta} = \frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta} = \frac{(1+\tan\theta)^2+(1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)} = \frac{2(1+\tan^2\theta)}{1-\tan^2\theta}$$

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$$= 2\frac{1}{\frac{1-\tan^2\theta}{1+\tan^2\theta}} = \frac{2}{\cos 2\theta} = \frac{2b}{a} \quad \left[\because\ \theta = \frac{1}{2}\cos^{-1}\frac{a}{b},\ \therefore\ \cos 2\theta = \frac{a}{b}\right]$$ **(Proved)**

**10. (i) Solution :** $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

or, $\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$ or, $\cos^{-1}\left(xy - \sqrt{(1-x^2)(1-y)^2}\right) = \pi - \cos^{-1}z$

or, $xy - \sqrt{(1-x^2)(1-y^2)} = \cos(\pi - \cos^{-1}z) = -\cos(\cos^{-1}z) = -z$

or, $xy + z = \sqrt{(1-x^2)(1-y^2)}$ or, $(xy+z)^2 = (1-x)^2(1-y^2)$

or, $x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$ or, $x^2 + y^2 + z^2 + 2xyz = 1$ **(Proved)**

**11. (ii) Solution :** Given, $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \theta$ or, $\cos^{-1}\left\{\frac{x}{a}\cdot\frac{y}{b} - \sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}\right\} = \theta$

or, $\frac{xy}{ab} - \sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)} = \cos\theta$ or, $\frac{xy}{ab} - \cos\theta = \sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}$

or, $\frac{x^2y^2}{a^2b^2} - \frac{2xy}{ab}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2}$ [squaring]

or, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\theta = 1 - \cos^2\theta$ or, $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$ **(Proved).**

**12. (i) Solution :** $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{zx}{yr} + \tan^{-1}\frac{xy}{zr} = \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{yz}{xr}\cdot\frac{zx}{yr}\cdot\frac{xy}{zr}}{1 - \frac{zx}{yr}\cdot\frac{xy}{zr} - \frac{xy}{zr}\cdot\frac{yz}{xr} - \frac{yz}{xr}\cdot\frac{zx}{yr}}\right)$

$$= \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{yz}{xr}\cdot\frac{zx}{yr}\cdot\frac{xy}{zr}}{1 - \frac{x^2}{r^2} - \frac{y^2}{r^2} - \frac{z^2}{r^2}}\right) = \tan^{-1}\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{yz}{xr}\cdot\frac{zx}{yr}\cdot\frac{xy}{zr}}{0} = \tan^{-1}\infty = \frac{\pi}{2}$$ **(Proved).**

$\left[\because\ r^2 = x^2 + y^2 + z^2 \text{ or, } 1 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\ \therefore\ 1 - \frac{x^2}{r^2} - \frac{y^2}{r^2} - \frac{z^2}{r^2} = 0\right]$

**12. (ii) Solution :** $\tan^{-1}\frac{yz}{xp} + \tan^{-1}\frac{zx}{yp} + \tan^{-1}\frac{xy}{zp} = \frac{\pi}{2}$ or, $\tan^{-1}\left(\frac{\frac{yz}{xp} + \frac{zx}{yp} + \frac{xy}{zp} - \frac{yz}{xp}\cdot\frac{zx}{yp}\cdot\frac{xy}{zp}}{1 - \frac{zx}{yp}\cdot\frac{xy}{zp} - \frac{xy}{zp}\cdot\frac{yz}{xp} - \frac{yz}{xp}\cdot\frac{zx}{yp}}\right) = \frac{\pi}{2}$

or, $\frac{\frac{yz}{xp} + \frac{zx}{yp} + \frac{xy}{zp} - \frac{xyz}{p^3}}{1 - \frac{x^2}{p^2} - \frac{y^2}{p^2} - \frac{z^2}{p^2}} = \tan\frac{\pi}{2}$ or, $\frac{1 - \frac{x^2}{p^2} - \frac{y^2}{p^2} - \frac{z^2}{p^2}}{\frac{yz}{xp} + \frac{zx}{yp} + \frac{xy}{zp} - \frac{xyz}{p^3}} = \cot\frac{\pi}{2} = 0$

or, $1 - \frac{x^2}{p^2} - \frac{y^2}{p^2} - \frac{z^2}{p^2} = 0$ or, $1 = \frac{x^2}{p^2} + \frac{y^2}{p^2} + \frac{z^2}{p^2}$ $\therefore\ x^2 + y^2 + z^2 = p^2$ **(Proved)**

**13. (ii).Solution :** Given, $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

or, $\tan^{-1}\left(\frac{x+y+z-xyz}{1-yz-zx-xy}\right) = \pi$ or, $\frac{x+y+z-xyz}{1-yz-zx-xy} = \tan\pi = 0$

or, $x + y + z - xyz = 0$ or, $x + y + z = xyz$ **(Proved)**

**14. (ii) Solution :** $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$

or $\cos(\pi\sin\theta) = \sin(\pi\cos\theta) = \cos\left(\frac{\pi}{2} - \pi\cos\theta\right)$ or, $\pi\sin\theta = 2n\pi \pm \left(\frac{\pi}{2} - \pi\cos\theta\right)$ $(n = 0, \pm 1, \pm 2, \ldots\ldots)$

Putting n = 0 we get, $\pi\sin\theta = \pm\left(\frac{\pi}{2} - \pi\cos\theta\right)$ or, $\sin\theta = \pm\frac{1}{2} \mp \cos\theta$

or, $\sin\theta \pm \cos\theta = \pm\frac{1}{2}$ or, $\sin^2\theta + \cos^2\theta \pm 2\sin\theta\cos\theta = \frac{1}{4}$ (squaring both sides)

or, $1 \pm \sin 2\theta = \frac{1}{4}$ or, $\mp\sin 2\theta = 1 - \frac{1}{4} = \frac{3}{4}$ or, $\sin 2\theta = \pm\frac{3}{4}$

or, $2\theta = \sin^{-1}\left(\pm\frac{3}{4}\right) = \pm\sin^{-1}\frac{3}{4}$ $\therefore\ \theta = \pm\frac{1}{2}\sin^{-1}\frac{3}{4}$ **(Proved)**

**15. (i) Solution :** $\tan(\cos^{-1}x) = \sin(\tan^{-1}2)$

or, $\tan\theta = \sin\alpha$ or, $\frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}}$

$\frac{1-x^2}{x^2} = \frac{4}{5}$ or, $\frac{1}{x^2} - 1 = \frac{4}{5}$ or, $\frac{1}{x^2} = 1 + \frac{4}{5}$

or, $\frac{1}{x^2} = \frac{5+4}{5}$ or, $\frac{1}{x^2} = \frac{9}{5}$

$\therefore\ x^2 = \frac{5}{9}$ $\therefore\ x = \pm\frac{\sqrt{5}}{3}$ **(Ans)**

Let $\cos^{-1}x = \theta$ $\therefore\ \cos\theta = x$ $\therefore\ \sin\theta = \sqrt{1-x^2}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}$

Again let $\tan^{-1}2 = \alpha$ $\therefore \tan\alpha = 2$

$\therefore \sec^2\alpha = 1 + \tan^2\alpha = 1 + 4 = 5$ $\therefore\ \cos^2\alpha = \frac{1}{5}$

$\therefore\ \sin^2\alpha = 1 - \cos^2\alpha = 1 - \frac{1}{5} = \frac{4}{5}$ $\therefore\ \sin\alpha = \frac{2}{\sqrt{5}}$

**15. (v) Solution :** $\tan^{-1}\left(\frac{1}{2}\sec x\right) + \cot^{-1}(2\cos x) = \frac{\pi}{3}$

or, $\tan^{-1}\left(\frac{1}{2}\sec x\right) + \tan^{-1}\left(\frac{1}{2\cos x}\right) = \frac{\pi}{3}$ or, $\tan^{-1}\left(\frac{1}{2}\sec x\right) + \tan^{-1}\left(\frac{1}{2}\sec x\right) = \frac{\pi}{3}$

or, $2\tan^{-1}\left(\frac{1}{2}\sec x\right) = \frac{\pi}{3}$ or, $\tan^{-1}\left(\frac{1}{2}\sec x\right) = \frac{\pi}{6}$

or, $\frac{1}{2}\sec x = \tan\frac{\pi}{6}$ or, $\sec x = 2\tan 30^0 = 2\times\frac{1}{\sqrt{3}}$

or, $\cos x = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ $\therefore\ x = \frac{\pi}{6}$ **(Ans)**

**16. (i) Solution :** $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

or, $\tan^{-1}\left(\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\cdot\frac{x+1}{x+2}}\right) = \frac{\pi}{4}$ or, $\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{(x-1)(x+1)}{(x-2)(x+2)}} = \tan\frac{\pi}{4}$

or, $\frac{\frac{(x-1)(x+2) + (x-2)(x+1)}{(x-2)(x+2)}}{\frac{(x-2)(x+2) - (x-1)(x+1)}{(x-2)(x+2)}} = 1$

or, $\frac{x^2 - x + 2x - 2 + x^2 - 2x + x - 2}{x^2 - 4 - x^2 + 1} = 1$ or, $\frac{2x^2 - 4}{-3} = 1$

or, $2x^2 - 4 = -3$ or, $2x^2 = 4 - 3$ or, $2x^2 = 1$ or, $x^2 = \frac{1}{2}$ $\therefore\ x = \pm\frac{1}{\sqrt{2}}$ **(Ans)**

**16. (iii) Solution :** $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{4x+1} = \tan^{-1}\frac{2}{x^2}$

or, $\cot^{-1}(2x+1) + \cot^{-1}(4x+1) = \cot^{-1}\frac{x^2}{2}$

or, $\cot^{-1}\left\{\frac{(2x+1)(4x+1) - 1}{4x+1+2x+1}\right\} = \cot^{-1}\frac{x^2}{2}$

or, $\frac{(2x+1)(4x+1)-1}{6x+2} = \frac{x^2}{2}$ or, $\frac{8x^2+4x+2x+1-1}{3x+1} = x^2$

or, $8x^2 + 6x = x^2(3x+1)$ or, $3x^3 + x^2 - 8x^2 - 6x = 0$

or, $3x^3 - 7x^2 - 6x = 0$ or, $x(3x^2 - 7x - 6) = 0$

or, $x\{3x^2 - 9x + 2x - 6\} = 0$ or, $x(x-3)(3x+2) = 0$

$\therefore$ either $x = 0$ or $x = 3, x = -\frac{2}{3}$

Hence the required solutions are $x = 0, 3, -\frac{2}{3}$ **(Ans)**

**17. (i) Solution :** $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$

or, $\sin^{-1}\frac{12}{x} = \frac{\pi}{2} - \sin^{-1}\frac{5}{x} = \cos^{-1}\frac{5}{x}$ $\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

or, $\sin^{-1}\frac{12}{x} = \sin^{-1}\sqrt{1-\frac{25}{x^2}}$

$\therefore \frac{12}{x} = \sqrt{1-\frac{25}{x^2}}$ $\therefore \frac{144}{x^2} = 1 - \frac{25}{x^2}$ [squaring both sides]

or, $\frac{144}{x^2} + \frac{25}{x^2} = 1$ or, $\frac{169}{x^2} = 1$ or, $x^2 = 169$ $\therefore x = \pm 13$ **(Ans)**

**17. (iii) Solution :** $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$ or, $\frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}2x = \frac{\pi}{3}$

or, $\pi - \left(\cos^{-1}x + \cos^{-1}2x\right) = \frac{\pi}{3}$ or, $\cos^{-1}x + \cos^{-1}2x = \frac{2\pi}{3}$

or, $\cos^{-1}\left\{x \cdot 2x - \sqrt{(1-x^2)(1-4x^2)}\right\} = \frac{2\pi}{3}$

or, $2x^2 - \sqrt{(1-x^2)(1-4x^2)} = \cos\frac{2\pi}{3} = -\frac{1}{2}$

or, $2x^2 + \frac{1}{2} = \sqrt{(1-x^2)(1-4x^2)}$ or, $4x^4 + \frac{1}{4} + 2x^2 = 1 - x^2 - 4x^2 + 4x^4$

or, $7x^2 = 1 - \frac{1}{4} = \frac{3}{4}$

or, $x^2 = \frac{3}{28}$ $\therefore x = \pm\frac{1}{2}\sqrt{\frac{3}{7}}$ **(Ans)**

**18. (ii) Solution :** Let $\sin^{-1}x = \alpha$ , $\sin^{-1}y = \beta$ $\therefore \sin^{-1}x - \sin^{-1}y = \frac{\pi}{3}$ or, $\alpha - \beta = \frac{\pi}{3}$-------(1)

Again, $\cos^{-1}x + \cos^{-1}y = \frac{2\pi}{3}$ or, $\frac{\pi}{2} - \sin^{-1}x + \frac{\pi}{2} - \sin^{-1}y = \frac{2\pi}{3}$

or, $\pi - \alpha - \beta = \frac{2\pi}{3}$ or, $\alpha + \beta = \pi - \frac{2\pi}{3}$ or, $\alpha + \beta = \frac{\pi}{3}$ --------- (2)

From (1) and (2) we get,

$\alpha + \beta = \frac{\pi}{3}, \alpha - \beta = \frac{\pi}{3} \Rightarrow 2\alpha = \frac{2\pi}{3}$ $\therefore \alpha = \frac{\pi}{3}$, $\beta = 0$

$\therefore x = \sin\alpha = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $y = \sin\beta = \sin 0^\circ = 0$

$\therefore$ the required solution is $x = \frac{\sqrt{3}}{2}, y = 0$ **(Ans)**

**19. Solution :** $\sin^{-1}\frac{2a}{1+a^2} + \sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}x$

or, $2\tan^{-1}a + 2\tan^{-1}b = 2\tan^{-1}x$ or, $\tan^{-1}a + \tan^{-1}b = \tan^{-1}x$

or, $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1}x$ or, $\frac{a+b}{1-ab} = x$ $\therefore x = \frac{a+b}{1-ab}$ **(Ans)**

**20. Solution :** Let $\tan^{-1}\alpha = \theta$ $\therefore$ $\tan\theta = \alpha$

or, $\tan(n\pi + \theta) = \alpha$, $n = 0, \pm 1, \ldots\ldots\ldots$ $\therefore$ $\tan^{-1}\alpha = n\pi + \theta$, where $\theta = \tan^{-1}\alpha$. Hence the result. **(Ans)**

**21. Solution :** L.H.S. = $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z)$

$= \tan\left\{\frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y + \frac{\pi}{2} - \cot^{-1}z\right\} = \tan\left\{\frac{3\pi}{2} - \left(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z\right)\right\}$

$= \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$ = **R. H. S. (Proved)**

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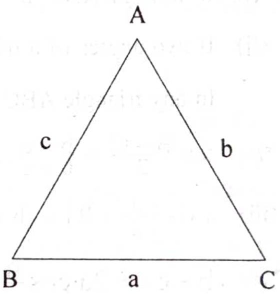
Chapter 5

# Properties of Triangle

**5.1** In $\Delta ABC$, let A, B, C be the measures of the angles at the vertices A, B, and C respectively, and the length of the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ are denoted by a, b and c.

Then $A + B + C = \pi$ and $2s = a + b + c$, where s denote the semi-perimeter of the $\Delta ABC$.

## Formulae:




(i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, R is the circum – radius $\Delta ABC$.

(ii) $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$, $c = a\cos B + b\cos A$

(iii) $a^2 = b^2 + c^2 - 2bc\cos A \qquad \therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$b^2 = c^2 + a^2 - 2ca\cos B \qquad \therefore \cos B = \frac{c^2 + a^2 - b^2}{2ca}$

$c^2 = a^2 + b^2 - 2ab\cos C \qquad \therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

(iv) $\tan A = \frac{abc}{R} \cdot \frac{1}{b^2 + c^2 - a^2} \qquad \tan B = \frac{abc}{R} \cdot \frac{1}{c^2 + a^2 - b^2}$

$\tan C = \frac{abc}{R} \cdot \frac{1}{a^2 + b^2 - c^2}$

$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \qquad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

Area of $\Delta ABC$,

$\Delta = \frac{1}{2}\, ab\, \sin C = \frac{1}{2}\, bc\, \sin A = \frac{1}{2} ca\, \sin B$

$\Delta = \frac{1}{4}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}$

$\Delta = \frac{abc}{4R} = rs$, r denote the in-radius of $\Delta ABC$

$r = 4R\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$

$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

# PROBLEM SET

[Problems with '*' marks are solved at the end of the problem set]

**1.** *(i) In a triangle the sides are in the ratio $\sqrt{2} : 2 : (\sqrt{3}+1)$. Find the possible values of the angles.

**[Ans.** $45^\circ$, $30^\circ$, $105^\circ$] **[HS – 78]**

(ii) If the angles of a triangle are in the ratio 1 : 2 : 3 and the circum-radius is 10 cm., find the lengths of its sides.

**[Ans.** 10 cm., $10\sqrt{3}$ cm., 20 cm.] **[HS – 90]**

*(iii) If the two sides and the included angle are respectively $a = \sqrt{3}+1$, $b = 2$, $\angle C = 60^\circ$, find the other angles and the third side.

**[Ans.** $\sqrt{6}$, $45^\circ$, $75^\circ$] **[HS – 95]**

(iv) Find $\angle B$ and $\angle C$ of a triangle ABC, if b = 2 cm., c = 1 cm. and $\angle A = 60^\circ$. **[Ans.** $90^\circ$, $30^\circ$] **[HS – 87]**

***2.** (i) In triangle ABC, a = 2b and A = 3B, find A, B, C **[WBSC – 83, 90]**

(ii) If two angles of a triangle are $\tan^{-1}2$ and $\tan^{-1}3$, find the third angle. **[Ans.** $45^\circ$] **[WBSC – 08]**

**3.** In any triangle ABC, prove that,

*(i) $\tan\frac{B-C}{2} = \frac{b-c}{b+c} \cdot \cot\frac{A}{2}$

*(ii) $\cos\frac{B-C}{2} = \frac{b+c}{a} \cdot \sin\left(\frac{A}{2}\right)$

(iii) $a\sin\left(\frac{A}{2}+B\right) = (b+c) \cdot \sin\frac{A}{2}$

(iv) $(b-c) \cdot \cos\frac{A}{2} = a\sin\frac{B-C}{2}$

*(v) $b+c == 2a\cos\frac{B-C}{2}$ when $A = 60^0$.

**4.** In any triangle ABC, prove that,

(i) $\frac{b^2-c^2}{a}\cos A + \frac{c^2-a^2}{b}\cos B + \frac{a^2-b^2}{c}\cos C = 0$

*(ii) $\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C = 0$

*(iii) $\frac{1+\cos C\cos(A-B)}{1+\cos B\cos(C-A)} = \frac{a^2+b^2}{a^2+c^2}$

(iv) $\frac{a^2\sin(B-C)}{\sin A} + \frac{b^2\sin(C-A)}{\sin B} + \frac{c^2\sin(A-B)}{\sin C} = 0$

**5.** If in a triangle ABC,

(i) $(a+b+c)(b+c-a) = 3bc$, show that, $A = \frac{\pi}{3}$

*(ii) $\frac{a-b+c}{a} = \frac{b}{b+c-a}$, show that $C = 60^0$. **[HS – 95]**

*(iii) $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, show that $C = 60^0$. **[WBSC – 86, 89]**

*(iv) $a^4 + b^4 + c^4 = 2c^2(a^2+b^2)$, show that $C = 45^\circ$ or $135^\circ$ **[WBSC – 87]**

(v) $a^4 + b^4 + c^4 + a^2b^2 = 2c^2(a^2+b^2)$, show that $C = 60^\circ$ or $120^\circ$.

(vi) $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A\sin B$, prove that $C = 60^\circ$

6. (i) The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$, Find the greatest angle of the triangle. **[Ans.** $120^\circ$**]**

*(ii) The sides of a triangle are $x^2 + 3x + 3$, $2x + 3$ and $x^2 + 2x$, where x is a given positive quantity. Find the greatest angle of the triangle. **[Ans.** $120^\circ$**]**

7. (i) In $\Delta ABC$, if $(a^2 + b^2)\sin(A - B) = (a^2 - b^2)\sin(A + B)$, where $a \neq b$. Prove that the triangle is right-angled.

*(ii) In $\Delta ABC$, if $\dfrac{a^2 + b^2}{a^2 - b^2} = \dfrac{\sin(A+B)}{\sin(A-B)}$, prove that the triangle is either isosceles or right-angled triangle. **[WBSC – 86]**

*(iii) If in $\Delta ABC$, $\cot A + \cot B + \cot C = \sqrt{3}$, show that the triangle is equilateral. **[WBSC – 91]**

*(iv) If $\cos A = \dfrac{\sin B}{2\sin C}$ show that the triangle is isosceles.

(v) If the cosines of two angles of a triangle are proportional to the opposite sides, show that the triangle is isosceles.

*(vi) If cosines of two of the angles of a triangle, which is not isosceles, are inversely proportional to the corresponding opposite sides, show that the triangle is right-angled.

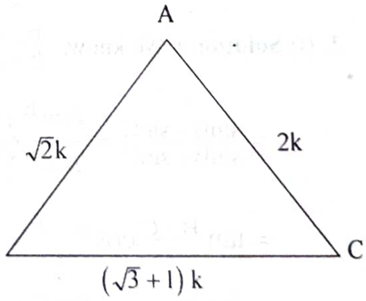
**8.** *(i) If $\tan\theta = \dfrac{2\sqrt{ab}}{a-b}\sin\dfrac{C}{2}$ show that $c = (a - b) \cdot \sec\theta$. **[WBSC – 92]**

(ii) In a triangle ABC, if a, b, c are in A.P., prove that, $\cos A + \cos C = 4(1 - \cos A)(1 - \cos C)$ **[WBSC – 92]**

*(iii) If $(\cos A + 2\cos C) : (\cos A + 2\cos B) = \sin B : \sin C$, then prove that the triangle is either isosceles or right-angled.

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

1. (i) **Solution :** Let the sides of the triangle are $\sqrt{2}k$, $2k$, $(\sqrt{3}+1)k$




$$\cos B = \frac{(\sqrt{2}k)^2 + \{(\sqrt{3}+1)k\}^2 - (2k)^2}{2\cdot\sqrt{2}k\cdot(\sqrt{3}+1)k} = \frac{2+3+1+2\sqrt{3}-4}{2\sqrt{2}(\sqrt{3}+1)} = \frac{2+2\sqrt{3}}{\sqrt{2}(2+2\sqrt{3})}$$

$$\therefore \cos B = \frac{1}{\sqrt{2}} = \cos 45^0$$

Therefore, $B = 45^0$

$$\cos C = \frac{(2k)^2 + \{(\sqrt{3}+1)k\}^2 - (\sqrt{2}k)^2}{2\cdot 2k\cdot(\sqrt{3}+1)k} = \frac{4+3+1+2\sqrt{3}-2}{4(\sqrt{3}+1)} = \frac{6+2\sqrt{3}}{4(\sqrt{3}+1)} = \frac{2\sqrt{3}(\sqrt{3}+1)}{4(\sqrt{3}+1)}$$

$$\therefore \cos C = \frac{\sqrt{3}}{2} = \cos 30^0$$

Therefore, $B = 30^0$

But $A + B + C = 180^0$ $\therefore A = 180^0 - (B + C) = 180^0 - (30^0 + 45^0) = 105^0$

$\therefore$ the required angles are $45^\circ$, $30^\circ$, $105^\circ$ **(Ans)**

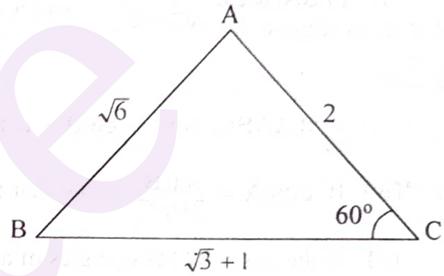
**1.** (iii) **Solution :** Let in $\Delta ABC$, $\angle C = 60^\circ$, $AC = b = 2$, $BC = a = \sqrt{3} + 1$

Let c be the third side.

$$\therefore c^2 = a^2 + b^2 - 2ab\cos C = \left(\sqrt{3}+1\right)^2 + (2)^2 - 2\cdot\left(\sqrt{3}+1\right)\cdot 2\cos 60^0$$

$$= 3+1+2\sqrt{3}+4-4\left(\sqrt{3}+1\right)\cdot\tfrac{1}{2} = 8+2\sqrt{3}-2\sqrt{3}-2 = 6$$

$$\therefore c = \sqrt{6}$$




$$\text{Now, } \cos A = \frac{\left(\sqrt{6}\right)^2 + 2^2 - \left(\sqrt{3}+1\right)^2}{2\cdot\sqrt{6}\cdot 2} = \frac{6+4-3-1-2\sqrt{3}}{4\sqrt{6}}$$

$$\cos A = \frac{2\sqrt{3}\left(\sqrt{3}-1\right)}{4\sqrt{6}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^0$$

$\therefore A = 75^\circ$ $\therefore B = 180^\circ - (75^\circ + 60^\circ) = 180^\circ - 135^\circ = 45^\circ$

$\therefore$ third side $= \sqrt{6}$ and other angles are $45^\circ$ and $75^\circ$. **(Ans)**

**2. Solution :** We know, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ or, $\dfrac{2b}{\sin 3B} = \dfrac{b}{\sin B}$ [since, a = 2b and A = 3B]

or, $\sin 3B = 2\sin B$ or, $3\sin B - 4\sin^3 B = 2\sin B$ or, $3\sin B - 2\sin B = 4\sin^3 B$ or, $\sin B = 4\sin^3 B$

or, $1 = 4\sin^2 B$ or, $\sin^2 B = \frac{1}{4}$ or, $\sin B = \frac{1}{2}$ or, $\sin B = \sin 30^\circ$ $\therefore B = 30^\circ$

$\therefore A = 3B = 3.30^\circ = 90^\circ$ and $A + B + C = 180^\circ$ $\therefore C = 180^\circ - (A + B) = 180^\circ - (90^\circ + 30^\circ) = 180^\circ - 120^\circ = 60^\circ$

$\therefore A = 90^\circ$, $B = 30^\circ$, $C = 60^\circ$ **(Ans)**

**3. (i) Solution :** We know, $\dfrac{b-c}{b+c} = \dfrac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}$ $\left[\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R\right]$

$$= \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} = \frac{\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{\cos\frac{B-C}{2}\sin\frac{B+C}{2}} = \tan\frac{B-C}{2}\cot\frac{B+C}{2}$$

$$= \tan\frac{B-C}{2}\cot\frac{\pi-A}{2} = \tan\frac{B-C}{2}\cot\left(\frac{\pi}{2}-\frac{A}{2}\right)$$

$$\therefore \frac{b-c}{b+c} == \tan\frac{B-C}{2}\tan\frac{A}{2} \quad \text{or, } \tan\frac{B-C}{2} = \frac{b-c}{b+c}\cdot\cot\frac{A}{2}$$ **(Proved)**

**3. (ii) Solution :** We know, $\dfrac{b+c}{a} = \dfrac{2R\sin B + 2R\sin C}{2R\sin A}$ $\left[\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R\right]$

$$= \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\sin A} = \frac{2\sin\frac{\pi-A}{2}\cos\frac{B-C}{2}}{\sin A}$$ [since $A + B + C = \pi$]

$$= \frac{2\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\frac{B-C}{2}}{\sin A} = \frac{2\cos\frac{A}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}} \quad \text{or, } \cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2}$$ **(Proved)**

**3. (v) Solution :** We know, $\frac{b+c}{a} = \frac{2R\sin B + 2R\sin C}{2R\sin A}$ $\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right]$

$$= \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\sin A} = \frac{2\sin\frac{120^0}{2}\cos\frac{B-C}{2}}{\sin 60^0}$$ [Since, A = $60^0$ therefore B + C = $120^0$]

$$= \frac{2\sin 60^0 \cos\frac{B-C}{2}}{\sin 60^0} = 2\cos\frac{B-C}{2}$$

or, $b + c == 2a\cos\frac{B-C}{2}$ **(Proved)**

**4. (ii) Solution :** L. H. S = $\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C$

1st part.= $\frac{b^2-c^2}{a^2}\sin 2A = \frac{4R^2\sin^2 B - 4R^2\sin^2 C}{4R^2\sin^2 A}\cdot 2\sin A\cos A$ $\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right]$

$$= \frac{\sin^2 B - \sin^2 C}{\sin A}\cdot 2\cos A = \frac{\sin(B+C)\sin(B-C)}{\sin A}\cdot 2\cos A$$

$$= \frac{\sin(\pi - A)\sin(B-C)}{\sin A}\cdot 2\cos\{\pi-(B+C)\} \quad [\because A + B + C = \pi]$$

$= \frac{\sin A\sin(B-C)}{\sin A}\cdot 2\{-\cos(B+C)\} = -2\sin(B-C).\cos(B+C) = -\{\sin 2B + \sin(-2C)\} = \sin 2C - \sin 2B$

Similarly, $\frac{c^2-a^2}{b^2}\sin 2B = \sin 2A - \sin 2C$ and $\frac{a^2-b^2}{c^2}\sin 2C = \sin 2B - \sin 2A$

$\therefore$ L. H. S = $\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C$

= sin 2C − sin 2B + sin 2A − sin 2C + sin 2B − sin 2A = 0 = R.H.S **(Proved)**

**4. (iii) Solution :** L. H. S = $\frac{1+\cos C\cos(A-B)}{1+\cos B\cos(C-A)} = \frac{1+\cos\{\pi-(A+B)\}\cos(A-B)}{1+\cos\{\pi-(C+A)\}\cos(C-A)}$

$$= \frac{1-\cos(A+B)\cos(A-B)}{1-\cos(C+A)\cos(C-A)} = \frac{1-\left(\cos^2 A - \sin^2 B\right)}{1-\left(\cos^2 C - \sin^2 A\right)}$$

$$= \frac{1-\cos^2 A + \sin^2 B}{1-\cos^2 C + \sin^2 A} = \frac{\sin^2 A + \sin^2 B}{\sin^2 C + \sin^2 A} = \frac{4R^2\sin^2 A + 4R^2\sin^2 B}{4R^2\sin^2 C + 4R^2\sin^2 A}$$

[multiplying numerator and denominator by $4R^2$]

$= \frac{a^2+b^2}{c^2+a^2} = \frac{a^2+b^2}{a^2+c^2}$ = R. H. S **(Proved)** $\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right]$

**5. (ii) Solution :** Given, $\frac{a-b+c}{a} = \frac{b}{b+c-a}$ or, $\{c + (a-b)\}\{c - (a-b)\} = ab$ or, $c^2 - (a-b)^2 = ab$

or, $c^2 - a^2 - b^2 + 2ab = ab$ or, $a^2 + b^2 - c^2 = ab$

Now, $\cos C = \frac{a^2+b^2-c^2}{2ab}$ or, $\cos C = \frac{ab}{2ab} = \frac{1}{2}$ or, $\cos C = \cos 60^\circ$ $\therefore$ C = $60^\circ$ **(Proved)**

**5 (iii) Solution :** Given, $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ or, $\left(\frac{1}{a+c}-\frac{1}{a+b+c}\right)+\left(\frac{1}{b+c}-\frac{1}{a+b+c}\right)=\frac{1}{a+b+c}$

or, $\frac{b}{(a+c)(a+b+c)}+\frac{a}{(b+c)(a+b+c)}=\frac{1}{a+b+c}$ or, $\frac{b}{a+c}+\frac{a}{b+c}=1$

or, $b^2+bc+a^2+ac=ab+bc+ac+c^2$ or, $a^2+b^2-c^2=ab$

Now, $\cos C=\frac{a^2+b^2-c^2}{2ab}$ or, $\cos C=\frac{ab}{2ab}=\frac{1}{2}$ or, $\cos C=\cos 60^\circ$ $\therefore$ $C=60^\circ$ **(Proved)**

**5. (iv) Solution :** Given, $a^4+b^4+c^4=2c^2(a^2+b^2)$ or, $a^4+b^4+c^4=2a^2c^2+2b^2c^2$

or, $a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2=2a^2b^2$ or, $(a^2+b^2-c^2)^2=2a^2b^2$ or, $a^2+b^2-c^2=\pm\sqrt{2}\,ab$

Now, $\cos C=\frac{a^2+b^2-c^2}{2ab}$ or, $\cos C=\frac{\pm\sqrt{2}\,ab}{2ab}=\pm\frac{1}{\sqrt{2}}$ or, $\cos C=\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}$

or, $\cos C=\cos 45^\circ, \cos 135^\circ$ $\therefore$ $C=45^\circ$ or $135^\circ$ **(Proved)**

**6. (ii) Solution :** Given sides are, $2x+3$, $x^2+3x+3$, $x^2+2x$

$\therefore$ $(x^2+3x+3)-(2x+3)=x^2+3x+3-2x-3=x^2+x>0$

and $(x^2+3x+3)-(x^2+2x)=x^2+3x+3-x^2-2x=x+3>0$

$\therefore$ $x^2+3x+3$ is the greatest sides.

Let $\theta$ be the greatest angle.

Then, $\cos\theta=\frac{(2x+3)^2+(x^2+2x)^2-(x^2+3x+3)^2}{2(2x+3)(x^2+2x)}$

or, $\cos\theta=\frac{4x^2+12x+9+x^4+4x^3+4x^2-x^4-9x^2-9-6x^3-6x^2-18x}{2(2x^3+4x^2+3x^2+6x)}$

or, $\cos\theta=-\frac{2x^3+7x^2+6x}{2(2x^3+7x^2+6x)}$ or, $\cos\theta=-\frac{1}{2}$ or, $\cos\theta=\cos 120^\circ$ $\therefore$ $\theta=120^\circ$

Hence the greatest angle is $120^\circ$ **(Proved)**

**7. (ii) Solution :** Given, $\frac{a^2+b^2}{a^2-b^2}=\frac{\sin(A+B)}{\sin(A-B)}$ or, $\frac{a^2+b^2}{a^2-b^2}=\frac{\sin A\cos B+\cos A\sin B}{\sin A\cos B-\cos A\sin B}$

By comp. and div. we get,

$\frac{a^2+b^2+a^2-b^2}{a^2+b^2-a^2+b^2}=\frac{\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B}{\sin A\cos B+\cos A\sin B-\sin A\cos B+\cos A\sin B}$

or, $\frac{2a^2}{2b^2}=\frac{2\sin A\cos B}{2\cos A\sin B}$ or, $\frac{a^2}{b^2}=\frac{\sin A\cos B}{\cos A\sin B}$

or, $\frac{\sin^2 A}{\sin^2 B}=\frac{\sin A\cos B}{\cos A\sin B}$ $\left[\because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \therefore \frac{a}{b}=\frac{\sin A}{\sin B}\right]$

or, $2\sin A\cos A=2\sin B\cos B$ or, $\sin 2A=\sin 2B$ ---------(1)

or, $2A = 2B \Rightarrow A = B \Rightarrow$ triangle is isosceles.

Again from (1), $\sin 2A = \sin(\pi - 2B) \Rightarrow 2A = \pi - 2B$

or, $2A + 2B = \pi$ or, $A + B = \frac{\pi}{2}$. Therefore $C = \frac{\pi}{2}$ [$\because A + B + C = \pi$ ]

$\Rightarrow$ triangle is right-angled. **(Proved)**

**7. (iii) Solution :** Given, $A + B + C = \pi$ or $B + C = \pi - A$

or, $\cot(B + C) = \cot(\pi - A)$ or, $\frac{\cot B \cot C - 1}{\cot B + \cot C} = -\cot A$

or, $\cot B \cdot \cot C - 1 = -\cot A \cdot \cot B - \cot C \cdot \cot A$ or, $\cot A \cdot \cot B + \cot B \cdot \cot C + \cot A = 1$ ------(1)

Now $\cot A + \cot B + \cot C = \sqrt{3}$ $(\cot A + \cot B + \cot C)^2 = 3$ squaring both sides

or, $\cot^2 A + \cot^2 B + \cot^2 C + 2(\cot A \cdot \cot B + \cot B \cdot \cot C + \cot C \cdot \cot A) = 3$

or, $\cot^2 A + \cot^2 B + \cot^2 C + 2.1 = 3$ or, $\cot^2 A + \cot^2 B + \cot^2 C = 1$ [from (1)]

or, $2(\cot^2 A + \cot^2 B + \cot^2 C) = 2$ or, $2(\cot^2 A + \cot^2 B + \cot^2 C) - 2 = 0$

or, $2\cot^2 A + 2\cot^2 B + 2\cot^2 C - 2(\cot A.\cot B + \cot B.\cot C + \cot C.\cot A) = 0$

or, $(\cot^2 A - 2\cot A \cdot \cot B + \cot^2 B) + (\cot^2 B - 2\cot B \cdot \cot C + \cot^2 C) + (\cot^2 C - 2\cot C \cdot \cot A + \cot^2 A) = 0$

or, $(\cot A - \cot B)^2 + (\cot B - \cot C)^2 + (\cot C - \cot A)^2 = 0$

$\Rightarrow \cot A - \cot B = 0, \cot B - \cot C = 0, \cot C - \cot A = 0 \Rightarrow \cot A = \cot B, \cot B = \cot C, \cot C = \cot A$

$\Rightarrow A = B, B = C, C = A \Rightarrow A = B = C \Rightarrow$ triangle ABC is equilateral. **(Proved)**

**7. (iv) Solution :** Given, $\cos A = \frac{\sin B}{2\sin C}$ or, $\frac{b^2 + c^2 - a^2}{2bc} = \frac{b}{2c}$

$$\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \therefore \frac{\sin B}{\sin C} = \frac{b}{c} \text{ and } \cos C = \frac{b^2 + c^2 - a^2}{2bc}\right]$$

or, $b^2 + c^2 - a^2 = b^2$ or, $c^2 - a^2 = 0$ or, $c^2 = a^2$ or, $c = a$

$\Rightarrow$ the triangle is isosecles. **(Proved)**

**7. (vii) Solution :** By the problem, $\frac{\cos A}{\cos B} = \frac{b}{a}$ or, $\frac{\cos A}{\cos B} = \frac{2R\sin B}{2R\sin A}$ $\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right]$

or, $\sin A \cos A = \sin B \cos B$ or, $2\sin A \cos A = 2\sin B \cos B$ or, $\sin 2A = \sin 2B$ ---------(1)

or, $\sin 2A = \sin(\pi - 2B)$ or, $2A = \pi - 2B$

$\therefore 2A + 2B = \pi \quad \therefore A + B = \frac{\pi}{2}$

$\therefore C = \pi - (A + B) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

Again from (1) $2A = 2B \Rightarrow A = B$, which is not the case here.

$\therefore$ the triangle is right-angled. **(Proved)**

**8. (i) Solution :** Given, $\tan\theta = \frac{2\sqrt{ab}}{a-b}\sin\frac{C}{2}$ or, $\tan^2\theta = \frac{4ab}{(a-b)^2}\sin^2\frac{C}{2}$

or, $\tan^2\theta = \frac{4ab}{(a-b)^2}\left(\frac{1-\cos C}{2}\right) = \frac{2ab}{(a-b)^2}(1-\cos C)$

or, $\tan^2\theta = \frac{2ab}{(a-b)^2}\left(1 - \frac{a^2+b^2-c^2}{2ab}\right) = \frac{2ab}{(a-b)^2}\left(\frac{2ab-a^2-b^2+c^2}{2ab}\right)$

or, $\tan^2\theta = \frac{2ab-a^2-b^2+c^2}{(a-b)^2} = \frac{c^2-(a-b)^2}{(a-b)^2} = \frac{c^2}{(a-b)^2} - 1$

or, $1+\tan^2\theta = \frac{c^2}{(a-b)^2}$ or, $\sec^2\theta = \frac{c^2}{(a-b)^2}$

or, $\sec\theta = \frac{c}{a-b}$ or, $c = (a-b)\sec\theta$. **(Proved)**

**8. (iii) Solution :** Given, (cosA + 2cosC) : (cosA + 2cosB) = sinB : sinC

or, $\frac{\cos A + 2\cos C}{\cos a + 2\cos B} = \frac{\sin B}{\sin C}$ or, cosA . sinC + 2sinC cosC = cosA sinB + 2 sinB cosB

or, cosA(sinB – sinC) + sin 2B – sin 2C = 0

or, $2\cos A\sin\frac{B-C}{2}\cos\frac{B+C}{2} + 2\sin(B-C)\cos(B+C) = 0$

or, $\cos A\sin\frac{B-C}{2}\cos\left(\frac{\pi}{2}-\frac{A}{2}\right) + \sin(B-C)\cos(\pi - A) = 0$ [$\because$ A + B + C = π]

or, $\cos A\sin\frac{B-C}{2}\sin\frac{A}{2} - \sin(B-C)\cos A = 0$ or, $\cos A\left\{\sin\frac{B-C}{2}\sin\frac{A}{2} - \sin(B-C)\right\} = 0$

or, $\cos A\left\{\sin\frac{B-C}{2}\sin\frac{A}{2} - 2\sin\frac{B-C}{2}\cos\frac{B-C}{2}\right\} = 0$ or, $\cos A\sin\frac{B-C}{2}\left\{\sin\frac{A}{2} - 2\cos\frac{B-C}{2}\right\} = 0$

or, $\cos A\sin\frac{B-C}{2}\left\{\sin\left(\frac{\pi}{2}-\frac{B+C}{2}\right) - 2\cos\frac{B-C}{2}\right\} = 0$ [$\because$ A + B + C = π]

or, $\cos A\sin\frac{B-C}{2}\left\{\cos\frac{B+C}{2} - 2\cos\frac{B-C}{2}\right\} = 0$

Now, $\frac{B-C}{2} < \frac{B+C}{2}$ $\therefore \cos\frac{B-C}{2} > \cos\frac{B+C}{2}$

$\therefore 2\cos\frac{B-C}{2} > \cos\frac{B+C}{2}$ $\therefore \cos\frac{B+C}{2} - 2\cos\frac{B-C}{2} \neq 0$

Hence, either cosA = 0 or, $\sin\frac{B-C}{2} = 0$ ⇒ either A = 90° or, B = C.

Hence, the triangle is either right angled or isosceles. **(Proved)**

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# Miscellaneous Exercise

## OBJECTIVE TYPE MULTIPLE CHOICE

1. Find the value of $\cos(-1170^0)$. [WBSC – 03]

2. Value of $\sin 1755^0$ is – (a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) none of these. [WBSC – 10]

3. Value of $\sin(-1755^0)$ is – (a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) none of these [WBSC – 16]

4. The value of $\sec(-945^0)$ is – (a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $-\frac{1}{\sqrt{2}}$ (d) none of these [WBSC – 18]

5. The value of $\sin 15^0 \cdot \sin 75^0$ is : (a) $\frac{1}{2}$, (b) 1, (c) $\frac{1}{4}$, (d) None. [WBSC – 03]

6. The value of $\cos 75^0 - \cos 15^0$ is – (a) $\sqrt{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $-\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{\sqrt{2}}$ [WBSC – 09]

7. Value of $\sin 105^0 + \cos 105^0$ is – (a) $\frac{1}{2}$ (b) 1 (iii) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$ [WBSC – 08, 10]

8. If $\sin\theta = \frac{1}{2}$ and $0 < \theta < \pi$ then the value of $\theta$ are – (a) $\frac{\pi}{6}, 0$ (b) $\frac{\pi}{6}, \frac{5\pi}{6}$ (c) $\frac{5\pi}{6}, \frac{\pi}{2}$ (d) none of these [WBSC – 09]

9. If $\cos x = p$, then the value of $\cos 2x$ is – (a) $2p$ (b) $2p^2$ (c) $2p^2 - 1$ (d) none of these. [WBSC – 09]

10. If $\cos 2\theta = -\frac{1}{2}$, find the value of $\cos\theta$. [WBSC – 03]

11. If $\cos 2\theta = -\frac{1}{2}$ then $\cos\theta$ is – (a) 2, (b) 1, (c) $\frac{1}{2}$ (d) 0 [WBSC – 15]

12. Given that $\tan\theta = -\frac{1}{\sqrt{3}}$ and $\theta$ lies in the 4th quadrant, then the value of $\cos\theta$ is –

    (a) $-\frac{1}{2}$; (b) $-\frac{\sqrt{3}}{2}$; (c) $\frac{\sqrt{3}}{2}$; (d) none of these. [WBSC – 08]

13. $\tan\theta = -\frac{1}{\sqrt{5}}$, $\sin\theta$ is negative then the value of $\cos\theta$ is – (a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{2}$ (c) $\sqrt{\frac{5}{6}}$ (d) $-\sqrt{\frac{5}{6}}$. [WBSC – 14]

14. The value of $\tan\frac{3\pi}{20} \cdot \tan\frac{5\pi}{20} \cdot \tan\frac{7\pi}{20}$ is : (a) 1, (b) –1, (c) 2, (d) none of these. [WBSC – 04, 11]

15. If $10\alpha = \frac{\pi}{2}$, then $\tan 3\alpha \tan 5\alpha \tan 7\alpha$ is (a) 0 (b) 1(c) 2 (d) none. [WBSC – 06, 16]

16. The value of $\sin 420^0 \cos 390^0 - \cos(-300^0)\sin(-330^0)$ is : (a) $-\frac{1}{2}$ (ii) $\frac{1}{2}$ (c) 0 (d) none of these. [WBSC – 04]

17. The value of $\cos 1^0 \cos 2^0 \cos 3^0 \ldots\ldots\ldots\cos 179^0$ is : (a) $\frac{1}{\sqrt{2}}$, (b) 0, (c) 1, (d) none of these. [WBSC – 05]

18. The value of $\cos1^0\cos2^0\cos3^0\cos4^0...\cos89^0...\cos95^0\cos96^0\cos97^0$ = (a) 0 (b) 1 (c) – 1 (d) none of these

[WBSC – 17]

19. The value of $\cos1^0$ $\cos2^0$ $\cos3^0$ .... $\cos89^0$ $\cos90^0$ .... $\cos120^0$ is – (a) 1 (b) (c) 0 (d) $-\frac{1}{2}$ [WBSC – 07]

20. The value of $\tan1^0$ $\tan2^0$ $\tan3^0$ ................. $\tan89^0$ is : (a) 1, (b) 0, (c) $\frac{1}{\sqrt{3}}$, (d) None of these. [WBSC – 06]

21. $\log\tan1^0 + \log\tan2^0 + ........ + \log\tan89^0$ = (a) 1 (b) 0 (c) $\tan1^0$ (d) $\tan89^0$ [WBSC – 08, 16]

22. The value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$ is : (a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{8}$ [WBSC – 05]

23. The minimum value of $\sin\theta + \cos\theta$ is – (a) 0, (b) –1, (c) 2, (d) $\sqrt{2}$ [WBSC – 06, 07, 11, 18]

24. The greatest value of $\sin x \cos x$ is – (a) 1 (b) 2 (c) $\sqrt{2}$ (d) $\frac{1}{2}$ [WBSC – 09]

25. The least value of $\sin^2\theta + \operatorname{cosec}^2\theta$ is – (a) 1 (b) 2 (c) 0 (d) 2.5 [WBSC – 14]

26. The rinimum value of $16\cos^2\theta + 9\sec^2\theta$ is – (a) 20 (b) 22 (c) 24 (d) none [WBSC – 15]

27. $\cos^2\theta + \sec^2\theta$ is – (a) less than 1 (b) equal to 1 (c) more than or equal to 2 (d) greater than 1 [WBSC – 09]

28. The value of $\sin^2\theta + \operatorname{cosec}^2\theta$ is (a) < 1 (b) > 1 (c) 1 (d) none. [WBSC – 12]

29. If $\cos\theta + \sin\theta = 2$, then $\sin2\theta$ = (a) 1 (b) 0 (c) 3 (d) 2 [WBSC – 17]

30. If $\tan x \tan2x = 1$, then the value of $\tan3x$ is – (a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (iii) $\frac{\pi}{2}$ (d) none of these. [WBSC – 07]

31. If $\tan x \tan3x = 1$, then the value of $\tan2x$ is: (a) $\sqrt{3}$, (b) $\frac{1}{\sqrt{3}}$, (c) 1, (d) None of these.

[WBSC – 06, 07, 10, 12, 18]

32. If $\sin\alpha + \cos\alpha = 1$, the value of $\sin^2\alpha$ is – (a) 0 (b) $\frac{1}{2}$ (iii) 1 (d) none [WBSC – 09]

33. The value of $\tan\left(\frac{\pi}{4}+\alpha\right)+\tan\left(\frac{\pi}{4}-\alpha\right)$ is – (a) $\sec2\alpha$ (b) $2\sec\alpha$ (c) $2\sec2\alpha$ (d) none of these. [WBSC – 09]

34. $\cot A + \operatorname{cosec} A = \frac{11}{2}$, then $\cot A$ is – (a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{117}{44}$ (d) $\frac{44}{117}$ [WBSC – 08]

35. If $\sin^2\theta + \sin^4\theta = 1$, then $\tan^4\theta - \tan^2\theta$ is – (a) 1 (b) –1 (c) 0 (d) $\frac{1}{\sqrt{2}}$ [WBSC – 08]

36. If $\sec\theta + \cos\theta = \sqrt{3}$, then the value of $\sec^3\theta + \cos^3\theta$ is – (a) 0 (b) 2 (c) –1 (d) 3 [WBSC – 17]

37. If $p\cos\theta - q\sin\theta + r = 0$, then the value of $p\sin\theta + q\cos\theta$ in terms of p, q and r is –

(a) $\pm\sqrt{p^2+q^2+r^2}$ (b) $\pm\sqrt{p^2+q^2-r^2}$ (c) $\pm\sqrt{p^2-q^2+r^2}$ (d) none of these. [WBSC – 17]

38. If x and y are two distinct real positive numbers, then the relation $\sec\theta = \frac{2xy}{x^2+y^2}$ betrue ? [WBSC – 15]

## SUBJECTIVE TYPE

1. If $x \sin^3 A + y \cos^3 A = \sin A \cos A$ and $x \sin A - y \cos A = 0$, prove that $x^2 + y^2 = 1$ [WBSC – 09]

2. If $\tan 25^0 = a$, show that $\dfrac{\tan 155^0 - \tan 115^0}{1 + \tan 155^0 \tan 115^0} = \dfrac{1-a^2}{2a}$ [WBSC – 10]

3. If $\sin^4 x + \sin^2 x = 1$, show that $\cot^4 x + \cot^2 x = 1$. [WBSC – 05, 06, 07, 09, 16, 17]

4. Prove that $\dfrac{\cos 7^0 + \sin 7^0}{\cos 7^0 - \sin 7^0} = \tan 52^0$ [WBSC – 07]

5. If $A + B + C = \pi$ and $\cos A = \cos B \cos C$, show that,

   (i) $\tan A = \tan B + \tan C$ (ii) $\cot B \cot C = \frac{1}{2}$ [WBSC – 09, `15]

6. If $2\tan\alpha = 3\tan\beta$, show that $\tan(\alpha - \beta) = \dfrac{\sin 2\beta}{5 - \cos 2\beta}$. [WBSC – 03, 07, 12, 17, 19]

7. If $\tan\alpha : \tan\beta = 1 : 4$, then prove that $\tan(\beta - \alpha) = \dfrac{3\sin 2\alpha}{5 - 3\cos 2\alpha}$ [WBSC – 14]

8. If $\tan^2\theta = 1 + 2\tan^2\phi$ show that $\cos 2\phi = 1 + 2\cos 2\theta$ [WBSC – 11]

9. If $\tan(\alpha - \beta) = \dfrac{\sin 2\beta}{5 - \cos 2\beta}$, find $\dfrac{\tan\alpha}{\tan\beta}$ [WBSC – 04]

10. If $A + B = 45^0$, prove that $(1 + \tan A)(1 + \tan B) = 2$. Hence show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$ [WBSC – 17]

11. If $\sin(\alpha + \beta) = 1$, $\sin(\alpha - \beta) = \frac{1}{2}$, $\alpha, \beta$ lie in the interval $\left[0, \frac{\pi}{2}\right]$,

    then find the value of $\tan(\alpha + 2\beta)\tan(2\alpha + \beta)$. [WBSC – 08]

12. If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$, show that: $\tan\dfrac{\theta - \phi}{2} = \pm\sqrt{\dfrac{4 - a^2 - b^2}{a^2 + b^2}}$ [WBSC – 03]

13. If $\alpha$ and $\beta$ are the acute angles and $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$ show that $\tan\alpha = \sqrt{2}\tan\beta$. [WBSC – 10, 18]

14. If $\tan\frac{\theta}{2} = \tan^3\frac{\phi}{2}$ and $\tan\phi = 2\tan\alpha$, then prove that $\theta + \phi = 2\alpha$ [WBSC – 06, 08, 16, '19]

# Inverse Circular Function :

## OBJECTIVE TYPE MULTIPLE CHOICE

1. Value of $\sin^{-1}x + \cos^{-1}x$ is (a) 1 (b) $\frac{\pi}{2}$ (c) 0 (d) none of these. [WBSC – 12]

2. If $\tan^{-1}\left(\frac{1}{2}\sec x\right) + \cot^{-1}(2\cos x) = \frac{\pi}{3}$ then x is : (a) $\frac{\pi}{6}$; (b) $\frac{\pi}{3}$; (c) $\frac{\pi}{4}$; (d) none of these. [WBSC – 04]

3. The value of $\sin\cot^{-1}x$ is – (a) $\sqrt{1+x^2}$, (b) x, (c) $\left(1+x^2\right)^{-\frac{3}{2}}$, (d) $\dfrac{1}{\sqrt{1+x^2}}$ [WBSC – 08]

4. The value of $\sin\left\{\cos^{-1}\left(-\frac{1}{2}\right)\right\}$ is : (a) $\frac{1}{2}$; (b) 1; (c) $\frac{\sqrt{3}}{2}$; (d) none. [WBSC – 03, 07, 09, 16]

5. The value of $\sin^{-1}\left(\frac{\sqrt{5}}{6}\right)+\cos^{-1}\left(\frac{\sqrt{5}}{6}\right)$ is – (a) $\frac{\sqrt{5}}{6}$ (b) 1 (c) $\frac{5}{36}$ (d) $\frac{\pi}{2}$ **[WBSC – 07]**

6. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$ = (a) $\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) none of these **[WBSC – 16]**

7. Value of $\sin\left[\tan^{-1}x+\tan^{-1}\frac{1}{x}\right]$ is – (a) 0 (b) 1 (c) – 1 (d) $\sqrt{2}$ **[WBSC – 18]**

## SUBJECTIVE TYPE

1. If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\theta$, prove that $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=\sin^2\theta$ **[WBSC – 03]**

2. If $\cos^{-1}x+\cos^{-1}y=\theta$, prove that $x^2-2xy\cos\theta+y^2=\sin^2\theta$. **[WBSC – 14]**

3. If $xy=1+a^2$, then prove that $\tan^{-1}\left(\frac{1}{a+x}\right)+\tan^{-1}\left(\frac{1}{a+y}\right)=\tan^{-1}\left(\frac{1}{a}\right), x+y+2a\neq 0$ **[WBSC – 04, 16]**

4. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, show that $x^2+y^2+z^2+2xyz=1$ **[WBSC – 07]**

5. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$ show that $xy+yz+zx=1$ **[WBSC – 07, 09, 11, 17, 18]**

6. If two angles of a triangle are $\tan^{-1}2$ and $\tan^{-1}3$, find the third angle. **[WBSC – 08, 12]**

7. Prove that, $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}.\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$. **[WBSC – 19]**

# General Values :

## SUBJECTIVE TYPE

1. Solve : $\cos\theta-\sin\theta=\frac{1}{\sqrt{2}}$; $0\leq\theta\leq 2\pi$ **[WBSC – 12]**

2. Solve : $\sin\theta-\sqrt{3}\cos\theta=1$, $0\leq\theta\leq 2\pi$ **[WBSC – 04]**

3. Solve : $\sqrt{3}\cos x+\sin x=1, 0<x<2\pi$. **[WBSC – 05, 10, 11]**

4. Solve : $\tan\theta+\tan 2\theta+\tan\theta\tan 2\theta=1$ **[WBSC – 06]**

5. Solve : $\sqrt{3}\cos x+\sin x=\sqrt{2}$, $0<x<2\pi$. **[WBSC – 07, 08, 09]**

6. Solve the trigonometrical equation $\sin\theta+\sin 5\theta=\sin 3\theta$, $0\leq\theta\leq\pi$ **[WBSC – 08]**

N.Tute

# UNIT – 4



N-Tute

Chapter 1

# Function

## 1.1 A Few Definitions :

### Constant :

In mathematical investigation or scientific experiment if the value of a symbol does not change in different situations, then the symbol is called a **constant**.

### There are two types of constants :

(i) Fixed or Absolute constant (ii) Parameter or Arbitrary constant.

### (i) Fixed or Absolute constant :

When the value of a symbol does not change with the change of initial conditions, then the symbol is called a **Fixed constant**.

e.g. the ratio of the circumference and diameter of any circle remains unaltered when the size of the circle changes. Therefore, this ratio is a fixed constant.

Any real number (e.g. 5, $-3$, $\frac{4}{3}$, $\pi$ etc) is a fixed constant.

### (ii) Parameter or Arbitrary constant :

If the value of a symbol does not change for a given set of initial conditions but with the change of initial conditions of investigation the value of the symbol also changes then the symbol is called an **Arbitrary constant**.

e.g., $y = mx$ represents the equation of straight line through the origin.

Let given initial condition be, the straight line make an angle $45^{O}$ with the positive x-axis, then $m = \tan 45^{O} = 1$. i.e., for a given initial condition m remains unaltered.

But if the initial condition changes, (i.e., if the line through the origin makes an angle $60^{O}$ with the positive x-axis, then $m = \tan 60^{O} = \sqrt{3}$ ) the value of m also changes. Hence, here m is a parameter.

## 1.2 Basic Concepts of set :

A set is a well-defined collection of distinct objects. By well-defined we mean some characteristics which determine whether or not a given object belongs to the set.

The examples of sets are :

(i) a set of mathematics books in the library. (ii) a pack of playing cards.

(iii) a set of elements which are vowels in English alphabets.

(iv) set of Natural numbers. (v) set of Integers. etc.

## 1.3 Variable :

Let X be a set of real numbers and x is its any element. Then x is called a variable in the set X. All the elements of the set X is called the **Domain** or **Range** of x.

- **Variables are of two types :**

(i) Discrete variable (ii) Continuous variable

### (i) DISCRETE VARIABLE :

Let x be the digit obtained when an unbised die is thrown. Then the **domain** of x is the set $\{1, 2, 3, 4, 5, 6\}$. Clearly, the variable x assume a finite number of values namely, $x = 1$, $x = 2$, $x = 3$, $x = 4$, $x = 5$ and $x = 6$. Again, if the variable x be a natural number, then the **domain** of x is the set $\{1, 2, 3, 4, 5, .....\}$. Clearly, x can assume countably infinite number of values.

In both the above cases, x is a discrete variable.

A variable is said to be Discrete if its domain contains a finite or countably infinite number of real values.

### (ii) CONTINUOUS VARIABLE :

Let a and b be two real numbers and $a < b$. Let X be the set of real mumbers lying between a and b and x be any element of X. We know that, there are uncountably infinite number of real values between the real numbers a and b ( $> a$).

Hence, the **domain** of x contains uncountably infinite number of real values. Here, x is a continuous variable. A variable is said to be continuous if its domain contains uncountably infinite number of real values.

## 1.4 Interval :

Suppose, a and b are two real numbers and $a < b$; then the set of all real values lying between a and b is called the interval of a and b.

If the end values a and b are excluded then the interval is said to be open and is denoted by $a < x < b$ or $(a, b)$.

If the end values a and b are included then the interval is said to be closed and is denoted by $a \leq x \leq b$ or $[a, b]$.

### Absolute value or Modulus of Real Number :

The absolute value or modulus of real number x is denoted by $|x|$ and is defined as follows :

$$|x| = \begin{cases} x, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -x, & \text{when } x < 0. \end{cases}$$

## 1.5 Function :

If x and y be two real variables, so related, that corresponding to every value of x within a defined domain we get a definite value of y, then y is said to be a function of x defined in its domain.

Here, x is called **independent variable**, and y is called **dependent variable.**

Functions are generally denoted by $f(x)$, $y(x)$, $F(x)$, $\varphi(x)$ etc.

- **Monotonic Function :**

Let $y = f(x)$ is defined in the interval $a \le x \le b$ and $a < x_1 < x_2 < b$. Then, $f(x)$ is said to be

**(i)** monotonic increasing in $a \le x \le b$ if $f(x_2) \ge f(x_1)$

**(ii)** strictly monotonic increasing in $a \le x \le b$ if $f(x_2) > f(x_1)$

**(iii)** monotonic decreasing in $a \le x \le b$ if $f(x_2) \le f(x_1)$

**(iv)** strictly monotonic decreasing in $a \le x \le b$ if $f(x_2) < f(x_1)$.

A function is said to be monotonic if it satisfies any one of the above conditions.

- **Even and odd Function :**

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$

For example, if $f(x) = 4x^2 + 3\cos x$,, then $f(-x) = 4(-x)^2 + 3\cos(-x) = 4x^2 + 3\cos x = f(x)$.

Hence, $f(x) = 4x^2 + 3\cos x$ is an even function.

A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$.

For example, if $f(x) = 6x^3 + 5\sin x$, then $f(-x) = 6(-x)^3 + 5\sin(-x) = -6x^3 - 5\sin x = -(6x^3 + 5\sin x) = -f(x)$

Hence, $f(x)$ is an odd function.

- **Periodic function :**

If p be a least positive real number such that $f(p + x) = f(x)$, for all x in the domain of definition of the function $f(x)$, then $f(x)$ is said to be a periodic function and p is called the period of the periodic function.

For example, if $f(x) = \sin x$, then $f(2\pi + x) = \sin(2\pi + x) = \sin x = f(x)$

$\therefore$ $f(x) = \sin x$ is a periodic function of period $2\pi$.

- **Inverse Function :**

Let $y = f(x)$ be a single–valued function of x, the domain of definition of $f(x)$ being A and its range B. If it is possible to find a single valued function $x = g(y)$ from the given functional relation $y = f(x)$ such that the domain of the function $g(y)$ is B and its range is A, then $x = g(y)$ is called the inverse function of $y = f(x)$ and is generally denoted by $f^{-1}(y)$

For example, let $y = f(x) = 4x - 5$

$\therefore$ $4x = y + 5$ or, $x = \frac{1}{4}(y + 5) = g(y)$

$\therefore$ $f^{-1}(y) = \frac{1}{4}(y + 5)$ is the required inverse function.

## PROBLEM SET

[.....Problems with '*' marks are solved at the end of the problem set.....]

1. (i) If $f(x) = e^{px+q}$, show that, $f(a).f(b).f(c) = f(a+b+c).e^{2q}$; p, q are real.

*(ii) If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$ [WBSC – 85, 04]

*(iii) If $f(x) = \frac{4^x}{4^x+2}$, show that, $f(x) + f(1-x) = 1$

(iv) If $f(x) = \frac{1-x}{1+x}$, find $f\left\{f\left(\frac{1}{x}\right)\right\}$, $x \neq 0$.

*(v) If $f(x) = \frac{ax-b}{bx-a}$, find the value of $f(x)\cdot f\left(\frac{1}{x}\right)$ [WBSC – 00, 06]

*(vi) If $y = f(x) = \frac{ax-b}{cx-a}$, show that $f(y) = x$ [WBSC – 97, 99]

(vii) If $F(x) = \frac{4x-5}{3x-4}$, show that, $F\{F(x)\} = x$

(viii) If $f(x) = \frac{1}{x-1}$, show that $f\left(\frac{a}{b}\right) + f\left(\frac{b}{a}\right) = -1$

*(ix) If $f(x) = \log_e x$ and $g(x) = e^x$, then show that $(gof)x = x$ where $(gof)x = g[f(x)]$ [WBSC – 05]

*(x) If $\phi(x) = \log\sin x$ and $\psi(x) = \log\cos x$, then show that $e^{2\phi(x)} + e^{2\psi(x)} = 1$ [WBSC – 07]

2. *(i) If $f(x) = x - |x|$, then find $f(-4)$ [WBSC – 03]

(ii) If $f(x) = x + |x-1|$, find $f(5)$ and $f(-5)$.

(iii) If $f(x) = x + |x|$, find $f(3)$ and $f(-3)$.

*(iv) If $f(x) = x - 3$, $g(x) = 4 - x$ and $|f(x) + g(x)| < |f(x)| + |g(x)|$. Find x.

*(v) If $f(x) = \frac{|x|}{x}$ and $c(\neq 0)$, a real number, show that $|f(c) - f(-c)| = 2$

3. Find the inverse function of the following

*(i) $y = x + \sqrt{x^2+1}$ (ii) $y = \frac{2x-1}{3-x}, x \neq 3$

*(iii) $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (iv) $y = e^x$, *(v) $y = x^2$

4. (a) Show that the following functions are strictly monotonic increasing function.

*(i) $\frac{x-1}{2x+5}, x \geq 0$ (ii) $\frac{3x+5}{x+2}, x \geq 0$

4 (b) Show that the following functions are strictly monotonic decreasing function

(i) $\cos x$, $0 < x < \frac{\pi}{2}$ *(ii) $\operatorname{cosec} x$, $0 < x < \frac{\pi}{2}$.

**5.** Examine the following functions are even or odd

*(i) $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$ *(ii) $f(x)=\log_e\left(x+\sqrt{x^2+1}\right)$

(iii) $f(x)=2\cos x+\sin^2 x-\frac{3}{x^2}+x^2$

6. Find the period of the following functions :

*(i) tanx *(ii) $f(x)=12\cos\frac{x}{2}+9\sin\frac{x}{2}+1$

**(iii)** $f(x)=12\cos\frac{3x}{2}+9\sin\frac{3x}{2}+3$ **(iv)** 3sin3x + 4cos3x + 1

## ANSWERS

2. (i) –8 (ii) 9, 1 (iii) 6, 0 (iv) x < 3,, x > 4

3. (i) $x=\frac{1}{2}\left(y-\frac{1}{y}\right)$, $y\neq 0$ (ii) $x=\frac{3y+1}{y+2}$, $y\neq -2$ (iii) $x=\sin^{-1}y$, $(-1\leq y\leq 1)$

(iv) $x=\log_e y$, $y>0$ (v) $x=\sqrt{y}$, $0\leq y<\infty$, $x=-\sqrt{y}$, $0<y<\infty$

5. (i) odd (ii) odd (iii) even 6.(i) $\pi$ (ii) $4\pi$ (iii) $\frac{4\pi}{3}$ (iv) $\frac{2\pi}{3}$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1. (ii) Solution :** Given $f(x)=\log\left(\frac{1+x}{1-x}\right)$

$$\therefore f\left(\frac{2x}{1+x^2}\right)=\log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)\quad\left[\text{replacing x by }\frac{2x}{1+x^2}\right]$$

$$=\log\left(\frac{1+x^2+2x}{1+x^2-2x}\right)=\log\left(\frac{1+x}{1+x}\right)^2=2\log\left(\frac{1+x}{1+x}\right)=2f(x)\qquad\left[\because f(x)=\log\left(\frac{1+x}{1-x}\right)\right]$$

$\therefore f\left(\frac{2x}{1+x^2}\right)=2f(x)$ **(Proved)**

**1. (iii) Solution :** Given, $f(x)=\frac{4^x}{4^x+2}$

$$\therefore f(1-x)=\frac{4^{1-x}}{4^{1-x}+2}=\frac{\frac{4}{4^x}}{\frac{4}{4^x}+2}=\frac{4}{4+2\times 4^x}\quad\text{[replacing x by 1 – x] or, }f(1-x)=\frac{2}{4^x+2}$$

$\therefore f(x)+f(1-x)=\frac{4^x}{4^x+2}+\frac{2}{4^x+2}=\frac{4^x+2}{4^x+2}=1$ **(Proved)**

**1. (v) Solution :** Given, $f(x)=\frac{ax-b}{bx-a}$

Replacing x by $\frac{1}{x}$ we get,

$f\left(\frac{1}{x}\right)=\frac{\frac{a}{x}-b}{\frac{b}{x}-a}=\frac{a-bx}{b-ax}=\frac{bx-a}{ax-b}$ $\quad\therefore f(x)\times f\left(\frac{1}{x}\right)=\frac{ax-b}{bx-a}\times\frac{bx-a}{ax-b}=1$ **(Ans)**

**1. (vi) Solution** :Given $y = f(x) = \frac{ax-b}{cx-a}$ ..... (1) $\therefore$ $f(y) = \frac{ay-b}{cy-a}$ ..... (2)

From (1) we get, $y = \frac{ax-b}{cx-a}$ or, $cxy - ay = ax - b$ or, $(cy-a)x = ay - b$

or, $x = \frac{ay-b}{cy-a} = f(y)$ [by (1)] $\therefore$ $f(y) = x$ **(Proved)**

**1. (ix) Solution** : Given $f(x) = \log_e x$ and $g(x) = e^x$

Therefore, $(gof)x = g[f(x)] = g(\log_e x) = e^{\log_e x} = x$ **(Proved)**

**1. (x) Solution** : Given, $\phi(x) = \log\sin x$, $\psi(x) = \log\cos x$

Therefore, $e^{2\phi(x)} + e^{2\psi(x)} = e^{2\log\sin x} + e^{2\log\cos x} = \sin^2 x + \cos^2 x = 1$ **(Proved)**

**2 (i) Solution** : $f(x) = x - |x|$ $\therefore$ $f(-4) = -4 - |-4| = -4 - 4 = -8$ **(Ans)**

**2.(iv) Solution** : Given $f(x) = x - 3$ and $g(x) = 4 - x$ $\therefore$ $f(x) + g(x) = x - 3 + 4 - x = 1$

$\therefore$ $|f(x) + g(x)| = |1| = 1$ ............................ (1)

Let us assume first $x < 3$.

Then $f(x) = x - 3 < 0$ $\therefore$ $|f(x)| = -(x-3) = 3 - x$

and $g(x) = 4 - x > 0$ $\therefore$ $|g(x)| = 4 - x$

$\therefore$ $|f(x)| + |g(x)| = 3 - x + 4 - x = 7 - 2x > 1$ [since $x < 3$] ............................ (2)

From (1) and (2) we get, $|f(x) + g(x)| < |f(x)| + |g(x)|$

Again, let us assume $x > 4$.

Then $f(x) = x - 3 > 0$ $\therefore$ $|f(x)| = x - 3$

and $g(x) = 4 - x < 0$ $\therefore$ $|g(x)| = -(4-x) = x - 4$

$\therefore$ $|f(x)| + |g(x)| = x - 3 + x - 4 = 2x - 7 > 1$ [since $x > 4$] ............................ (3)

From (1) and (3) we get, $|f(x) + g(x)| < |f(x)| + |g(x)|$

Now, if $3 \le x \le 4$ then,

$f(x) = x - 3 \ge 0$ $\therefore$ $|f(x)| = x - 3$

$g(x) = 4 - x \ge 0$ $\therefore$ $|g(x)| = 4 - x$

$\therefore$ $|f(x)| + |g(x)| = x - 3 + 4 - x = 1 = |f(x) + g(x)|$ [by (1)]

Hence, $|f(x) + g(x)| < |f(x)| + |g(x)|$ when $x < 3$ or, $x > 4$. **(Ans)**

**2.(v) Solution** : We know, $|x| = x$ when $x > 0$;

$= 0$ when $x = 0$

$= -x$ when $x < 0$

Now, if $c > 0$ then $f(c) = \frac{|c|}{c} = \frac{c}{c} = 1$ [since $c \neq 0$] and $f(-c) = \frac{|-c|}{-c} = \frac{|c|}{-c} = \frac{c}{-c} = -1$ [since $c \neq 0$]

Therefore, for $c > 0$, $f(c) - f(-c) = 1 - (-1) = 1 + 1 = 2$ ............................ (1)

Again, if $c < 0$ then $f(c) = \frac{|c|}{c} = \frac{-c}{c} = -1$ [since $c \neq 0$] and $f(-c) = \frac{|-c|}{-c} = \frac{|c|}{-c} = \frac{-c}{-c} = 1$ [since $c \neq 0$]

Therefore, for $c < 0$, $f(c) - f(-c) = -1 - 1 = -2$ ............................ (2)

From (1) and (2) we get, $|f(c) - f(-c)| = 2$ **(Proved)**

**3. (i) Solution :** Given, $y = x + \sqrt{x^2+1}$ ............................................. (1)

$\therefore \frac{1}{y} = \frac{1}{x+\sqrt{x^2+1}} = \frac{x-\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)\left(x-\sqrt{x^2+1}\right)} = \frac{x-\sqrt{x^2+1}}{x^2-x^2-1} = \sqrt{x^2+1} - x$ ............................................. (2)

From (1) and (2) we get, $y - \frac{1}{y} = x + \sqrt{x^2+1} - \sqrt{x^2+1} + x = 2x$

$\therefore x = \frac{1}{2}\left(y - \frac{1}{y}\right)$, $y \neq 0$ is the required inverse function. **(Ans)**

**3. (iii) Solution :** $y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Clearly, the set of values of y lies in $-1 \le y \le 1$

Therefore, the inverse of $y = \sin x$ exists and is given by $x = \sin^{-1} y,\ -1 \le y \le 1$. **(Ans)**

**3. (v) Solution :** $y = x^2$

The given function is continuous in $-\infty < x < \infty$ but it is neither monotonic increasing nor decreasing in the interval.

If $y = x^2$ is defined in $0 \le x < \infty$, then it is continuous and monotonic increasing in the interval.

Therefore, the inverse of the function exists and is given by $x = \sqrt{y}$, $0 \le y < \infty$.

Similarly, for $y = x^2$ $(-\infty < x < 0)$, the function is continuous and monotonic decreasing in the interval.

Therefore, the inverse function of $y = x^2$ exists and is given by $x = -\sqrt{y}$, $0 < y < \infty$. **(Ans)**

**4.(a) (i) Solution :** Let $f(x) = \frac{x-1}{2x+5}$, $x \ge 0$

Let $x_1$ and $x_2$ be two values of x such that, $0 \le x_1 < x_2$

Now, $f(x_1) = \frac{x_1-1}{2x_1+5}$, $f(x_2) = \frac{x_2-1}{2x_2+5}$

$\therefore f(x_2) - f(x_1) = \frac{x_2-1}{2x_2+5} - \frac{x_1-1}{2x_1+5} = \frac{(x_2-1)(2x_1+5)-(x_1-1)(2x_2+5)}{(2x_1+5)(2x_2+5)}$

$= \frac{2x_1x_2+5x_2-2x_1-5-2x_1x_2-5x_1+2x_2+5}{(2x_1+5)(2x_2+5)} = \frac{7(x_2-x_1)}{(2x_1+5)(2x_2+5)} > 0 [\because 0 \le x_1 < x_2]$

$\therefore f(x_2) - f(x_1) > 0$, when $x_2 > x_1$. $\therefore f(x_2) > f(x_1)$, when $x_2 > x_1$.

Therefore, f(x) is a strictly monotonic increasing function. **(Proved)**

**4. (b) (ii) Solution :** Let $f(x) = \operatorname{cosec} x,\ 0 < x < \frac{\pi}{2}$.

Let $x_1, x_2$ be two values of x such that $0 < x_1 < x_2 < \frac{\pi}{2}$

Now, $f(x_1) = \operatorname{cosec} x_1$, $f(x_2) = \operatorname{cosec} x_2$

$\therefore f(x_2) - f(x_1) = \operatorname{cosec} x_2 - \operatorname{cosec} x_1 = \frac{1}{\sin x_2} - \frac{1}{\sin x_1} = \frac{\sin x_1 - \sin x_2}{\sin x_1 \sin x_2} = \frac{2\sin\frac{x_1-x_2}{2}\cdot\cos\frac{x_1+x_2}{2}}{\sin x_1 \sin x_2} < 0$

$[\because 0 < x_1 < x_2 < \frac{\pi}{2}, \therefore 0 < \frac{x_1+x_2}{2} < \frac{\pi}{2}, x_1 - x_2 < 0, \sin x_1 > 0, \sin x_2 > 0]$

$\therefore$ for, $0 < x_1 < x_2 < \frac{\pi}{2}$, $f(x_2) - f(x_1) < 0$. or, $f(x_2) < f(x_1)$, when $x_2 > x_1$.

Therefore, $f(x) = \operatorname{cosec} x$, $(0 < x < \frac{\pi}{2})$ is a strictly monotonic decreasing function. **(Proved)**

**5. (i) Solution :** Given, $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$

Rreplacing x by $-x$ we get,

$\therefore\ f(-x)=\sqrt{1+(-x)+(-x)^2}-\sqrt{1-(-x)+(-x)^2}=\sqrt{1-x+x^2}-\sqrt{1+x+x^2}=-\left\{\sqrt{1+x+x^2}-\sqrt{1-x+x^2}\right\}$

$\therefore\ f(-x)=-f(x).$

Therefore, from definition of odd function $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$ is an odd function. **(Ans)**

**5. (ii) Solution :** Given $f(x)=\log_e\left(x+\sqrt{x^2+1}\right)$

Replacing x by $-x$ we get,

$$f(-x)=\log_e\left(-x+\sqrt{(-x)^2+1}\right)=\log_e\left(\sqrt{x^2+1}-x\right)=\log_e\left\{\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}\right\}=\log_e\left\{\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right\}$$

$$=\log_e\left(\frac{1}{\sqrt{x^2+1}+x}\right)=\log_e\left(\sqrt{x^2+1}+x\right)^{-1}=-\log_e\left(\sqrt{x^2+1}+x\right)=-f(x)$$

$\therefore\ f(-x)=-f(x).$

Therefore, from definition of odd function, $f(x)=\log_e\left(x+\sqrt{x^2+1}\right)$ is an odd function.. **(Ans)**

**6. (i) Solution :** Let $f(x)=\tan x$ $\quad\because f(\pi+x)=\tan(\pi+x)$

or, $f(\pi+x)=\tan(90^0\times 2+x)=\tan x=f(x)$ $\therefore$ $\pi$ is the required period.. **(Ans)**

**6.(ii) Solution :** Given, $f(x)=12\cos\frac{x}{2}+9\sin\frac{x}{2}+1$ .................................... (1)

Let, $12=r\sin\theta,\ 9=r\cos\theta$

$\therefore\ 12^2+9^2=r^2$ or, $r^2=144+81=225$ $\therefore$ $r=15$ and $\frac{r\sin\theta}{r\cos\theta}=\frac{12}{9}$ or, $\tan\theta=\frac{4}{3}$ or, $\theta=\tan^{-1}\frac{4}{3}$

From (1) we get, $f(x)=12\cos\frac{x}{2}+9\sin\frac{x}{2}+1=r\left(\sin\theta\cos\frac{x}{2}+\cos\theta\sin\frac{x}{2}\right)+1=r\sin\left(\frac{x}{2}+\theta\right)+1$ ...................... (2)

Now, for a periodic function f(x), $f(x+p)=f(x)$, where p is the period.

From (2) we get, $r\sin\left(\frac{x+p}{2}+\theta\right)+1=r\sin\left(\frac{x}{2}+\theta\right)+1$

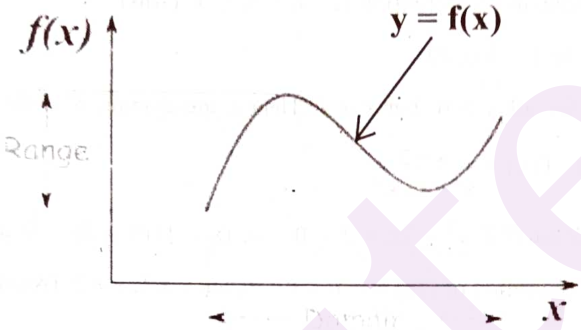
$r\sin\left(\frac{x+p}{2}+\theta\right)=r\sin\left(\frac{x}{2}+\theta\right)$ or, $\sin\left(\frac{x+p}{2}+\theta\right)=\sin\left(2\pi+\frac{x}{2}+\theta\right)$ [since, $\sin(2\pi+\theta)=\sin\theta$]

$\frac{x+p}{2}+\theta=2\pi+\frac{x}{2}+\theta$ or, $\frac{x+p}{2}=2\pi+\frac{x}{2}$ or, $\frac{p}{2}=2\pi$ or, $p=4\pi$ is the required period. **(Ans)**

## • Domain and Range of a Function :

The **domain** of a function f(x) is the set of all values for which the function is defined.

The **range** of the function is the set of all values that the function takes, i. e., the set of all output values of a function.




## PROBLEM SET

[.....Problems with '*' marks are solved at the end of the problem set.....]

**1.** Find the domain of definition of f(x), where

*(i) $f(x) = \sqrt{4-x}$

*(ii) $f(x) = \log_x 10$ *(iii) $f(x) = \dfrac{x-2}{x^2-3x+2}$

(iv) $f(x) = \sqrt{x^2-7x+10}$ *(v) $f(x) = \dfrac{1}{\sqrt{(x-2)(3-x)}}$

(vi) $f(x) = \sqrt{x+1} + \sqrt{4-x}$ *(vii) $f(x) = \sqrt{\log_e\left(\dfrac{4x-x^2}{3}\right)}$

**2.** Indicte the range of the following functions :

**(i)** $y = \sin x, \ -\frac{\pi}{4} \le x \le \pi$ ***(ii)** $y = \cos x, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

***(iii)** $y = \dfrac{x}{1+x^2}$ **[HS – 96, 99]** **(iv)** $y = \dfrac{x}{x^2-5x+9}$

**(v)** $y = \dfrac{1}{2-\cos 3x}$ **(vi)** $y = \dfrac{1}{2-\sin 3x}$

***(vii)** $y = \dfrac{1}{3-\cos 2x}$ **[HS – 00]** **(viii)** $y = \tan x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

## ANSWERS

**1.(i)** $-\infty < x \le 4$ **(ii)** $0 < x < \infty, x \ne 1$ **(iii)** $-\infty < x < \infty, x \ne 1, x \ne 2$ **(iv)** $-\infty < x \le 2$ & $5 \le x < \infty$. **(v)** $2 < x < 3$ **(vi)** $-1 \le x \le 4$ **(vii)** $1 \le x \le 3$. **2.(i)** $0 \le y \le 1$ **(ii)** $0 \le y \le 1$ **(iii)** $-\frac{1}{2} \le y \le \frac{1}{2}$ **(iv)** $-\frac{1}{11} \le y \le 1$ **(v)** $\frac{1}{3} \le y \le 1$ **(vi)** $\frac{1}{3} \le y \le 1$ **(vii)** $\frac{1}{4} \le y \le \frac{1}{2}$ **(viii)** $-\infty < y < \infty$

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## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1 (i) Solution :** Given, $f(x) = \sqrt{4-x}$

Clearly, the given function will be defined when $4 - x$ ³ 0 i.e., when $x \le 4$.

Therefore, the required domain of definition is $-\infty < x \le 4$. **(Ans)**

**1. (ii) Solution :** Given $f(x) = \log_x 10$

The function is defined for all $x > 0$, but $x \neq 1$. Hence, the domain of definition is $0 < x < \infty$ , $x \neq 1$. **(Ans)**

**1. (iii) Solution :** Given, $f(x) = \dfrac{x-2}{x^2 - 3x + 2}$

Clearly, f(x) will be undefined if $x^2 - 3x + 2 = 0$ or, $(x - 1)(x - 2) = 0$ or, $x = 1, 2$

Therefore, the domain of definition of f(x) is, $-\infty < x < \infty$, but $x \neq 1, x \neq 2$. **(Ans)**

**1. (v) Solution :** Given $f(x) = \dfrac{1}{\sqrt{(x-2)(3-x)}}$

Clearly, the function f(x) will be defined if $(x - 2)(3 - x) > 0$

**Case I :** When $x \ge 3$, $3 - x \le 0$ and $(x - 2) > 0$, $\therefore (x - 2)(3 - x) \le 0$

**Case II :** When $x \le 2$, $x - 2 \le 0$ and $3 - x > 0$, $\therefore (x - 2)(3 - x) \le 0$ and

**Case III :** When $2 < x < 3$, $x - 2 > 0$ and $3 - x > 0$ $\therefore (x - 2)(3 - x) > 0$

Therefore, when $2 < x < 3$, $(x - 2)(3 - x) > 0$

Hence, the domain of definition for the given function is $2 < x < 3$ **(Ans)**

**1. (vii) Solution :** Given $f(x) = \sqrt{\log_e\left(\dfrac{4x - x^2}{3}\right)}$

The given function will be defined if

$\log_e\left(\dfrac{4x - x^2}{3}\right) \ge 0$ or, $\dfrac{4x - x^2}{3} \ge e^0 = 1$ or, $4x - x^2 \ge 3$

or, $x^2 - 4x + 3 \le 0$ or, $(x - 1)(x - 3) \le 0$

**Case I :** When $x > 3$, $x - 3 > 0$ and $x - 1 > 0$ $\therefore (x - 1)(x - 3) > 0$

**Case II :** When $x < 1$, $x - 1 < 0$ and $x - 3 < 0$ $\therefore (x - 1)(x - 3) > 0$

**Case III :** When $1 \le x \le 3$, $x - 1 \ge 0$ and $x - 3 \le 0$ $\therefore (x - 1)(x - 3) \le 0$

Therefore, the required domain of definition is $1 \le x \le 3$ **(Ans)**

**2.(ii) Solution :** Given, $y = \cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Clearly, for all x in $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ we get, $0 \le \cos x \le 1$

Therefore, the required range of y is $0 \le y \le 1$ **(Ans)**

**2.(iii) Solution :** Given, $y = \frac{x}{1+x^2}$

Therefore, $y = 0$ at $x = 0$.

For $x \neq 0$, $x^2y - x + y = 0$

$$\text{or, } x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$$

Since, x is real, the range of the function y is determined from the relation,

$$1 - 4y^2 \geq 0 \Rightarrow 4y^2 - 1 \leq 0 \Rightarrow y^2 - \frac{1}{4} \leq 0 \Rightarrow \left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \leq 0$$

**Case I** When $y > \frac{1}{2}$, $y - \frac{1}{2} > 0$ and $y + \frac{1}{2} > 0$ $\quad \therefore \left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) > 0$

**Case II** When $y < -\frac{1}{2}$, $y + \frac{1}{2} < 0$ and $y - \frac{1}{2} < 0$ $\quad \therefore \left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) > 0$

**Case III** When $-\frac{1}{2} \leq y \leq \frac{1}{2}$, $y - \frac{1}{2} \leq 0$ and $y - \frac{1}{2} \leq 0$ $\therefore \left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \leq 0$

Hence, the required range is $-\frac{1}{2} \leq y \leq \frac{1}{2}$ **(Ans)**

**2.(vii) Solution :** Given $y = \frac{1}{3-\cos 2x}$

$$\text{or, } 3 - \cos 2x = \frac{1}{y} \quad \text{or, } \cos 3x = 3 - \frac{1}{y}$$

Now, for real values of x, we must have,

$$-1 \leq \cos 3x \leq 1 \Rightarrow -1 \leq 3 - \frac{1}{y} \leq 1 \Rightarrow -4 \leq -\frac{1}{y} \leq -2 \Rightarrow 4 \geq \frac{1}{y} \geq 2$$

$$\Rightarrow \frac{1}{4} \leq y \leq \frac{1}{2} \quad \left[\because y = \frac{1}{3-\cos 3x} > 0 \text{ for } -1 \leq \cos 3x \leq 1\right]$$

Therefore, the required range of y is $\frac{1}{4} \leq y \leq \frac{1}{2}$ **(Ans)**

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# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. The period of the function tanx is: (a) 0; (b) $\pi$; (c) $2\pi$; (d) none of these. **[WBSC – 03]**

2. If $f(x) = x - |x|$, then $f(-4)$ is : (a) $-4$; (b) $-8$; (c) 0; (d) none of these. **[WBSC – 03]**

3. The function $f(x) = \sqrt{1-x^2}$ is not define for – (a) $x = 1$ (b) $x = 0$ (c) $|x| > 1$ (d) none of these. **[WBSC – 11]**

4. Find the domain of definition of $f(x) = \sqrt{6-x}$ **[WBSC – 12]**

5. Find the domain of the function f(x) where $f(x) = \frac{1}{\sqrt{3-x}}$ **[WBSC – 14]**

6. The domain of the function $f(x) = \frac{1}{\sqrt{6-x}}$ is – (a) $[-\infty, 6]$ (b) $(-\infty, 6)$ (c) $(-\infty, 6]$ (d) none of these **[WBSC – 18]**

7. If $f(x) = e^x$, show that $f(a) \cdot f(b) = f(a+b)$ **[WBSC – 11]**

8. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to : (a) f(x), (b) 2f(x), (c) $f(x^2)$, (d) none of these. **[WBSC – 04]**

9. If $f(x) = \frac{ax-b}{bx-a}$, then $f(x).f\left(\frac{1}{x}\right)$ is – (a) 1, (b) 2, (c) 3, (d) none of these. **[WBSC – 06, 07, 08, 12, 15]**

10. If $g(x) = \frac{1-x}{1+x}$, then find $g\left(\frac{1}{x}\right) + g(x)$ – (a) $-1$ (b) 0 (c) 1 (d) $\frac{1}{2}$ **[WBSC – 14]**

11. If $\phi(x) = \log\sin x$ and $\psi(x) = \log\cos x$ : then $e^{2\phi(x)} + e^{2\psi(x)}$ is (a) 0, (b) 1, (c) 2, (d) none of these **[WBSC – 07, 09, 15]**

12. If $f(x) = \log_e x$ and $g(x) = e^x$, then (gof)x is – (a) $e^x$, (b) $\log_e x$, (c) x, (d) none of these. **[WBSC – 05]**

13. If $f(x) = \log_3 x$ and $\phi(x) = x^2$, then $f\{\phi(3)\}$ is – (a) 2 (b) 3 (c) 9 (d) 4 **[WBSC – 11]**

14. If $f(x) = \frac{4^x}{4^x+2}$, then the value of $f(x) + f(1-x)$ is – (a) 1 (b) 4 (c) $4^x$ (d) $4^x + 4^{-x}$ **[WBSC – 07, 09]**

15. Examine the following function as even or odd : $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$ **[WBSC – 15]**

16. If $f(x) = |x-5|$ then the value of f(3) is – (a) $-8$ (b) $-2$ (c) 2 (d) $-5$ **[WBSC – 17]**

17. If $\alpha(x) = \log_e \sin x$ and $\beta(x) = \log_e \cos x$, then find the value of $e^{\alpha(x)} + e^{\beta(x)}$ **[WBSC – 17]**

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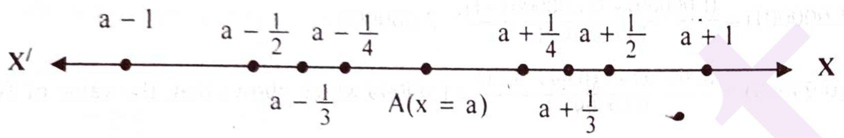
Chapter 2

# Limit, Continuity

## 2.1 Concept of Limit :




Let A represent the real number x = a and P, the variable point on the real axis $\overleftrightarrow{X'X}$ represents a real variable x.

Let $A_1$, $A_2$, $A_3$,..............represent the real number $a + 1$, $a + \frac{1}{2}$, $a + \frac{1}{3}$, ............ We consider the infinite sequence $a + 1$, $a + \frac{1}{2}$, $a + \frac{1}{3}$, .......... when the variable x will gradually approach towards a by assuming the successive values $a + 1$, $a + \frac{1}{2}$, $a + \frac{1}{3}$, ...... of the infinite sequence, then the point P will gradually approach towards the point A from its right side after passing successively through the points $A_1$, $A_2$, $A_3$, ...... ; but it will never meet the point A.

In this case we shall say that the point P approaches A from the left and denote it by the symbol $x \to a + 0$ or, $x \to a+$.

Similarly, we can illustrate $x \to a - 0$ or $x \to a-$

When $x \to a-$ and $x \to a+$, i.e., when the value of $|x - a|$ will be less then any pre-assigned positive quantity, however small, then we shall say that P approaches A and symbolically denoted by $x \to a$.

## 2.2 Division by '0' :

We know, $\frac{24}{2} = 12$ or, $24 = 2 \times 12$

If a = 24, b = 12, c = 2 then above relation will be a = b.c, which shows that for two given numbers a and b the quotient, denoted by $\frac{a}{b}$ (or $a \div b$) is an unique number c, such that a = bc .................................. (1)

[This is the definition of division]

Let a = 4, b = 0, then (1) becomes $4 = 0 \times c = 0$, which is not possible.

Hence, $\frac{4}{0}$ is meaningless or undefined.

Again, let a = 0, b = 0, then (1) becomes, 0 = 0.c, which is true for any value of c and therefore c is not an unique number. Hence $\frac{0}{0}$ has no meaning, hence undefined.

Therefore, division by 0 is undefined.

## 2.3 Limit of a function :

Let $f(x)=\frac{x^2-1}{x-1}$ $\therefore$ $f(1)=\frac{1^2-1}{1-1}=\frac{0}{0}$

Therefore, value of f(x) at x = 1 is indeterminate, i.e., the value of the function f(x) does exist at x = 1.

Now let x = 1·000001 then, $f(1.000001)=\frac{(1.000001+1)(1.000001-1)}{1.000001-1}=2.000001$

Again, let x = 0·99999 then, $f(0.99999)=\frac{(0.99999+1)(0.99999-1)}{0.99999-1}=1.99999$ which shows that, the value of f(x) is close to 2 when the value of x is very close to 1 (but $x \neq 1$).

If the value of x closer to 1, then the value of f(x) will be more close to 2.

Now, for x = 1·000001, 1·0000001, 1·00000001 etc.

f(x) = 2·000001, 2·0000001, 2·00000001 etc, which is in general denoted by

$\underset{x\to a+}{\text{Lt}} f(x)=l_1$ [here, a = 1, $l_1$ = 2], known as right hand limiting value of f(x) at x = a.

Similarly, in the 2nd case, $\underset{x\to a-}{\text{Lt}} f(x)=l_2$, known as left hand limiting value of f(x) at x = a.

When $l_1 = l_2 = l$ (say), i.e., $\underset{x\to a+}{\text{Lt}} f(x)=\underset{x\to a-}{\text{Lt}} f(x)=l$, then limiting value of f(x) at x = a exist and is equal to l and is denoted by $\underset{x\to a}{\text{Lt}} f(x)=l$

## 2.4 Distinction between $\underset{x\to a}{\text{Lt}} f(x)$ and f(a).

$\underset{x\to a}{\text{Lt}} f(x)$ stands for the value of f(x) when x is sufficiently close to a, but not for x = a.

On the contrary f(a) stands for the value of f(x) when x is exactly equato a.

## 2.5 Continuity

For a given function y = f(x), if $\underset{x\to a}{\text{Lt}} f(x)=f(a)$

i.e., $\underset{x\to a-}{\text{Lt}} f(x)=\underset{x\to a+}{\text{Lt}} f(x)=f(a)$ then y = f(x) will be continuous at x = a.

The function f(x) is said to be discontinuous at x = a if

(i) f(a) is undefined or, (ii) $\underset{x\to a}{\text{Lt}} f(x)$ does not exist

or, (iii) $\underset{x\to a}{\text{Lt}} f(x)$ exist but $\underset{x\to a}{\text{Lt}} f(x)\neq f(a)$.

The function f(x) is said to be continuous in the interval $a \le x \le b$ if it is continuous at every point in the interval.

## PROBLEM SET

[.....Problems with '*' marks are solved at the end of the problem set.....]

**1.** A function f(x) is defined as

$$f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$$

Find the value of $\underset{x \to 0}{\text{Lt}}\, f(x)$.

***2.** A function f(x) is defined as

$$f(x) = \begin{cases} x^2 & \text{when } x < 1 \\ 2\cdot5 & \text{when } x = 1 \\ x^2 + 2 & \text{when } x > 1 \end{cases}$$

Examine if $\underset{x \to 1}{\text{Lt}}\, f(x)$ exist. **[WBSC – 87]**

***3.** Discuss the continuity of the following function at x = 2

$$f(x) = \begin{cases} x^2 + 2, & 0 \le x \le 2 \\ x + 4, & 2 \le x \le 3 \\ 6, & x = 2 \end{cases}$$

**[WBSC – 84]**

**4.** A function f(x) is defined in the following way

$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & 0 < x < 1 \\ 2 - x, & x \ge 1, \end{cases}$$

Show that, f(x) is continuous at x = 0 and x = 1. **[WBSC – 85]**

***5.** A function f(x) is defined as follows :

$$f(x) = \begin{cases} 3 + 2x & \text{for } -\frac{3}{2} \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{for } x \ge \frac{3}{2}, \end{cases}$$

Examine the continuity of f(x) at x = 0 and $x = \frac{3}{2}$ **[WBSC – 96, 03]**

**6.** A function is defined as follows :

$$f(x) = \begin{cases} 1 + \sin x, & 0 < x < \frac{\pi}{2} \\ 1 + \left(x - \frac{\pi}{2}\right), & \frac{\pi}{2} \le x < \infty, \end{cases}$$

Discuss the continuity of f(x) at $x = \frac{\pi}{2}$. **[WBSC – 88]**

***7.** If f(x) is defined as

$$f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x^2 - 1} & \text{for } x \ne 1 \\ 2, & \text{for } x = 1, \end{cases}$$

Test the continuity of the function at x = 1. **[WBSC – 89]**

***8.** $f(x)=\dfrac{x^2-16}{x-4}$ is undefined at x = 4. What value must be assigned to f(x) so as to be continuous at x = 4 ?

**[WBSC – 87, 95]**

***9.** Find k, if the function defined by $f(x)=\dfrac{1-\sin x}{(\pi-2x)^2}$, $x \neq \frac{\pi}{2}$ and f(x) = k, $x=\frac{\pi}{2}$ is continuous at $x=\frac{\pi}{2}$.

**[WBSC – 93]**

**10.** What is the domain and range of the function

$f(x)=\dfrac{x^2-3x+2}{x^2+x-2}$. Find the limit of f(x) as x approaches to 2. **[WBSC – 90]**

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**2. Solution :** Left hand limit,

$$\underset{x\to 1-}{\mathrm{Lt}}\ f(x)=\underset{x\to 1-}{\mathrm{Lt}}\left(x^2\right)=1^2=1 \qquad [\because\ f(x)=x^2,\ \text{when } x<1]$$

Right hand limit.

$$\underset{x\to 1+}{\mathrm{Lt}}\ f(x)=\underset{x\to 1+}{\mathrm{Lt}}\left(x^2+2\right)=1^2+2=3 \qquad [\because\ f(x)=x^2+2,\ \text{when } x>1]$$

$\because\ \underset{x\to 1-}{\mathrm{Lt}}\ f(x)\neq \underset{x\to 1+}{\mathrm{Lt}}\ f(x)$. Hence, $\underset{x\to 1}{\mathrm{Lt}}\ f(x)$ does not exist. **(Ans)**

**3. Solution :** Here,

$$\underset{x\to 2-}{\mathrm{Lt}}\ f(x)=\underset{x\to 2-}{\mathrm{Lt}}\left(x^2+2\right)=2^2+2=6 \qquad [\because f(x)=x^2+2,\ \text{when } x<2]$$

$$\underset{x\to 2+}{\mathrm{Lt}}\ f(x)=\underset{x\to 2+}{\mathrm{Lt}}\ (x+4)=2+4=6 \qquad [\because f(x)=x+4,\ \text{when } x>2]$$

and f(2) = 2 + 4 = 6

$\because\ \underset{x\to 2-}{\mathrm{Lt}}\ f(x)=\underset{x\to 2+}{\mathrm{Lt}}\ f(x)=f(2$ i.e., $\underset{x\to 2}{\mathrm{Lt}}\ f(x)=f(2)$

Hence, from definition of continuity f(x) is continuous at x = 2. **(Ans)**

**5. Solution : Test of continuity at x = 0**

$$\underset{x\to 0-}{\mathrm{Lt}}\ f(x)=\underset{x\to 0-}{\mathrm{Lt}}\ (3+2x)=3+0=3 \qquad [\because\ f(x)=3+2x \text{ for } x<0]$$

$$\underset{x\to 0+}{\mathrm{Lt}}\ f(x)=\underset{x\to 0+}{\mathrm{Lt}}\ (3-2x)=3-0=3 \qquad [\because f(x)=3-2x,\ \text{for } x>0]$$

and f(0) = 3 – 2 × 0 = 3

$\because\ \underset{x\to 0-}{\mathrm{Lt}}\ f(x)=\underset{x\to 0+}{\mathrm{Lt}}\ f(x)=f(0)$ i.e., $\underset{x\to 0}{\mathrm{Lt}}\ f(x)=f(0)$

Hence, from definition of continuity, f(x) is continuous at x = 0. **(Ans)**

**Test of continuity at** $x = \frac{3}{2}$

$$\underset{x \to \frac{3}{2}-}{\text{Lt}} f(x) = \underset{x \to \frac{3}{2}-}{\text{Lt}} (3-2x) = 3 - 2 \times \frac{3}{2} = 0 \qquad [\because f(x) = 3 - 2x \text{ for } x < \tfrac{3}{2}]$$

$$\underset{x \to \frac{3}{2}+}{\text{Lt}} f(x) = \underset{x \to \frac{3}{2}+}{\text{Lt}} (-3-2x) = -3 - 2 \times \frac{3}{2} = -6 \; [\because f(x) = -3 - 2x \text{ for } x > \tfrac{3}{2}]$$

$$\because \underset{x \to \frac{3}{2}-}{\text{Lt}} f(x) \neq \underset{x \to \frac{3}{2}+}{\text{Lt}} f(x) \qquad \therefore \underset{x \to \frac{3}{2}}{\text{Lt}} f(x) \text{ does not exist.}$$

Therefore, $f(x)$ is not continuous at $x = \frac{3}{2}$. **(Ans)**

**7. Solution : Test of continuity at x = 1**

Here, $\underset{x \to 1}{\text{Lt}} f(x) = \underset{x \to 1}{\text{Lt}} \frac{x^2 - 4x + 3}{x^2 - 1} = \underset{x \to 1}{\text{Lt}} \frac{(x-1)(x-3)}{(x+1)(x-1)}$ $[\because x \to 1, \therefore x \neq 1]$

$$= \underset{x \to 1}{\text{Lt}} \frac{x-3}{x+1} = \frac{1-3}{1+1} = -1$$

and $f(1) = 2$ $\therefore \underset{x \to 1}{\text{Lt}} f(x) \neq f(1)$

Therefore, $f(x)$ is not continuous at $x = 1$. **(Ans)**

**8. Solution :** By the problem,

$$f(x) = \frac{x^2 - 16}{x - 4}, \; x \neq 4$$

Now, $\underset{x \to 4}{\text{Lt}} f(x) = \underset{x \to 4}{\text{Lt}} \frac{x^2 - 16}{x - 4} = \underset{x \to 4}{\text{Lt}} \frac{(x+4)(x-4)}{x-4}$ $[\because x \to 4 \therefore x \neq 4]$

$$= \underset{x \to 4}{\text{Lt}} (x+4) = 4 + 4 = 8$$

If $f(x) = 8$, at $x = 4$, then $\underset{x \to 4}{\text{Lt}} f(x) = f(4)$

Hence, from definition of continuity, $f(x)$ is continuous at $x = 4$

$\therefore f(x) = 8$ when $x = 4$ **(Ans)**

**9. Solution :** Here $\underset{x \to \frac{\pi}{2}}{\text{Lt}} f(x) = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{1 - \sin x}{(\pi - 2x)^2} = \underset{z \to 0}{\text{Lt}} \frac{1 - \sin\left(\frac{\pi}{2} - \frac{z}{2}\right)}{z^2} = \underset{z \to 0}{\text{Lt}} \frac{1 - \cos\frac{z}{2}}{z^2}$

$$\left[\text{let, } \pi - 2x = z \because x \to \tfrac{\pi}{2}, \; z \to 0 \text{ and } x = \tfrac{\pi}{2} - \tfrac{z}{2}\right]$$

$$= \underset{z \to 0}{\text{Lt}} \frac{2\sin^2\frac{z}{4}}{z^2} = \frac{1}{8}\left(\underset{\frac{z}{4} \to 0}{\text{Lt}} \frac{\sin\frac{z}{4}}{\frac{z}{4}}\right)^2 = \frac{1}{8} \times 1^2 = \frac{1}{8} \qquad \left[\because z \to 0, \tfrac{z}{4} \to 0\right]$$

By the problem, $f(x)$ will be continuous at $x = \frac{\pi}{2}$

if $\underset{x \to \frac{\pi}{2}}{\text{Lt}} f(x) = f\left(\frac{\pi}{2}\right)$ or, $\quad$ or, $\frac{1}{8} = k \quad \therefore k = \frac{1}{8}$ **(Ans)**

## 2.6 Fundamental Theorem on Limit :

Let x be a real variable and $f(x)$ and $\varphi(x)$ are two single-valued function of x. Then,

**(i)** $\underset{x \to a}{\text{Lt}} \{c \cdot f(x)\} = c \cdot \underset{x \to a}{\text{Lt}} f(x)$, c is a constant.

**(ii)** $\underset{x \to a}{\text{Lt}} \{f(x) \pm \phi(x)\} = \underset{x \to a}{\text{Lt}} f(x) \pm \underset{x \to a}{\text{Lt}} \phi(x)$ **(iii)** $\underset{x \to a}{\text{Lt}} \{f(x) \times \phi(x)\} = \underset{x \to a}{\text{Lt}} f(x) \times \underset{x \to a}{\text{Lt}} \phi(x)$

**(iv)** $\underset{x \to a}{\text{Lt}} \left[\dfrac{f(x)}{\phi(x)}\right] = \dfrac{\underset{x \to a}{\text{Lt}} f(x)}{\underset{x \to a}{\text{Lt}} \phi(x)}, \ \underset{x \to a}{\text{Lt}} \phi(x) \neq 0$ **(v)** $\underset{x \to a}{\text{Lt}} \phi\{f(x)\} = \phi\left\{\underset{x \to a}{\text{Lt}} f(x)\right\}$

## 2.7 Some Important Limits :

**(i)** $\underset{x \to a}{\text{Lt}} \dfrac{x^n - a^n}{x - a} = n \cdot a^{n-1}$

**(ii)** $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} = 1$ **(iii)** $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - 1}{x} = 1$ **(iv)** $\underset{x \to 0}{\text{Lt}} \dfrac{1}{x} \log_e(1 + x) = 1$

**(v)** $\underset{x \to 0}{\text{Lt}} (1 + x)^{\frac{1}{x}} = e$ **(vi)** $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - 1}{x} = \log_e a$, $(a > 0)$ **(vii)** $\underset{x \to \infty}{\text{Lt}} \left(1 + \dfrac{1}{x}\right)^x = e$

# PROBLEM SET

[.....Problems with '*' marks are solved at the end of the problem set.....]

**1.** (i) $\underset{x \to 4}{\text{Lt}} \dfrac{x^4 - 256}{x - 4}$ *(ii) $\underset{x \to -2}{\text{Lt}} \dfrac{x^3 + 8}{x^5 + 32}$

(iii) $\underset{x \to y}{\text{Lt}} \dfrac{x^{\frac{9}{2}} - y^{\frac{9}{2}}}{x^{\frac{5}{2}} - y^{\frac{5}{2}}}$ (iv) $\underset{x \to 3}{\text{Lt}} \dfrac{x^{-6} - 3^{-6}}{x^{-4} - 3^{-4}}$

**2.** (i) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 5x}{x}$ (ii) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin \frac{x}{3}}{x}$ **[WBSC – 82]**

*(iii) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \sin 2x}{x - \sin 3x}$ (iv) $\underset{x \to 0}{\text{Lt}} \dfrac{\tan 2x - x}{3x - \sin x}$

**3.** (i) $\underset{x \to 0}{\text{Lt}} \dfrac{e^{mx} - 1}{nx}$ (ii) $\underset{x \to a}{\text{Lt}} \dfrac{e^{x-a} - 1}{x - a}$

(iii) $\underset{x \to 0}{\text{Lt}} \dfrac{e^{\alpha x} - e^{\beta x}}{x}$ *(iv) $\underset{x \to 0}{\text{Lt}} \dfrac{e^{\alpha x} + e^{\beta x} - 2}{x}$

**4.** (i) $\underset{x \to 0}{\text{Lt}} \dfrac{a^{\alpha x} - b^{\beta x}}{x}$ *(ii) $\underset{x \to 0}{\text{Lt}} \dfrac{2^{2x} - 3^{3x}}{x}$ **[WBSC – 84]**

(iii) $\underset{x \to 0}{\text{Lt}} \dfrac{2^x - 3^x}{x}$ (iv) $\underset{x \to 0}{\text{Lt}} \dfrac{5^x - 4^x}{4^x - 3^x}$

**5.** (i) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1 + 3x)}{x}$ (ii) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1 + 2x)}{3x}$

*(iii) $\underset{x \to 2}{\text{Lt}} \dfrac{\log(2x - 3)}{2(x - 2)}$ *(iv) $\underset{x \to e}{\text{Lt}} \dfrac{\log x - 1}{x - e}$

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**6.** (i) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} (1+ax)^{\frac{1}{x}}$

*(ii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} (1+4x)^{\frac{x+2}{x}}$

(iii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 1} x^{\frac{1}{1-x}}$

*(iv) $\displaystyle \mathop{\mathrm{Lt}}_{x \to \infty} \left(\frac{x+6}{x+1}\right)^{x+4}$

**7.** *(i) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{x}{\sqrt{1+x}-1}$ [WBSC – 82]

(ii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

(iii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{a-\sqrt{a^2-x^2}}{x^2}$

*(iv) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 3} \frac{3-\sqrt{6+x}}{\sqrt{3}-\sqrt{6-x}}$

*(v) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{2^x-1}{(1+x)^{\frac{1}{2}}-1}$ [WBSC – 06, 07, 08]

(vi) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\sqrt{1+x}-1}{\log(1+x)}$

(vii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to a} \frac{x\sin a - a\sin x}{x-a}$ [WBSC – 15, 19]

*(viii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to a} \frac{xe^a - ae^x}{x-a}$

*(ix) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{e^{\sin x}-1}{\log(1+3x)}$ [WBSC – 96]

(x) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{e^{3x}-1}{\log(1+5x)}$ [WBSC – 97]

*(xi) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{e^{ax}-1}{\log(1+ax)}$ [WBSC – 06]

*(xii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{e^{2x}-1}{\sin 3x}$ [WBSC – 07]

(xiii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\log(1+\alpha x)}{e^{2x}-1}$

*(xiv) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{e^{\sin x}-1}{\log(1+3x)}$ [WBSC– 00, 08]

*(xv) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{e^{x^2}-\cos x}{x^2}$ [WBSC – 04]

*(xvi) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{e^{x^2}-1}{\sin^2 x}$ [WBSC – 99, 02]

(xvii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\log(1+\sin x)}{x}$

(xviii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to \infty} \frac{3x^2-4x+5}{12x^2+x+3}$ [WBSC – 82]

(xix) $\displaystyle \mathop{\mathrm{Lt}}_{n \to \infty} \frac{2n^2+3n-9}{3n^2+2n+7}$

*(xx) $\displaystyle \mathop{\mathrm{Lt}}_{x \to \infty} \left(\sqrt{1+x+x^2}-x\right)$

*(xxi) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\cos 5x-\cos 7x}{\cos x-\cos 5x}$ [WBSC – 97]

(xxii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\sin 6x+\sin 8x}{\sin 4x+\sin 6x}$

*(xxiii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{1-\sin x}$ [WBSC – 83]

(xxiv) $\displaystyle \mathop{\mathrm{Lt}}_{h \to 0} \frac{1-\cos h}{h\sin h}$

(xxv) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\sin x(1-\cos x)}{x^3}$

(xxvi) $\displaystyle \mathop{\mathrm{Lt}}_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

*(xxvii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \left(\frac{1}{\sin x}-\frac{1}{\tan x}\right)$

*(xxviii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\tan x-\sin x}{x^3}$ [WBSC – 91]

(xxix) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\tan x-\sin 3x}{x^3}$

(xxx) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\cos x-\sec x}{x^2}$

*(xxxi) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\tan 2x-2\sin x}{x^3}$ [WBSC – 92]

*(xxxii) $\displaystyle \mathop{\mathrm{Lt}}_{x \to 0} \frac{\sin x^3\left(1-\cos x^3\right)}{x^9}$

(xxxiii) $\underset{x \to 0}{\text{Lt}} \frac{1-\cos x}{x^2}$ [WBSC – 08]

## ANSWERS

1. (i) 256 (ii) $\frac{3}{20}$ (iii) $\frac{9}{5}y^2$ (iv) $\frac{1}{6}$ 2. (i) 5 (ii) $\frac{1}{3}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$

3. (i) $\frac{m}{n}$ (ii) 1 (iii) a – b (iv) a + b 4.(i) $a\log a - b\log b$ (ii) $\log_e\left(\frac{4}{27}\right)$ (iii) $\log_e\left(\frac{2}{3}\right)$ (iv) $\frac{\log_e\left(\frac{5}{4}\right)}{\log_e\left(\frac{4}{3}\right)}$

5. (i) 3 (ii) $\frac{2}{3}$ (iii) 1 (iv) $\frac{1}{e}$ 6. (i) $e^a$ (ii) $e^8$ (iii) $e^{-1}$ (iv) $e^5$

7. (i) 2 (ii) $\frac{1}{2}$ (iii) $\frac{1}{2a}$ (iv) $-\frac{\sqrt{3}}{3}$ (v) $2\log_e 2$ (vi) $\frac{1}{2}$ (vii) $\sin a - a\cos a$ (viii) $e^a(1-a)$ (ix) $\frac{1}{3}$ (x) $\frac{3}{5}$ (xi) 1 (xii) $\frac{2}{3}$ (xiii) $\frac{\alpha}{2}$ (xiv) $\frac{m}{3}$ (xv) $\frac{3}{2}$ (xvi) 1 (xvii) 1 (xviii) $\frac{1}{4}$ (xix) $\frac{2}{3}$ (xx) $\frac{1}{2}$ (xxi) 1 (xxii) $\frac{7}{5}$ (xxiii) 2 (xxiv) $\frac{1}{2}$ (xxv) $\frac{1}{2}$ (xxvi) 0 (xxvii) 0 (xxviii) $\frac{3}{2}$ (xxix) $\frac{29}{6}$ (xxx) –1 (xxxi) $\frac{3}{2}$ (xxxii) $\frac{1}{2}$ (xxxiii) $\frac{1}{2}$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1. (ii) Solution :** $\underset{x \to -2}{\text{Lt}} \frac{x^3+8}{x^5+32} = \underset{x \to -2}{\text{Lt}} \left[\frac{\frac{x^3+8}{x+2}}{\frac{x^5+32}{x+2}}\right] = \frac{\underset{x \to -2}{\text{Lt}} \frac{x^3-(-2)^3}{x-(-2)}}{\underset{x \to -2}{\text{Lt}} \frac{x^5-(-2)^5}{x-(-2)}}$ [$\because$ x + 2 ≠ 0]

$= \frac{3(-2)^{3-1}}{3(-2)^{5-1}} \left[\because \underset{x \to a}{\text{Lt}} \frac{x^n - a^n}{x-a} = n \cdot a^{n-1}\right] = \frac{3 \times 4}{5 \times 16} = \frac{3}{20}$ **(Ans)**

**2. (iii) Solution :** $\underset{x \to 0}{\text{Lt}} \frac{x-\sin 2x}{x-\sin 3x} = \underset{x \to 0}{\text{Lt}} \left[\frac{\frac{x-\sin 2x}{x}}{\frac{x-\sin 3x}{x}}\right] = \frac{\underset{x \to 0}{\text{Lt}} \left(1-\frac{\sin 2x}{x}\right)}{\underset{x \to 0}{\text{Lt}} \left(1-\frac{\sin 3x}{x}\right)}$ [$\because$ x ≠ 0]

$= \frac{1-2 \cdot \underset{2x \to 0}{\text{Lt}} \frac{\sin 2x}{2x}}{1-3 \cdot \underset{3x \to 0}{\text{Lt}} \frac{\sin 3x}{3x}}$ [$\because x \to 0 \therefore 2x \to 0,\ 3x \to 0$] $= \frac{1-2 \times 1}{1-3 \times 1} = \frac{-1}{-2} = \frac{1}{2}$ **(Ans)** $\left[\because \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1\right]$

**3. (iv) Solution :** $\underset{x \to 0}{\text{Lt}} \frac{e^{\alpha x} - e^{\beta x} - 2}{x} = \underset{x \to 0}{\text{Lt}} \frac{\left(e^{\alpha x}-1\right)+\left(e^{\beta x}-1\right)}{x} = \alpha \cdot \underset{\alpha x \to 0}{\text{Lt}} \frac{e^{\alpha x}-1}{\alpha x} + \beta \cdot \underset{\beta x \to 0}{\text{Lt}} \frac{e^{\beta x}-1}{\beta x}$

[$\because x \to 0 \therefore ax \to 0,\ bx \to 0$]

$= \alpha \times 1 + \beta \times 1 = \alpha + \beta \left[\because \underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{x} = 1\right]$ **(Ans)**

**4. (ii) Solution :** $\underset{x \to 0}{\text{Lt}} \frac{2^{2x} - 3^{3x}}{x} = \underset{x \to 0}{\text{Lt}} \frac{4^x - 27^x}{x} = \underset{x \to 0}{\text{Lt}} \frac{\left(4^x-1\right)-\left(27^x-1\right)}{x}$

$= \underset{x \to 0}{\text{Lt}} \frac{4^x-1}{x} - \underset{x \to 0}{\text{Lt}} \frac{27^x-1}{x} = \log_e 4 - \log_e 27 \left[\because \underset{x \to 0}{\text{Lt}} \frac{a^x-1}{x} = \log_e a\right] = \log_e\left(\frac{4}{27}\right)$ **(Ans)**

**5 (iii) Solution :** $\underset{x \to 2}{\text{Lt}} \frac{\log(2x-3)}{2(x-2)} = \underset{x \to 2}{\text{Lt}} \frac{\log(2x-4+1)}{2x-4} = \underset{z \to 0}{\text{Lt}} \frac{1}{z}\log(1+z) = 1$ **(Ans)**

[Let z = 2x – 4 $\therefore x \to 2,\ z \to 0$]

**5 (iv) Solution :** $\underset{x \to e}{\text{Lt}} \frac{\log x - 1}{x-e} = \underset{z \to 0}{\text{Lt}} \frac{\log(e+z)-1}{z}$ [let x – e = z, $\therefore$ x = z + e, $\because x \to e,\ z \to 0$]

$= \underset{z\to 0}{Lt}\dfrac{\log(e+z)-\log e}{z} = \underset{z\to 0}{Lt}\dfrac{\log\left(\frac{e+z}{e}\right)}{z}$ $[\because \log e = 1]$

$= \underset{\frac{z}{e}\to 0}{Lt}\dfrac{\log\left(1+\frac{z}{e}\right)}{\frac{z}{e}}\times\dfrac{1}{e}$ $[\because z\to 0 \therefore \frac{z}{e}\to 0] = 1\times\frac{1}{e} = \times\frac{1}{e}$ **(Ans)** $\left[\because \underset{x\to 0}{Lt}\frac{1}{x}\log(1+x)=1\right]$

**6 (ii) Solution :** $\underset{x\to 0}{Lt}(1+4x)^{\frac{x+2}{x}} = \underset{x\to 0}{Lt}(1+4x)^{1+\frac{2}{x}} = \underset{x\to 0}{Lt}(1+4x)\times\underset{x\to 0}{Lt}(1+4x)^{\frac{2}{x}}$

$= \underset{x\to 0}{Lt}(1+4x)\times\underset{4x\to 0}{Lt}\left\{(1+4x)^{\frac{1}{4x}}\right\}^8$ $[\because x\to 0, \therefore 4x\to 0]$

$= (1+4\times 0)\times\left\{\underset{4x\to 0}{Lt}(1+4x)^{\frac{1}{4x}}\right\}^8 = e^8$ **(Ans)** $\left[\because \underset{x\to 0}{Lt}(1+x)^{\frac{1}{x}} = e\right]$

**6 (iv) Solution :** $\underset{x\to\infty}{Lt}\left(\dfrac{x+6}{x+1}\right)^{x+4}$ $[\text{let } x=\frac{1}{y}, \because x\to\infty, y\to 0] = \underset{y\to 0}{Lt}\left(\dfrac{\frac{1}{y}+6}{\frac{1}{y}+1}\right)^{\frac{1}{y}+4}$

$= \underset{y\to 0}{Lt}\left(\dfrac{1+6y}{1+y}\right)^{4+\frac{1}{y}} = \underset{y\to 0}{Lt}\left(\dfrac{1+6y}{1+y}\right)^4\times\underset{y\to 0}{Lt}\left(\dfrac{1+6y}{1+y}\right)^{\frac{1}{y}} = \left(\dfrac{1+0}{1+0}\right)^4\times\dfrac{\left\{\underset{6y\to 0}{Lt}(1+6y)^{\frac{1}{6y}}\right\}^6}{\underset{y\to 0}{Lt}(1+y)^{\frac{1}{y}}} = \dfrac{e^6}{e} = e^5$

$[\because y\to 0, \therefore 6y\to 0]$ $= \dfrac{e^6}{e} = e^5$ **(Ans)**

**7 (i) Solution :** $\underset{x\to 0}{Lt}\dfrac{x}{\sqrt{x+1}-1} = \underset{x\to 0}{Lt}\dfrac{x\left(\sqrt{x+1}+1\right)}{\left(\sqrt{x+1}+1\right)\left(\sqrt{x+1}-1\right)} = \underset{x\to 0}{Lt}\dfrac{x\left(\sqrt{x+1}+1\right)}{x+1-1}$

$= \underset{x\to 0}{Lt}\dfrac{x\left(\sqrt{x+1}+1\right)}{x} = \underset{x\to 0}{Lt}\left(\sqrt{x+1}+1\right), [\because x\neq 0] = \sqrt{0+1}+1 = 2$ **(Ans)**

**7 (iv) Solution :** $\underset{x\to 3}{Lt}\dfrac{3-\sqrt{6+x}}{\sqrt{3}-\sqrt{6-x}} = \underset{x\to 3}{Lt}\left[\dfrac{\left(3-\sqrt{6+x}\right)\left(3+\sqrt{6+x}\right)}{3+\sqrt{6+x}}\times\dfrac{\left(\sqrt{3}+\sqrt{6-x}\right)}{\left(\sqrt{3}+\sqrt{6-x}\right)\left(\sqrt{3}-\sqrt{6-x}\right)}\right]$

$= \underset{x\to 3}{Lt}\left[\dfrac{(9-6-x)}{3+\sqrt{6+x}}\times\dfrac{\sqrt{3}+\sqrt{6-x}}{3-6+x}\right] = \underset{x\to 3}{Lt}\left[-\dfrac{(3-x)}{3+\sqrt{6+x}}\times\dfrac{\sqrt{3}+\sqrt{6-x}}{3-x}\right]$ $\because x\neq 3$

$= \underset{x\to 3}{Lt}\left[-\dfrac{\sqrt{3}+\sqrt{6-x}}{3+\sqrt{6+x}}\right] = -\dfrac{\sqrt{3}+\sqrt{6-3}}{3+\sqrt{6+3}} = -\dfrac{\sqrt{3}+\sqrt{3}}{3+3} = -\dfrac{2\sqrt{3}}{6} = -\dfrac{\sqrt{3}}{3}$ **(Ans)**

**7 (v) Solution :** $\underset{x\to 0}{Lt}\dfrac{2^x-1}{(1+x)^{\frac{1}{2}}-1} = \underset{x\to 0}{Lt}\left[\dfrac{\frac{2^x-1}{x}}{\frac{\sqrt{1+x}-1}{x}}\right] = \dfrac{\underset{x\to 0}{Lt}\frac{2^x-1}{x}}{\underset{x\to 0}{Lt}\frac{\sqrt{1+x}-1}{x}}$ $[\because x\neq 0]$ $= \dfrac{\log_e 2}{\underset{x\to 0}{Lt}\frac{\sqrt{1+x}-1}{x}}$

$= \dfrac{\log_e 2}{\underset{x\to 0}{Lt}\frac{\left(\sqrt{1+x}+1\right)\left(\sqrt{1+x}-1\right)}{x\left(\sqrt{1+x}+1\right)}} = \dfrac{\log_e 2}{\underset{x\to 0}{Lt}\frac{1+x-1}{x\left(\sqrt{1+x}+1\right)}} = \dfrac{\log_e 2}{\underset{x\to 0}{Lt}\frac{x}{x\left(\sqrt{1+x}+1\right)}} = \dfrac{\log_e 2}{\underset{x\to 0}{Lt}\left(\frac{1}{\sqrt{1+x}+1}\right)} = \dfrac{\log_e 2}{\frac{1}{1+1}} = 2\log_e 2$ **(Ans)**

**7 (viii) Solution :** $\underset{x\to a}{Lt}\dfrac{xe^a-ae^x}{x-a} = \underset{x\to a}{Lt}\dfrac{(x-a)e^a-ae^x+ae^a}{x-a} = \underset{x\to a}{Lt}e^a - \underset{x\to a}{Lt}\dfrac{e^x-e^a}{x-a}$ $[\because x\neq a]$

$= e^a - ae^a \cdot \underset{x\to a}{\text{Lt}} \frac{e^{x-a}-1}{x-a} = e^a - ae^a \cdot \underset{z\to 0}{\text{Lt}} \frac{e^z-1}{z}$ [Let $z = x - a \therefore x \to a, z \to 0$]

$= e^a - ae^a \times 1 = e^a(1-a)$ **(Ans)**

**7(ix) Solution :** $\underset{x\to 0}{\text{Lt}} \frac{e^{\sin x}-1}{\log(1+3x)} = \underset{x\to 0}{\text{Lt}} \left[\frac{e^{\sin x}-1}{\sin x} \times \frac{\sin x}{x} \times \frac{x}{\log(1+3x)}\right]$ [$\because x \neq 0 \therefore \sin x \neq 0$]

$= \underset{x\to 0}{\text{Lt}} \frac{e^{\sin x}-1}{\sin x} \times \underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} \times \underset{x\to 0}{\text{Lt}} \frac{x}{\log(1+3x)}$ [$\because x \to 0, 3x \to 0, \sin x \to 0$]

$= \underset{\sin x\to 0}{\text{Lt}} \frac{e^{\sin x}-1}{\sin x} \times \underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} \times \frac{1}{\underset{3x\to 0}{\text{Lt}} \frac{1}{3x}\log(1+3x)} \times \frac{1}{3} = 1\times 1\times \frac{1}{1}\times\frac{1}{3} = \frac{1}{3}$ **(Ans)**

**7(xi) Solution :** $\underset{x\to 0}{\text{Lt}} \frac{e^{ax}-1}{\log(1+ax)} = \underset{x\to 0}{\text{Lt}} \left[\frac{e^{ax}-1}{x} \times \frac{x}{\log(1+ax)}\right]$ [$\because x \neq 0$]

$= \underset{ax\to 0}{\text{Lt}} \frac{e^{ax}-1}{ax} \times a \times \underset{x\to 0}{\text{Lt}} \frac{x}{\log(1+ax)}$ [$\because x \to 0, \therefore ax \to 0$] $= 1\times a \times \frac{1}{\underset{ax\to 0}{\text{Lt}} \frac{1}{ax}\log(1+ax)} \times \frac{1}{a} = a\times\frac{1}{1}\times\frac{1}{a} = 1$ **(Ans)**

**7 (xii) Solution :** $\underset{x\to 0}{\text{Lt}} \frac{e^{2x}-1}{\sin 3x} = \underset{2x\to 0}{\text{Lt}} \frac{e^{2x}-1}{2x} \times \frac{1}{\underset{3x\to 0}{\text{Lt}} \frac{\sin 3x}{3x}} \times \frac{2}{3} = \frac{2}{3}$ **(Ans)**

**7 (xiv) Solution :** $\underset{x\to 0}{\text{Lt}} \frac{e^{mx}-1}{\log(1+3x)} = \underset{x\to 0}{\text{Lt}} \left[\frac{e^{mx}-1}{x} \times \frac{x}{\log(1+3x)}\right]$ [$\because x \neq 0$]

$= \underset{mx\to 0}{\text{Lt}} \frac{e^{mx}-1}{mx} \times m \times \underset{x\to 0}{\text{Lt}} \frac{x}{\log(1+3x)}$ [$\because x \to 0, \therefore mx \to 0, 3x \to 0$]

$= 1\times m \times \frac{1}{\underset{3x\to 0}{\text{Lt}} \frac{1}{3x}\log(1+3x)} \times \frac{1}{3} = m\times\frac{1}{1}\times\frac{1}{3} = \frac{m}{3}$ **(Ans)**

**7 (xv) Solution :** $\underset{x\to 0}{\text{Lt}} \frac{e^{x^2}-\cos x}{x^2} = \underset{x\to 0}{\text{Lt}} \frac{\left(e^{x^2}-1\right)+(1-\cos x)}{x^2}$

$= \underset{x\to 0}{\text{Lt}} \frac{e^{x^2}-1}{x^2} + \underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{x^2} = \underset{x^2\to 0}{\text{Lt}} \frac{e^{x^2}-1}{x^2} + \underset{x\to 0}{\text{Lt}} \frac{2\sin^2\frac{x}{2}}{x^2}$ [$\because x \to 0 \therefore x^2 \to 0$]

$= 1 + \underset{x\to 0}{\text{Lt}} \frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2} = 1 + \frac{1}{2}\cdot\left(\underset{\frac{x}{2}\to 0}{\text{Lt}} \frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$ $\left[\because x \to 0 \therefore \frac{x}{2} \to 0\right]$ $= 1+\frac{1}{2}\cdot 1 = \frac{3}{2}$ **(Ans.)**

**7 (xvi) Solution :** $\underset{x\to 0}{\text{Lt}} \frac{e^{x^2}-1}{\sin^2 x} = \underset{x\to 0}{\text{Lt}} \left[\frac{e^{x^2}-1}{x^2} \times \frac{x^2}{\sin^2 x}\right]$ [$\because x \neq 0$]

$= \underset{x^2\to 0}{\text{Lt}} \frac{e^{x^2}-1}{x^2} \times \underset{x\to 0}{\text{Lt}} \frac{x^2}{\sin^2 x}$ [$\because x \to 0, x^2 \to 0$] $= 1\times\frac{1}{\left(\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x}\right)^2} = 1\times\frac{1}{1^2} = 1$ **(Ans)**

**7(xx) Solution :** $\underset{x\to\infty}{\text{Lt}} \left[\sqrt{1+x+x^2} - x\right]$ [let $x = \frac{1}{y}$, $\because x \to \infty, y \to 0$]

$$= \underset{y\to 0}{\text{Lt}}\left[\sqrt{1+\frac{1}{y}+\frac{1}{y^2}}-\frac{1}{y}\right] = \underset{y\to 0}{\text{Lt}}\left[\frac{\sqrt{1+y+y^2}}{y}-\frac{1}{y}\right] = \underset{y\to 0}{\text{Lt}}\left[\frac{\sqrt{1+y+y^2}-1}{y}\right]$$

$$= \underset{y\to 0}{\text{Lt}}\left[\frac{\left(\sqrt{1+y+y^2}-1\right)\left(\sqrt{1+y+y^2}+1\right)}{y\left(\sqrt{1+y+y^2}+1\right)}\right] = \underset{y\to 0}{\text{Lt}}\left[\frac{y(1+y)}{y\left(\sqrt{1+y+y^2}+1\right)}\right] = \underset{y\to 0}{\text{Lt}}\left[\frac{1+y}{\sqrt{1+y+y^2}+1}\right] = \frac{1}{1+1} = \frac{1}{2}\ \textbf{(Ans)}$$

**7 (xxi) Solution :** $\underset{x\to 0}{\text{Lt}}\dfrac{\cos 5x-\cos 7x}{\cos x-\cos 5x} = \underset{x\to 0}{\text{Lt}}\dfrac{2\sin\frac{5x+7x}{2}\cdot\sin\frac{7x-5x}{2}}{2\sin\frac{x+5x}{2}\cdot\sin\frac{5x-x}{2}} = \underset{x\to 0}{\text{Lt}}\dfrac{2\sin 6x\cdot\sin x}{2\sin 3x\cdot\sin 2x}$

$= \underset{x\to 0}{\text{Lt}}\dfrac{2\sin 3x\cos 3x\sin x}{2\sin 3x\,2\sin x\cos x} = \underset{x\to 0}{\text{Lt}}\dfrac{\cos 3x}{\cos x} = \dfrac{1}{1} = 1$ **(Ans)** $[\because\ x\neq 0,\ \therefore\ \sin x\neq 0,\ \sin 3x\neq 0]$

**7 (xxiii) Solution :** $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\dfrac{\cos^2 x}{1-\sin x} = \underset{x\to\frac{\pi}{2}}{\text{Lt}}\dfrac{1-\sin^2 x}{1-\sin x} = \underset{x\to\frac{\pi}{2}}{\text{Lt}}\dfrac{(1-\sin x)(1+\sin x)}{1-\sin x}$

$= \underset{x\to\frac{\pi}{2}}{\text{Lt}}(1+\sin x) = 1+\sin\frac{\pi}{2} = 1+1 = 2$ **(Ans)** $[\because\ x\neq\frac{\pi}{2},\ \therefore\ 1-\sin x\neq 0]$

**7 (xxvii) Solution :** $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{\sin x}-\dfrac{1}{\tan x}\right) = \underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{\sin x}\right) = \underset{x\to 0}{\text{Lt}}\dfrac{1-\cos x}{\sin x}$

$= \underset{x\to 0}{\text{Lt}}\dfrac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \underset{x\to 0}{\text{Lt}}\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\ \left[\because x\neq 0\ \therefore \sin\frac{x}{2}\neq 0\right] = \dfrac{0}{1} = 0$ **(Ans)**.

**7 (xxviii) Solution :** $\underset{x\to 0}{\text{Lt}}\dfrac{\tan x-\sin x}{x^3} = \underset{x\to 0}{\text{Lt}}\dfrac{\frac{\sin x}{\cos x}-\sin x}{x^3} = \underset{x\to 0}{\text{Lt}}\dfrac{\sin x\left(\frac{1}{\cos x}-1\right)}{x^3} = \underset{x\to 0}{\text{Lt}}\dfrac{\sin x}{x}\times\underset{x\to 0}{\text{Lt}}\dfrac{1-\cos x}{x^2\cos x}$

$= 1\times\underset{x\to 0}{\text{Lt}}\dfrac{1}{\cos x}\times\underset{x\to 0}{\text{Lt}}\dfrac{2\sin^2\frac{x}{2}}{x^2} = \dfrac{1}{1}\times\left(\underset{\frac{x}{2}\to 0}{\text{Lt}}\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\times\dfrac{1}{2} = \dfrac{1}{1}\times 1^2\times\dfrac{1}{2} = \dfrac{1}{2}\ \left[\because x\to 0\ \therefore \dfrac{x}{2}\to 0\right]$ **(Ans)**

**7 (xxxi) Solution :** $\underset{x\to 0}{\text{Lt}}\dfrac{\tan 2x-2\sin x}{x^3} = \underset{x\to 0}{\text{Lt}}\dfrac{\frac{\sin 2x}{\cos 2x}-2\sin x}{x^3} = \underset{x\to 0}{\text{Lt}}\dfrac{\frac{2\sin x\cos x}{\cos 2x}-2\sin x}{x^3}$

$= 2\cdot\underset{x\to 0}{\text{Lt}}\dfrac{\sin x}{x}\times\underset{x\to 0}{\text{Lt}}\dfrac{\cos x-\cos 2x}{x^2\cos 2x} = 2\times 1\times\underset{x\to 0}{\text{Lt}}\dfrac{1}{\cos 2x}\times\underset{x\to 0}{\text{Lt}}\dfrac{2\sin\frac{3x}{2}\sin\frac{x}{2}}{x^2} = 2\times\dfrac{1}{1}\times 2\times\underset{x\to 0}{\text{Lt}}\dfrac{\sin\frac{3x}{2}}{x}\times\underset{x\to 0}{\text{Lt}}\dfrac{\sin\frac{x}{2}}{x}$

$= \underset{\frac{3x}{2}\to 0}{\text{Lt}}\dfrac{\sin\frac{3x}{2}}{\frac{3x}{2}}\times\dfrac{3}{2}\times\underset{\frac{x}{2}\to 0}{\text{Lt}}\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\times\dfrac{1}{2}\ \left[\because x\to 0\ \therefore \dfrac{3x}{2}\to 0,\ \dfrac{x}{2}\to 0\right] = 1\times\dfrac{3}{2}\times 1\times\dfrac{1}{2} = \dfrac{3}{4}$ **(Ans)**

**7 (xxxii) Solution :** $\underset{x\to 0}{\text{Lt}}\dfrac{\sin x^3\left(1-\cos x^3\right)}{x^9}$ $[\text{let } x^3 = y,\ \because\ x\to 0,\ y\to 0]$

$= \underset{y\to 0}{\text{Lt}}\dfrac{\sin y(1-\cos y)}{y^3} = \underset{y\to 0}{\text{Lt}}\dfrac{\sin y}{y}\times\underset{y\to 0}{\text{Lt}}\dfrac{1-\cos y}{y^2} = 1\times\underset{y\to 0}{\text{Lt}}\dfrac{2\sin^2\frac{y}{2}}{y^2}$

$$= \frac{1}{2}\times\left(\underset{\frac{y}{2}\to 0}{\text{Lt}}\ \frac{\sin\frac{y}{2}}{y}\right)^2 = \frac{1}{2}\times 1^2 = \frac{1}{2}\left[\because y\to 0 \therefore \frac{y}{2}\to 0\right] \textbf{(Ans)}$$

# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. $\underset{x\to 0}{\text{Lt}}\ \frac{a^x-1}{x}$ is (a) 1 (b) loge (c) $\log_e a$ (d) none of these. **[WBSC – 12]**

2. $\underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{x}$ is 1 – (a) True (b) False. **[WBSC – 08]**

3. Evaluate : $\underset{x\to 0}{\text{Lt}}\ \frac{\sin 4x}{x}$. **[WBSC – 12]**

4. If $f(x)=\frac{x^3-8}{x^2-4}$, then $\underset{x\to 2}{\text{Lt}}\ f(x)$ is – (a) 2 (b) 4 (c) 8 (d) none of these **[WBSC – 09]**

5. $\underset{x\to 0}{\text{Lt}}\ \frac{\sin 3x}{\sin 2x}$ is – (a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 1 (d) none of these. **[WBSC – 09, 15]**

6. $\underset{x\to 0}{\text{Lt}}\ \frac{\tan px}{\tan qx}$ is – (a) $\frac{q}{p}$ (b) $\frac{p}{q}$ (c) 1 (d) 0 **[WBSC – 10]**

7. $\underset{x\to 0}{\text{Lt}}\ \frac{\tan 3x}{\tan x}$ is (a) $\frac{1}{3}$ (b) 1 (c) 3 (d) 0 **[WBSC – 14, 16]**

8. The value of $\underset{x\to \log_e 3}{\text{Lim}}\ e^{3x+2}$ is – (a) $e^2$ (b) $9e^2$ (c) $27e^2$ (d) none of these. **[WBSC – 07]**

9. $\underset{x\to 0}{\text{Lt}}\ \frac{1-\cos x}{x^2}$ is – (a) 2, (b) $\frac{1}{2}$, (c) 1, (d) None is true. **[WBSC – 07, 08]**

10. $\underset{x\to 0}{\text{Lt}}\ \frac{e^{2x}-1}{\sin 3x}$ is - (a) 3, (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) none of these. **[WBSC – 07]**

11. Evaluate : $\underset{x\to 0}{\text{Lt}}\ \frac{e^{3x}-1}{\sin 5x}$ **[WBSC – 16]**

12. $\underset{x\to 0}{\text{Lt}}\ \frac{e^{ax}-1}{\log(1+ax)}$ is (a) 1, (b) a, (c) –a, (d) none of these. **[WBSC – 06, 15]**

13. $\underset{x\to 0}{\text{Lt}}\dfrac{\cos x-\cos 3x}{x^2}$ is – (a) $\frac{1}{4}$ (b) 1 (c) 4 (d) none of these. [WBSC – 11]

14. If $\underset{x\to 0}{\text{Lt}}\dfrac{\log_e(1+mx)}{3x}=1$ then the value of m is – (a) 1 (b) $\frac{1}{3}$ (c) 3 (d) none of these [WBSC – 17]

15. The value of $\underset{x\to 0}{\text{Lt}}\dfrac{\cos x}{\frac{\pi}{2}-x}$ is – (a) 1 (b) 0 (c) 2 (d) –1 [WBSC – 18]

## SUBJECTIVE TYPE

1. $\underset{x\to 0}{\text{Lim}}\dfrac{e^{mx}-1}{\log_e(1+3x)}$ [WBSC – 08]

2. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{e^{3x}-1}{\log(1+5x)}$ [WBSC – 12]

3. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{e^{x^2}-\cos x}{x^2}$ [WBSC – 04]

4. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{e^{2x}-1}{\sin 3x}$ [WBSC – 14]

5. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{e^{2x}-e^{3x}}{\log_e(1+x)}$ [WBSC – 17]

6. Evaluate $\underset{x\to 0}{\text{Lt}}\dfrac{2^x-1}{\sqrt{1+x}-1}$ [WBSC – 06, 07, 08, 09, 11, 17]

7. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{\sqrt{1+x}-1}{\log_e(1+x)}$ [WBSC – 14]

8. Evaluate – $\underset{x\to 0}{\text{Lt}}\dfrac{\cos 5x-\cos 7x}{\cos x-\cos 5x}$ [WBSC – 09]

9. Evaluate : $\underset{x\to \frac{\pi}{2}}{\text{Lt}}\dfrac{1-\sin^3 x}{\cos^2 x}$ [WBSC – 09]

10. Evaluate $\underset{x\to a}{\text{Lt}}\dfrac{x\sin a-a\sin x}{x-a}$ [WBSC – 15, 19]

11. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{(e^x+1)\log(1+x)}{\sin x}$ [WBSC – 18]

12. Find the value of k for which $f(x)=\dfrac{\sin 5x}{3x}$, if $x\neq 0$

$= k$, if $x = 0$, is continuous at $x = 0$. [WBSC – 09]

13. A function f(x) is defined as follows:

N.Tute

$f(x) = 3 + 2x$ for, $-\frac{3}{2} \le x < 0$

$= 3 - 2x$ for, $0 \le x < \frac{3}{2}$

$= -3 - 2x$ for, $x \ge \frac{3}{2}$

Show that $f(x)$ is continuous at $x = 0$ and discontinuous at $x = \frac{3}{2}$ [WBSC – 03]

**14.** Discuss the continuity of the following function at $x = 2$.

$$f(x) = x^2 + 2, \quad 0 \le x \le 2$$
$$= x + 4, \quad 2 \le x \le 3$$
$$= 6, \quad x = 2$$

[WBSC – 15]

**15.** A function is defined by

$$F(x) = |x|, \quad x \ne 0$$
$$= 0, \quad x = 0$$

Test the continuity of the function at $x = 0$. [WBSC – 16]

**16.** Discuss the continuity of the following function at $x = 2$,

$$f(x) = \begin{cases} x^2 + 2, 0 \le x \le 2 \\ x + 4, \quad 2 \le x \le 3 \\ 2, \quad x = 2 \end{cases}$$

[WBSC – 17]

**17.** Find wheather the function

$$f(x) = \frac{|x|}{x}, x \ne 0,$$

$= 0,\ x = 0$ is continuous at $x = 0$ [WBSC – 18]

================================

# Derivative

## 3.1 Increment :

Let x be a real variable and its assumed value changes from $x_0$ to $x_1$ ; then $x_0$ is called the **initial** value and $x_1$ the **final** value of the variable x.

Then, increment of the variable x = final value of x – its initial value = $x_1 - x_0$

Then increment of the variable x is usually denoted by $\Delta x$ or h.

$\therefore \Delta x = x_1 - x_0$ or, $x_1 = x_0 + \Delta x$

$\therefore$ if x be the initial value and $\Delta x$ (or h), the increment of the variable x, then its final value = $x + \Delta x = x + h$.

Let $y = f(x)$ be a single valued function of x. Now, if $\Delta y$ (or k) be the increment in y corresponding to the increment $\Delta x$ (or h) in x, then $y + \Delta y = f(x + \Delta x)$ = the final value of y corresponding to the final value $x + \Delta x$ of x.

Therefore, Increment of y = final value of y – its initial value.

or, $\Delta y = (y + \Delta y) - y$ or, $\Delta y = f(x + \Delta x) - f(x)$

## 3.2 Definition of Derivative or Differential Coefficient :

Let $y = f(x)$ be a single valued function of x defined in the interval $a \le x \le b$. Let x be a particular value in the interval.

Let $\Delta x$ (or h) be the increment of x and let $\Delta y = f(x + \Delta x) - f(x)$ be the corresponding increment of y.

If the ratio $\frac{\Delta y}{\Delta x}$ tends to a definite finite limit as $\Delta x \to 0$, then this limit is called the **differential coefficient** or **derivative** of f(x) (or y) for the particular value of x, and is denoted by $\frac{dy}{dx}$ or $f'(x)$.

Thus, symbolically, the differential coefficient of y[= f(x)] with respect to x is

$$\frac{dy}{dx} = f'(x) = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \underset{h \to 0}{\mathrm{Lt}} \frac{f(x+h) - f(x)}{h}$$, provided this limit exists.

**NOTE :**

i) The process of obtaining the derivative of a function is called **differentiation.**

ii) The derivative of the function $y = f(x)$ at $x = a$ is denoted by $f'(a)$ or, $\left(\frac{dy}{dx}\right)_{x=a}$

and, $$f'(a) = \left(\frac{dy}{dx}\right)_{x=a} = \underset{h \to 0}{\mathrm{Lt}} \frac{f(a+h) - f(a)}{h}$$

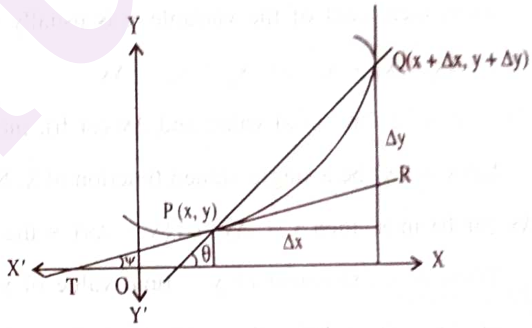
iii) From the definition of limit of a function we know, $\underset{h \to 0}{\text{Lt}} \frac{f(a+h)-f(a)}{h}$ exists if $\underset{h \to 0+}{\text{Lt}} \frac{f(a+h)-f(a)}{h}$ and $\underset{h \to 0-}{\text{Lt}} \frac{f(a+h)-f(a)}{h}$ both exists and if they are equal.

Here, $\underset{h \to 0+}{\text{Lt}} \frac{f(a+h)-f(a)}{h}$ is called the right hand derivative at $x = a$ and is denoted by $Rf'(a)$ and $\underset{h \to 0-}{\text{Lt}} \frac{f(a+h)-f(a)}{h}$ is called the left hand derivative at $x = a$ and is denoted by $Lf'(a)$.

Therefore, the derivative of the function $y = f(x)$ at $x = a$ exists if $Rf'(a)$ and $Lf'(a)$ both exist and if $Rf'(a) = Lf'(a)$.

### 3.3 Physical meaning or Geometrical Interpretation of derivative.

Let P be a point (x, y) and Q a neighbouring point $(x + \Delta x, y + \Delta y)$ on the curve.




Then $\tan\theta = \frac{\Delta y}{\Delta x}$, $\theta$ be the inclination of the line PQ to the x-axis.

Then as $Q \to P$ along the curve so that $\Delta x \to 0$, the chord PQ tends to the tangent TPR and $\theta \to \psi$, where y is the inclination of the tangent TPR at the point P(x, y) of the curve $y = f(x)$ to the x-axis. $\therefore \tan\psi = \frac{dy}{dx}$

Hence, the derivative $\frac{dy}{dx}$ for any value of x, when it exists, is the trigonometrical tangent of the inclination of the tangent line at the corresponding point P on the curve $y = f(x)$.

## PROBLEM SET

1. A function f(x) is defined as follows :

$$f(x) = 2x + 6, \quad -3 \le x \le 0$$
$$= 6, \quad 0 < x < 2$$
$$= 2x - 6, \quad 2 \le x \le 5$$

Draw the graph of the function (without using graph paper) and determine the point of discontinuity **[WBSC – 91]**

*2. A function f(x) is defined as follows :

$$f(x) = \frac{|x|}{x}, \quad \text{when } x \ne 0$$
$$= 0, \quad \text{when } x = 0$$

(i) Draw the graph. (ii) Discuss the continuity.

*3. A function f(x) is defined as follows :

$$f(x) = -x, \quad x \le 0$$
$$= x, \quad x > 0$$

(i) Draw the graph. (ii) Discuss the continuity. (iii) Discuss differentiability.

*4. A function f(x) is defined as follows :

$$f(x) = -x \quad \text{when } x < 0$$
$$= x^2 \quad \text{when } 0 \le x \le 1$$
$$= x^3 - x + 1 \quad \text{when } x > 1$$

Discuss continuity at $x = 0$ and $x = 1$. Discuss differentiability at $x = 0$ and at $x = 1$.

*5. Given $y = |x - 1| + |x - 2|$.

(i) Draw the graph in the interval [0, 3]. (ii) Discuss the continuity in the interval [0, 3]

(iii) Discuss the differentiability in the interval [0, 3]

*6. Given $y = f(x) = |x| + |x - 1|$

(i) Draw the graph. (ii) Discuss the coutinuity. (iii) Discuss the differentiability.

*7. Draw the graph of

$f(x) = |x - 1| + |x + 1|$ without using graph paper. From the graph show that f(x) is continuous at $x = 1$

*8. Given, $f(x) = \frac{\sin 3x}{2x}$, when $x \ne 0$

$= \frac{2}{3}$, when $x = 0$. Examine the countinuity of f(x) at $x = 0$.

9. Given $f(x) = \frac{1 - \cos x}{x^2}$, $x \ne 0$

$= \frac{1}{2}$, $x = 0$. Prove that f(x) is continuous at $x = 0$.

10. Given $f(x) = x^2 - 2x + 3, \quad x < 1$

$= 2 \quad x = 1$

$= 2x^2 - 5x + 5, \quad x > 1.$

Examine whether the function is continuous and differentiable at $x = 1$.

## ANSWERS

1. Discontinous at $x = 2$. 2. Discontinuous at $x = 0$.
3. Continuous in $-\infty < x < \infty$. Not differentiable at $x = 0$
4. Continuous at $x = 0$ and $x = 1$. Not differentiable at $x = 0$ Differentiable at $x = 1$.
5. Continuous at $x = 1$ and at $x = 2$ Not differentiable at $x = 1$ and $x = 2$.
6. Continuous in $-\infty < x < \infty$. Not differentiable at $x = 0$ and $x = 1$.
8. Discoutinuous at $x = 0$ 10. Continuous at $x = 1$ Not differentiable at $x = 1$.

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

2. **Solution :** To draw the graph of the given function

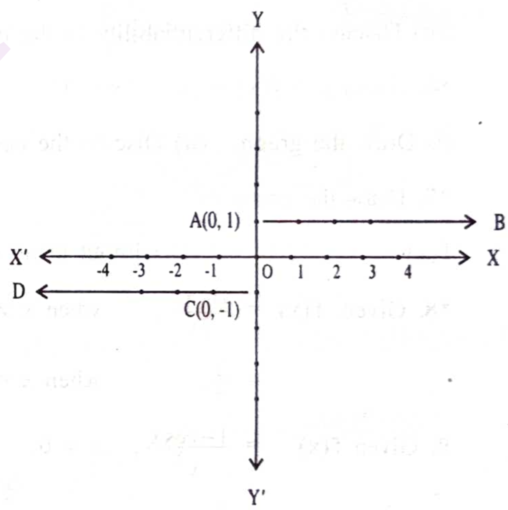
$$y = f(x) = \frac{|x|}{x}, \qquad x \neq 0$$
$$= 0, \qquad x = 0 \qquad \text{..... (1)}$$

We prepare the following table showing the values of the independent variable x and the dependent variable y or f(x).

| x | 0 | ·01 | 1 | 2 | 3 | –0·1 | –1 | –2 | –3 | ...... |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 0 | 1 | 1 | 1 | 1 | –1 | –1 | –1 | –1 | .... |

### Test of coutinuity : Geometrically

The graph of the given function has been drawn and shown in the figure using the above table.




(i) From the graph we see that it consists of two straight lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

(ii) The straight line $\overrightarrow{AB}$ is parallel to $\overrightarrow{OX}$ and $\overrightarrow{CD}$ is parallel to $\overrightarrow{OX}$.

(iii) The straight lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$ each at a distance 1 unit from the x axis.

(iv) The straight line $\overrightarrow{AB}$ does not contain the point (0, 1) and $\overrightarrow{CD}$ does not contain the point (0, –1).

(v) From the graph we see that it breaks at x = 0.

Therefore, the given function is discontinuous at x = 0; it is continuous at all other points in $-\infty < x < \infty$.

### Test of continuity :

We re-write the given function as follows :

$y = f(x) = 1$ when $x > 0$.

$= 0$ when $x = 0$ [$\because$ from definition $|x| = x, x > 0$

$= -1$ when $x < 0$ $= -x, x < 0$]

Therefore, at x = 0,

$$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} (-1) = -1 \qquad [\because f(x) = -1, \text{ when } x < 0]$$

and, $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} (1) = 1$ [$\because$ $f(x) = 1$, when $x > 0$] and $f(0) = 0$

Therefore, $\lim\limits_{x \to 0-} f(x) \neq \lim\limits_{x \to 0+} f(x) \neq f(0)$

$\therefore$ the function f(x) is discontinuous at $x = 0$.

**3. Solution :** To draw the graph of the given function

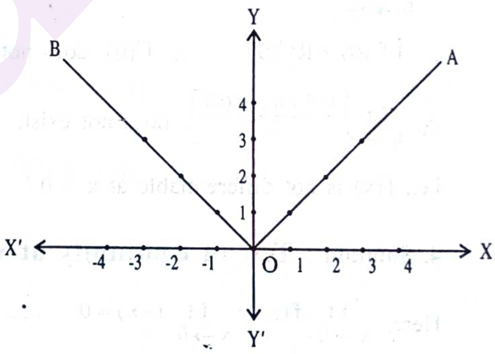
$y = f(x) = -x$ when $x \leq 0$

$= x$ when $x > 0$ ..... (1)

We prepare the following table showing the values of the independent variable x and the dependent variable y or f(x).

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| x | 0 | ·01 | 1 | 2 | 3 | –0·1 | –1 | –2 | –3 | ...... |
| y | 0 | ·01 | 1 | 2 | 3 | ·01 | –1 | –2 | –3 | .... |

**Test of continuity from graph :**




The graph of the given function has been drawn and shown in the figure using the above table.

(i) The graph of the given function consists of two straight lines $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(ii) The straight line $\overrightarrow{OA}$ lies in the first quadrant and makes an angle $45^{O}$ with $\overrightarrow{OX}$ while the line $\overrightarrow{OB}$ lies in the second quadrant and makes an angle of $135^{O}$ with $\overrightarrow{OX}$.

(iii) The graph does not break anywhere in the interval $-\infty < x < \infty$; hence it is continuous everywhere in $-\infty < x < \infty$.

Therefore, it is continuous at $x = 0$.

**Test of continuity :**

Here, $\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} (-x) = 0$ [$\because$ $f(x) = -x$, $x < 0$]

and $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} (x) = 0$ [$\because$ $f(x) = x$, $x > 0$]

and $f(0) = 0$

Therefore, $\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0+} f(x) = f(0)$

Therefore, the given function is continuous at $x = 0$ and hence it is continuous everywhere in $-\infty < x < \infty$.

**Test of Differentiability :**

Since the slopes of the graph at the point O (x = 0) are different on the left hand and right hand sides, the function f(x) is not differentiable at x = 0.

**Test of differentiability :**

**At x = 0**

$$Lf'(0) = \underset{h \to 0-}{Lt}\left[\frac{f(0+h)-f(0)}{h}\right] = \underset{h \to 0-}{Lt}\left[\frac{(-h)-(0)}{h}\right] \quad [\because h \to 0- \therefore 0+h<0]$$

$$= \underset{h \to 0-}{Lt}(-1) = -1 \quad [\text{and } f(x) = -x, \text{ when } x < 0]$$

$$Rf'(0) = \underset{h \to 0+}{Lt}\left[\frac{f(0+h)-f(0)}{h}\right] = \underset{h \to 0+}{Lt}\left[\frac{h-0}{h}\right]$$

$[\because h \to 0+, \therefore 0 + h > 0$ and $f(x) = x$, when $x > 0]$

$$= \underset{h \to 0+}{Lt}(1) = 1$$

$\therefore$ $Lf'(0) \neq Rf'(0)$ $\quad\therefore$ $f'(0)$ does not exist

$\therefore$ $\underset{h \to 0}{Lt}\left[\dfrac{f(0+h)-f(0)}{h}\right]$, does not exist.

i.e., f(x) is not differentiable at x = 0

**4. Solution : Test of continuity at x = 0**

Here, $\underset{x \to 0-}{Lt} f(x) = \underset{x \to 0-}{Lt}(-x) = 0$ $\quad[\because$ for $x < 0$, $f(x) = -x]$

$\underset{x \to 0+}{Lt} f(x) = \underset{x \to 0+}{Lt}\left(x^2\right) = 0$ $\quad[\because f(x) = x^2$, when $x > 0]$

and $f(0) = 0^2 = 0$ $\quad[\because f(x) = x^2$, when $x = 0]$

$\underset{x \to 0-}{Lt} f(x) = \underset{x \to 0+}{Lt} f(x) = f(0)$

Therefore, f(x) is continuous at x = 0.

**Test of continuity at x = 1**

Here, $\underset{x \to 1-}{Lt} f(x) = \underset{x \to 1-}{Lt}\left(x^2\right) = 1$ $\quad[\because f(x) = x^2$, when $x < 1]$

$\underset{x \to 1+}{Lt} f(x) = \underset{x \to 1+}{Lt}\left(x^3 - x + 1\right)$ $\quad[\because f(x) = x^3 - x + 1$, when $x > 1]$

$= 1^3 - 1 + 1 = 1$

and $f(1) = 1^2 = 1$ $\quad[\because f(x) = x^2$, when $x = 1]$

$\underset{x \to 1-}{Lt} f(x) = \underset{x \to 1+}{Lt} f(x) = f(1)$ Hence, f(x) is continuous at x = 1.

**Test of differentiability at x = 0**

Here $Lf'(0) = \underset{h\to 0-}{Lt}\left[\frac{f(0+h)-f(0)}{h}\right] = \underset{h\to 0-}{Lt}\left[\frac{-h-0}{h}\right]$ $[\because\ h \to 0-\ \therefore\ 0 + h < 0]$

$= \underset{h\to 0-}{Lt}\left[\frac{-h}{h}\right] = \underset{h\to 0-}{Lt}\ (-1) = -1$ $[\because h \neq 0]$

and, $Rf'(0) = \underset{h\to 0+}{Lt}\left[\frac{f(0+h)-f(0)}{h}\right]$

$= \underset{h\to 0+}{Lt}\left[\frac{h^2-0}{h}\right]$ $[\because\ h \to 0+,\ \therefore\ 0 + h > 0]$

$= \underset{h\to 0+}{Lt}\ [h] = 0$ $[\because\ h \neq 0]$

$\therefore\ Lf'(0) \neq Rf'(0)$. Hence, f(x) is not differentiable at x = 0.

**Test of differentiability at x = 1**

Here,

$Lf'(1) = \underset{h\to 0-}{Lt}\left[\frac{f(1+h)-f(1)}{h}\right] = \underset{h\to 0-}{Lt}\left[\frac{(1+h)^2-1^2}{h}\right]$ $[\because\ h \to 0-,\ \therefore\ 1 + h < 1]$

$= \underset{h\to 0-}{Lt}\left[\frac{1+2h+h^2-1}{h}\right] = \underset{h\to 0-}{Lt}\left[\frac{h(2+h)}{h}\right]$ $[\because\ h \neq 0]$

$= \underset{h\to 0-}{Lt}\ (2+h) = 2$

$Rf'(1) = \underset{h\to 0+}{Lt}\left[\frac{f(1+h)-f(1)}{h}\right]$

$= \underset{h\to 0+}{Lt}\left[\frac{(1+h)^3-(1+h)+1-1}{h}\right]$ $[\because\ h \to 0+,\ \therefore\ 1 + h > 1]$

$\underset{h\to 0+}{Lt}\left[\frac{1+3h+3h^2+h^3-1-h}{h}\right] = \underset{h\to 0+}{Lt}\left[\frac{h(3+3h+h^2-1)}{h}\right]$

$= \underset{h\to 0+}{Lt}\ (3+3h+h^2-1) = = 2 + 0 = 2$ $[\because\ h \neq 0]$ $Lf'(1) = Rf'(1) \Rightarrow$

f(x) is differentiable at x = 1.

**5. Solution :** Here 1 and 2 are critical points.


**Case I** For $x < 1$, $x - 1 < 0$ and $x - 2 < 0$

$\therefore\ y = -(x - 1) - (x - 2)$ if $x < 1 = -2x + 3$ if $x < 1$

or, $y = 3 - 2x$ if $x < 1$

**Case II** For $x \geq 2$, $x - 1 > 0$ and $x - 2 \geq 0$

$\therefore\ y = x - 1 + x - 2$ if $x \geq 2$ or, $y = 2x - 3$ if $x \geq 2$

**Case III** For $1 \le x < 2$, $x - 1 \ge 0$ and $x - 2 < 0$

$\therefore$ $y = x - 1 - (x - 2)$ if $1 \le x < 2$ or, $y = 1$ if $1 \le x < 2$

$\therefore$ the given function is re-defined as

$$y = f(x) = 3 - 2x \quad \text{if } x < 1$$
$$= 1 \quad \text{if } 1 \le x < 2$$
$$= 2x - 3 \quad \text{if } x \ge 2$$

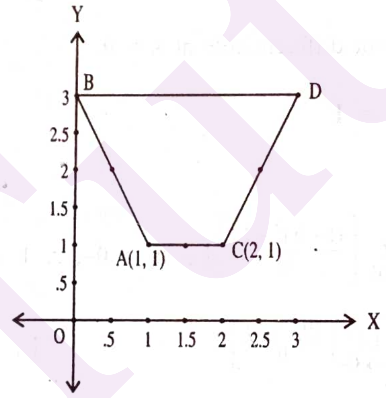
To draw the graph of the above function we prepare the following tables showing the values of the independent variable x and the dependent variable y:

**Test of Continuity :** from graph

| x | 0 | ·5 | 1 |
|---|---|---|---|
| y | 3 | 2 | 1 |

| x | 1 | 1·5 | 2 |
|---|---|---|---|
| y | 1 | 1 | 1 |

| x | 2 | 2·5 | 3 |
|---|---|---|---|
| y | 1 | 2 | 3 |




The graph of the given function has been drawn and shown in the above figure using the above tables.

From the graph we see that it consists of three lines segments $\overline{AB}$ (containing the extreme points A and B), $\overline{AC}$ (containing the extreme points A and C) and $\overline{CD}$ (containing the extreme points C and D) in the interval [0, 3]

The graph does not break anywhere in the interval [0, 3]; hence it is continuous everwhere in [0, 3]

Hence the function possesses no point of discontinuity throughout the interval [0, 3].

**Test of continuity :**

**At x = 1**

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (3 - 2x) = 3 - 2 \times 1 = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (1) = 1$$

$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x)$ Therefore, $\lim_{x \to 1} f(x)$ exist.

and $f(1) = 1$

$\therefore$ $\lim_{x \to 1} f(x) = f(1)$ Therefore, the function f(x) is continuous at x = 1.

**At x = 2**

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (2x-3) = 4-3=1$$

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} (2x-3) = 2\times 2-3=4-3=1$$

$\therefore \lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x)$ Therefore, $\lim_{x\to 2} f(x)$ exist.

and $f(2) = 2\cdot 2 - 3 = 1$

$\therefore \lim_{x\to 2} f(x) = f(2)$

Therefore, the function $f(x)$ is continuous at $x = 2$

Hence, $f(x)$ is continuous throughout the interval [0, 3]

**Test of Differentiability :** from graph

Since the slopes at the points A (x = 1) and C (x = 2) are different on the left hand and right hand sides, the function $f(x)$ is not differentiable at $x = 1$ and $x = 2$.

**Test of Differentiability :**

**At x = 1**

$$Lf'(1) = \lim_{h\to 0-}\left[\frac{f(1+h)-f(1)}{h}\right] = \lim_{h\to 0-}\left[\frac{3-2(1+h)-1}{h}\right] \quad [\because h\to 0- \Rightarrow 1+h<1]$$

$$= \lim_{h\to 0-}\left[\frac{3-2-2h-1}{h}\right] = \lim_{h\to 0-}\left[\frac{-2h}{h}\right] \lim_{h\to 0-}(-2) = -2 \quad [\because h\neq 0]$$

$$Rf'(1) = \lim_{h\to 0+}\left[\frac{f(1+h)-f(1)}{h}\right] = \lim_{h\to 0+}\left[\frac{1-1}{h}\right] \quad [\because h\to 0+ \Rightarrow 1+h>1]$$

$$= 0 \quad [\because h\neq 0]$$

$\because Lf'(1)\neq Rf'(1)$ Hence $f(x)$ is not differentiable at $x = 1$

**At x = 2**

$$Lf'(2) = \lim_{h\to 0-}\left[\frac{f(2+h)-f(2)}{h}\right] = \lim_{h\to 0-}\left[\frac{1-1}{h}\right] \quad [\because h\to 0- \Rightarrow 2+h<2]$$

$$= 0 \qquad \because f(x) = 2.2-3=1 \ [\because h\neq 0]$$

$$Rf'(2) = \lim_{h\to 0+}\left[\frac{f(2+h)-f(2)}{h}\right]$$

$$= \lim_{h\to 0+}\left[\frac{2(2+h)-3-1}{h}\right] \quad [\because h\to 0+ \Rightarrow 2+h>2]$$

$$= \lim_{h\to 0+}\left[\frac{4+2h-4}{h}\right] = \lim_{h\to 0+}\left[\frac{2h}{h}\right] \lim_{h\to 0+}(2) = 2 \quad [\because h\neq 0]$$

$\therefore Lf'(2)\neq Rf'(2)$. Hence, $f(x)$ is not differentiable at $x = 2$.

**6. Solution :** Here, the critical points are 0 and 1. Mark 0 and 1 on the real line.

Now we re-define the given function as follows :


**Case I** For $x \geq 1$, $x > 0$ and $x - 1 \geq 0$

$\therefore$ for $x \geq 1$, $|x| = x$ and $|x - 1| = x - 1$

$\therefore$ $f(x) = x + x - 1$ if $x \geq 1 = 2x - 1$, if $x \geq 1$

**Case II** For $x < 0$, $x < 0$ and $x - 1 < 0$

$\therefore$ for $x < 0$, $|x| = -x$ and $|x - 1| = -x + 1$

$\therefore$ $f(x) = -x - x + 1$ if $x < 0 = -2x + 1$ if $x < 0$

**Case III** For $0 \leq x < 1$, $x \geq 0$ and $x - 1 < 0$

$\therefore$ for $0 \leq x < 1$, $|x| = x$ and $|x - 1| = -x + 1$

$\therefore$ $f(x) = x - x + 1$ if $0 \leq x < 1 = 1$ if $0 \leq x < 1$

Therefore, the given function is re-written as follows :

$$f(x) = -2x + 1 \quad \text{if } x < 0$$
$$= 1 \quad \text{if } 0 \leq x < 1$$
$$= 2x - 1 \quad \text{if } x \geq 1$$

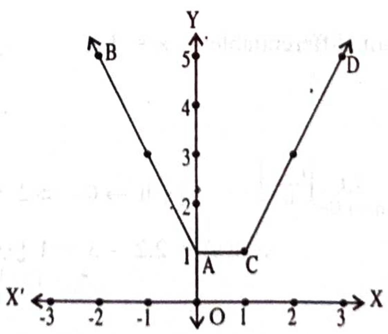
**Test of continuity** : from graph

To draw the graph of the above function we prepare the following tables showing the values of the independent variable x and the dependent variable y.

| | | | |
|---|---|---|---|
| x | 0 | –1 | –2 |
| y | 1 | 3 | 5 |

| | | | |
|---|---|---|---|
| x | 0 | ·5 | 1 |
| y | 1 | 1 | 1 |

| | | | |
|---|---|---|---|
| x | 1 | 2 | 3 |
| y | 1 | 3 | 5 |




The graph of the above function has been drawn and shown in the above figure using the above three tables. From the graph we see that it consists of three straight lines $\overrightarrow{AB}$ (containing the point A), $\overline{AC}$ (containing the points A and C) and $\overrightarrow{CD}$ (containing the point C)

The graph does not break anywhere in the interval $-\infty < x < \infty$.

Hence, the function possesses no point of discontinuity.

N-Tute

**Test of continuity :**

**At x = 1**

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (1) = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2x - 1) = 2 \times 1 - 1 = 1$$

$\therefore \lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) \Rightarrow \lim_{x \to 1} f(x)$ exists.

Again, $f(1) = 1 \quad \therefore \lim_{x \to 1} f(x) = f(1)$

$\Rightarrow$ f(x) is continuous at x = 1.

**At x = 0**

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-2x + 1) = 1$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (1) = 1$$

$\therefore \lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) \Rightarrow \lim_{x \to 0} f(x)$ exist.

Again $f(0) = 1 \Rightarrow \lim_{x \to 0} f(x) = f(0)$

Hence f(x) is continuous at x = 0.

Hence, f(x) is continuous throughout the interval $-\infty < x < \infty$.

**Test of Differentiability** : from graph

Since the slopes at the points A (x = 0) and C (x = 1) are different on the left hand and right hand sides, the function f(x) is not differentiable at x = 0 and x = 1.

**Test of differentiability :**

**At x = 0**

$$Lf'(0) = \lim_{h \to 0-} \left[\frac{f(0+h) - f(0)}{h}\right]$$

$$= \lim_{h \to 0-} \left[\frac{-2(0+h) + 1 - 1}{h}\right] \qquad [\because h \to 0- \Rightarrow 0 + h < 0]$$

$$= \lim_{h \to 0-} \left[\frac{-2h}{h}\right] = \lim_{h \to 0-} (-2) = -2 \qquad [\because h \neq 0]$$

$$Rf'(0) = \lim_{h \to 0+} \left[\frac{f(0+h) - f(0)}{h}\right]$$

$$= \lim_{h \to 0+} \left[\frac{1-1}{h}\right] = 0 \quad [\because h \neq 0] \qquad [\because h \to 0+, \therefore 0 + h > 0]$$

$\therefore Lf'(0) \neq Rf'(0)$ and hence f(x) is not differentiable at x = 0.

**At x = 1**

$$Lf'(1) = \underset{h\to 0-}{\text{Lt}}\left[\frac{f(1+h)-f(1)}{h}\right] = \underset{h\to 0-}{\text{Lt}}\left[\frac{1-1}{h}\right] \qquad [\because h \to 0- \Rightarrow 1+h<1]$$

$$= 0 \; [\because h \neq 0]$$

$$Rf'(1) = \underset{h\to 0+}{\text{Lt}}\left[\frac{f(1+h)-f(1)}{h}\right] \underset{h\to 0+}{\text{Lt}}\left[\frac{2(1+h)-1-1}{h}\right] [\because h \to 0+, \therefore 1+h>1]$$

$$= \underset{h\to 0+}{\text{Lt}}\left[\frac{2h}{h}\right] = \underset{h\to 0+}{\text{Lt}}(2) = 2 \qquad [\because h \neq 0]$$

$\therefore Lf'(1) \neq Rf'(1)$ Hence $f(x)$ is not differentiable at $x = 1$.

**7. Hints :**

Here, the critical points are 1 and –1. Mark 1 and –1 on the real line.

Now, we re-define the given function as follows :

**Case I** For, $x \geq 1$, $x - 1 \geq 0$ and $x + 1 > 0$.

$\therefore$ for, $x \geq 1$, $|x - 1| = x - 1$ and $|x + 1| = x + 1$

$\therefore$ $f(x) = x - 1 + x + 1$ if $x \geq 1 = 2x$ if $x \geq 1$

**Case II** For, $x < -1$, $x - 1 < 0$ and $x + 1 < 0$

$\therefore$ for, $x < -1$, $|x - 1| = -(x - 1)$ and $|x + 1| = -(x + 1)$

$\therefore$ $f(x) = -x + 1 - x - 1$ if $x < -1 = -2x$ if $x < -1$

**Case III** For, $-1 \leq x < 1$, $x - 1 < 0$ and $x + 1 \geq 0$

$\therefore$ for, $-1 \leq x < 1$, $|x - 1| = -x + 1$ and $|x + 1| = x + 1$

$\therefore$ $f(x) = -x + 1 + x + 1$ if $-1 \leq x < 1 = 2$ if $-1 \leq x < 1$

Therefore; the given function is re-written as

$$f(x) = \begin{cases} -2x & \text{if } x < -1 \\ 2 & \text{if } -1 \leq x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$$

**8. Solution :** Here,

$$\underset{x\to 0}{\text{Lt}} f(x) = \underset{x\to 0}{\text{Lt}} \frac{\sin 3x}{2x} \qquad \left[\because f(x) = \frac{\sin 3x}{2x} \text{ when } x \neq 0\right]$$

$$= \underset{3x\to 0}{\text{Lt}} \frac{\sin 3x}{3x} \times \frac{3}{2} = 1 \times \frac{3}{2} = \frac{3}{2} \qquad [\because x \to 0, \therefore 3x \to 0]$$

and $f(x) = \frac{2}{3}$ when $x = 0$

$\therefore f(0) = \frac{2}{3}$

$\underset{x\to 0}{\text{Lt}} f(x) \neq f(0)$ Hence, $f(x)$ is discontinuous at $x = 0$. **(Ans)**

# First Order Derivative

## 4.1 Derivative of some standard functions :

(i) $\frac{d}{dx}\left(x^{n}\right)=nx^{n-1}$ (ii) $\frac{d}{dx}(x)=1$ (iii) $\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}$

(iv) $\frac{d}{dx}\left(e^{x}\right)=e^{x}$ (v) $\frac{d}{dx}\left(a^{x}\right)=a^{x}\log_{e}a$ (vi) $\frac{d}{dx}(\log x)=\frac{1}{x}$

(vii) $\frac{d}{dx}(\log_{a}x)=\frac{1}{x}\cdot\log_{a}e$ (viii) $\frac{d}{dx}(\sin x)=\cos x$ (ix) $\frac{d}{dx}(\cos x)=-\sin x$

(x) $\frac{d}{dx}(\tan x)=\sec^{2}x$ (xi) $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^{2}x$ (xii) $\frac{d}{dx}(\sec x)=\sec x\cdot\tan x$

(xiii) $\frac{d}{dx}(\operatorname{cosec}x)=-\operatorname{cosec}x\cdot\cot x$ (xiv) $\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^{2}}}$ (xv) $\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^{2}}}$

(xvi) $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^{2}}$ (xvii) $\frac{d}{dx}\left(\cot^{-1}x\right)=-\frac{1}{1+x^{2}}$

(xviii) $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^{2}-1}}$ (xix) $\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=-\frac{1}{x\sqrt{x^{2}-1}}$

## 4.2 Fundamental Theorems on Differentiations :

**Theorem 1.** The derivative of a constant is zero i.e., if c is a constant, then $\frac{d}{dx}(c)=0$.

**Theorem 2.** If $f(x)$ is a differentiable function of x, then

$\frac{d}{dx}[c\cdot f(x)]=c\cdot\frac{d}{dx}[f(x)]=c\cdot f'(x)$, where c is a constant.

**Theorem 3.** If $f(x)$ and $g(x)$ are two differentiable functions of x, then

$\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[g(x)]=f'(x)\pm g'(x)$.

**Theorem 4.** If u and v are two differentiable functions of x, then $\frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx}$.

This theorem is true for more than two functions. For three functions u, v, w

$\frac{d}{dx}(u\cdot v\cdot w)=uv\frac{dw}{dx}+uw\frac{dv}{dx}+vw\frac{du}{dx}$.

**Theorem 5.** If u and v are two differentiable functions of x, then

$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\cdot\frac{du}{dx}-u\cdot\frac{dv}{dx}}{v^{2}}$, where $v\neq 0$.

## 4.3 Derivative of a function of function (Chain Rule).

Let us assume that $y = \sin x^2$

To find $\frac{dy}{dx}$ we cannot use $\frac{d}{dx}(\sin x) = \cos x$ directly.

We know the derivative of sinx with respect to x is cosx. Here, sinx is a function of x. But $\sin x^2$ is a function of $x^2$. Hence, the derivative of $\sin x^2$ with respect to x is obviously not equal to $\cos x^2$. But, the derivative of $\sin x^2$ with respect to $x^2$ is equal to $\cos x^2$.

However, we want to differentiate $y = \sin x^2$ with respect to x.

Given, $y = \sin x^2$ and let $z = x^2$ then $y = \sin z$, is a function of z.

Differentiating both sides with respect to z we get, $\frac{dy}{dz} = \cos z$

Again $z = x^2$ is a function of x.

Differentiating both sides with respect to x we get, $\frac{dz}{dx} = 2x$

Now, $\frac{dy}{dz} \cdot \frac{dz}{dx} = \cos z \cdot 2x$ or, $\frac{dy}{dx} = 2x \cos x^2$

Hence, if $y = f(z)$; $z = f(x)$, then, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

This is true for more than two functions; i.e., if $y = f(u)$; $u = f(v)$; $v = y(x)$, then, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$ etc.

## PROBLEM SET

[.....Problems with '*' marks are solved at the end of the problem set.....]

**1.** Find $\frac{dy}{dx}$ :

*(i) $y = \log \tan x$ **[WBSC – 05]**

(ii) $y = \log(\sec x + \tan x)$

(iii) $y = \log(\operatorname{cosec} x - \cot x)$

*(iv) $y = \log\left(x^2 - \sin x\right)$ **[WBSC – 84]**

(v) $y = \log\left(\sqrt{x+a} + \sqrt{x+b}\right)$

(vi) $y = \frac{1}{\sqrt{x+a} - \sqrt{x+b}}$

*(vii) $y = \left(x + \sqrt{x^2 + a^2}\right)^n$ **[WBSC – 89]**

*(viii) $y = \frac{5x}{(1-x)^{\frac{2}{3}}} + \cos^2(2x+1)$ **[WBSC – 90]**

(ix) $y = \log\left(x + \sqrt{x^2 - a^2}\right)$ **[HS – 91, 99]**.

*(x) $y = \tan^{-1}\left(\frac{a+bx}{a-bx}\right)$ **[WBSC – 04]**

**2.** Find $\frac{dy}{dx}$ :

*(i) $y = \sec^{-1}\frac{\sqrt{x}+1}{\sqrt{x}-1} + \sin^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1}$ **[WBSC – 90]**

*(ii) $y = \sin^{-1}\left(\frac{a + b\cos x}{b + a\cos x}\right)$

*(iii) $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ [WBSC – 19]

(iv) $y = \cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

*(v) $y = \tan^{-1}\left(\frac{\alpha + \beta x}{\beta - \alpha x}\right)$,

(vi) $y = \tan^{-1}\frac{1-x}{1+x}$

(vii) $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

*(viii) $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ [WBSC – 03]

*(ix) $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ [WBSC – 06]

3. Find $\frac{dy}{dx}$ :

*(i) $y = \sin^{-1}\frac{2x}{1+x^2}$ [WBSC – 82, 07]

(ii) $y = \sin^{-1}(3x - 4x^3)$

*(iii) $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$ [WBSC – 97]

(iv) $y = \cos^{-1}(8x^4 - 8x^2 + 1)$

*(v) $y = \cot^{-1}\left[\sqrt{1+x^2} - x\right]$

*(vi) $y = \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$ [WBSC – 86]

(vii) $y = \cos^{-1}\frac{1-x}{1+x}$

(viii) $y = \tan^{-1}\sqrt{\frac{x+1}{x-1}}$ [WBSC – 07, 08]

## 4.4 Derivatives of Implicit Functions :

Assume that the equation $f(x, y) = 0$ represents y as an implicit function of x. For such function, differentiate each term with respect to x [note that $\frac{d}{dx}\{\varphi(y)\} = \frac{d}{dy}\{\varphi(y)\}\frac{dy}{dx} = \varphi'(y)\cdot\frac{dy}{dx}$ ] and the value of $\frac{dy}{dx}$ is obtained by solving this equation.

4. Find $\frac{dy}{dx}$:

(i) $x^4 + 3x^2y^2 - 2y^4 = 5$

(ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

*(iii) $x^3 + y^3 = 3axy$

(iv) $x + y = \sin(x + y)$

*(v) $e^{xy} - 4xy = 4$ [HS – 96, 99]

*(vi) $xy = \tan(x + y)$

(vii) $\log(xy) = x + y$

*(viii) $\log(x + y) = x^2 + y^2$; $x = 1$, $y = 1$.

## 4.5 Logarithmic Differentiation :

To differentiate a function of the forms

(i) $y = \{f(x)\}^{\varphi(x)}$ (ii) $y = f_1(x).f_2(x).f_3(x) \ldots\ldots\ldots f_n(x)$ (iii) $y = \frac{f_1(x).f_2(x).f_3(x)\ldots\ldots\ldots f_n(x)}{\varphi_1(x).\varphi_2(x).\varphi_3(x)\ldots\ldots\ldots\varphi_m(x)}$

We first take the logarithm of the function and then we obtain the derivative by differentiating both sides with respect to x. [note that $\frac{d}{dx}(\log y) = \frac{d}{dy}(\log y)\frac{dy}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$ ]

**5.** Find $\frac{dy}{dx}$:

(i) $y=(1+x)^x$

(ii) $y=x^{\sin x}$

(iii) $y=(\sin x)^{\log x}$ **[HS – 00]**

*(iv) $y=(\tan x)^{\sin x}$, at $x=\frac{\pi}{4}$ **[HS – 94]**

(v) $y=x^{\log x}+(\log x)^x$

*(vi) $y=x^{1+\frac{1}{x}}+\left(1+\frac{1}{x}\right)^x$

*(vii) $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$ **[WBSC – 87]**

(viii) $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$ **[JEE – 80]**

**6.** Find $\frac{dy}{dx}$:

*(i) $x^y=y^x$ **[WBSC – 82, 84, 95]**

*(ii) $x^m y^n=(x+y)^{m+n}$ **[WBSC – 05, 08]**

(iii) $(\cos x)^y=(\sin y)^x$

*(iv) $x^y=e^{x-y}$ **[WBSC – 88, 03 HS – 01]**

(v)

*(vi) $x^y+y^x=1$ **[HS – 94]**

*(vii) $x^y.y^x=x+y$ **[WBSC – 04, 06]**

(viii) $x^{\log y}=\log x$

**7.** Find $\frac{dy}{dx}$:

(i) $y=(x-a)(x-b)(x-c)(x-d)(x-e)$

(ii) $y=\sqrt{\frac{x^2+1}{x^2-1}}$

*(iii) $y=x\cdot\sqrt{\frac{x^2+4}{x^2+3}}$ **[WBSC – 82]**

(iv) $y=x^2\cdot\sqrt{\frac{x^2-x+1}{x^2+x+1}}$

*(v) $y=\sqrt{\frac{(x-a)(x-b)}{(x-c)(x-d)}}$

(vi) $y=\frac{x+2}{(x-1)(x+5)}$

## 4.6 Derivative from Parametric Equations :

If $x=f(t)$ and $y=g(t)$ be the parametric equation, where t is the parameter, then, $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{g'(t)}{f'(t)}$ where, $f'(t)\neq 0$

**8.** Find $\frac{dy}{dx}$:

(i) $x=a\cos^3\theta,\ y=b\sin^3\theta$

*(ii) $x=a\sec^2\theta,\ y=a\tan^3\theta$ **[WBSC – 84, HS – 00]**

(iii) $x=a(\cos t+t\sin t),\ y=a(\sin t-t\cos t)$ at $t=\frac{3\pi}{4}$

*(iv) $x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$

(v) $x=a(\theta+\sin\theta),\ y=a(1+\cos\theta)$ at $\theta=\frac{\pi}{2}$

*(vi) $x=\sec^{-1}\frac{1}{2v^2-1},\ y=\tan^{-1}\frac{v}{\sqrt{1-v^2}}$

(vii) $x=\sin^{-1}\frac{2t}{1+t^2},\ y=\tan^{-1}\frac{2t}{1-t^2}$

*(viii) $\tan y=\frac{2t}{1-t^2},\ \sin x=\frac{2t}{1+t^2}$ **[WBSC – 05]**

*(ix) $x=\sin\theta\cdot\sqrt{\cos 2\theta},\ y=\cos\theta\cdot\sqrt{\sin 2\theta}$ at $\theta=\frac{\pi}{4}$

*(x) $x=a\left(\cos t+\log\tan\frac{t}{2}\right),\ y=a\sin t$ **[WBSC – 11]**

(xi) $x=t\log t,\ y=\frac{\log t}{t}$ at $t=1$

## 4.7 Derivative of one function with respect to another function.

To find the derivative of f(x) with respect to $\varphi(x)$, let $u = f(x)$ and $v = \varphi(x)$.

Then, the derivative of f(x) with respect to $\varphi(x)$ is $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x)}{\varphi'(x)}$ [where $\varphi'(x) \neq 0$]

**9.** Find the derivative of

(i) $\cos^{-1}\frac{1-x^2}{1+x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$ **[HS – 98]**

*(ii) $\sin^{-1}\frac{2x}{1+x^2}$ with respect to $\tan^{-1}\frac{2x}{1-x^2}$ **[WBSC – 96, 98, 00, 02]**

(iii) $\tan^{-1}\frac{t}{\sqrt{1-t^2}}$ with respect to $\sec^{-1}\left(\frac{1}{2t^2-1}\right)$ **[HS – 95]**

*(iv) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}x$ **[WBSC – 09, 12, 16]**

*(v) $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$ **[WBSC – 11]**

(vi) $\sec^{-1}\left(\frac{1}{8x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = \frac{1}{2}$. **[HS – 01]**

*(vii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ with respect to $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$. **[WBSC – 94]**

(viii) $x^5$ with respect to $x^2$.

(ix) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = 0$

(x) $\tan x$ with respect to $\log x$. *(xi) $y = x^6$ with respect to $x^3$. **[WBSC – 05]**

## ANSWERS

1(i) 2cosec2x (ii) sec x (iii) cosec x (iv) $\frac{2x-\cos x}{x^2-\sin x}$ (v) $\frac{1}{2\sqrt{(x+a)(x+b)}}$ (vi) $\frac{\sqrt{x+a}+\sqrt{x+b}}{2(a-b)\sqrt{(x+a)(x+b)}}$

(vii) $\frac{ny}{\sqrt{x^2+a^2}}$ (viii) $\frac{5(3-x)}{3(1-x)^{\frac{5}{3}}}-2\sin(4x+2)$ (ix) $\frac{1}{\sqrt{x^2-a^2}}$ (x) $\frac{ab}{a^2+b^2x^2}$

2 (i) 0 (ii) $-\frac{\sqrt{b^2-a^2}}{b+a\cos x}$ (iii) $-1$ (iv) 1 (v) $\frac{1}{1+x^2}$ (vi) $-\frac{1}{1+x^2}$ (vii) 0 (viii) $\frac{1}{2}$ (ix) $-\frac{1}{2}$

3 (i) $\frac{2}{1+x^2}$ (ii) $\frac{3}{\sqrt{1-x^2}}$ (iii) $-\frac{1}{2\sqrt{1-x^2}}$ (iv) $-\frac{4}{\sqrt{1-x^2}}$ (v) $\frac{1}{2(1+x^2)}$ (vi) $-\frac{x}{\sqrt{1-x^2}}$ (vii) $\frac{1}{(x+1)\sqrt{x}}$ (viii) $-\frac{1}{2x\sqrt{x^2-1}}$

4 (i) $\frac{2x^3+3xy^2}{4y^3-3x^2y}$ (ii) $-\left(\frac{y}{x}\right)^{\frac{1}{3}}$ (iii) $\frac{ay-x^2}{y^2-ax}$ (iv) $-1$, (v) $-\frac{y}{x}$ (vi) $\frac{-y+1+x^2y^2}{-1+x-x^2y^2}$ (vii) $\frac{y(x-1)}{x(1-y)}$ (viii) $-1$.

5(i) $(1+x)^x\left[\frac{x}{1+x}+\log(1+x)\right]$ (ii) $x^{\sin x}\left[\frac{\sin x}{x}+\log x\cdot\cos x\right]$ (iii) $(\sin x)^{\log x}\left[\log x\cdot\cot x+\frac{\log(\sin x)}{x}\right]$

(iv) $\sqrt{2}$ (v) $x^{\log x-1}\cdot 2\log x+(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}$ (vi) $x^{1+\frac{1}{x}}\left(\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^2}\log x\right)+\left(1+\frac{1}{x}\right)^x\left\{\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right\}$

(vii) $(\sin x)^{\cos x}\{\cos x\cdot\cot x-\sin x\cdot\log(\sin x)\}+(\cos x)^{\sin x}\{\cos x\cdot\log(\cos x)-\tan x\cdot\sin x\}$

(viii) $(\tan x)^{\cot x}\left\{\operatorname{cosec}^2 x(1-\log\tan x)\right\}+(\cot x)^{\tan x}\left\{\sec^2 x(\log\cot x-1)\right\}$

6.(i) $\frac{y(x\log y-y)}{x(y\log x-x)}$ (ii) $\frac{y}{x}$ (iii) $\frac{y\tan x+\log(\sin y)}{\log(\cos x)-x\cot y}$ (iv) $\frac{\log x}{\{\log(ex)\}^2}$ (v) $x\cdot x^x\left[\frac{1}{x}+\log(1+\log x)\right]$

(vi) $-\frac{y^x\log y+x^{y-1}y}{x^y\log x+y^{x-1}x}$ (vii) $\frac{y\{x-y(x+y)-x(x+y)\log y\}}{x\{y(x+y)\log x+x(x+y)-y\}}$ (viii) $\frac{y(1-\log x\cdot\log y)}{x(\log x)^2}$

7(i) $y\left[\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\frac{1}{x-d}+\frac{1}{x-e}\right]$ (ii) $\sqrt{\frac{x^2+1}{x^2-1}}\left[\frac{-2x}{x^4-1}\right]$ (iii) $x\sqrt{\frac{x^2+4}{x^2+3}}\left[\frac{1}{x}-\frac{x}{(x^2+4)(x^2+3)}\right]$

(iv) $y\left[\frac{2}{x}+\frac{x^2-1}{x^4+x^2+1}\right]$ (v) $\frac{1}{2}y\left[\frac{1}{x-a}+\frac{1}{x-b}-\frac{1}{x-c}-\frac{1}{x-d}\right]$ (vi) $y\left[\frac{1}{x+2}-\frac{1}{x-1}-\frac{1}{x+5}\right]$

8(i) $-\frac{b}{a}\tan\theta$ (ii) $\frac{3}{2}\tan\theta$ (iii) $-1$ (iv) $-\cot\frac{\theta}{2}$ (v) $-1$ (vi) $-\frac{1}{2}$ (vii) 1 (viii) 1 (ix) 0 (x) $\tan t$ (xi) 1

9(i) 1 (ii) 1 (iii) $-\frac{1}{2}$ (iv) $\frac{1}{2}$ (v) $x^{\sin^{-1}x}\left[\frac{\sqrt{1-x^2}}{x}\cdot\sin^{-1}x+\log x\right]$ (vi) 4

(vii) $\frac{2\sqrt{1+x^2}}{x}$ (viii) $\frac{5x^3}{2}$ (ix) $\frac{1}{4}$ (x) $x\sec^2 x$. (xi) $2x^3$

## SOLUTION OF THE PROBLEMS WITH '*' MARKS

1. (i) **Solution :**

Given, $y = \log\tan x$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\{\log\tan x\}=\frac{1}{\tan x}\times\frac{d}{dx}(\tan x))=\frac{1}{\tan x}\times\sec^2 x=\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x}=\frac{2}{2\sin x\cos x}=\frac{2}{\sin 2x}=2\operatorname{cosec}2x \quad \textbf{(Ans)}$$

1. (iv) **Solution :**

Given, $y = \log(x^2-\sin x)$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\log\left(x^2-\sin x\right)\right\}=\frac{1}{x^2-\sin x}\times\frac{d}{dx}\left(x^2-\sin x\right)=\frac{1}{x^2-\sin x}\times(2x-\cos x)=\frac{2x-\cos x}{x^2-\sin x} \quad \textbf{(Ans)}$$

1 (vii) **Solution :**

Given, $y=\left(x+\sqrt{x^2+a^2}\right)^n$

Differentiating both sides with respect to x we get

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(x+\sqrt{x^2+a^2}\right)^n\right\}=n\left(x+\sqrt{x^2+a^2}\right)^{n-1}\times\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$

$$= n\left(x+\sqrt{x^2+a^2}\right)^{n-1}\times\left(1+\frac{1}{2\sqrt{x^2+a^2}}\times 2x\right) = n\left(x+\sqrt{x^2+a^2}\right)^{n-1}\times\left(\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$$

$$= \frac{n\left(x+\sqrt{x^2+a^2}\right)^{n}}{\sqrt{x^2+a^2}} = \frac{ny}{\sqrt{x^2+a^2}} \quad \textbf{(Ans)} \qquad \left[\because y=\left(x+\sqrt{x^2+a^2}\right)^{n}\right]$$

1(viii) **Solution :**

Given, $y=\dfrac{5x}{(1-x)^{\frac{2}{3}}}+\cos^2(2x+1)$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx}=5\frac{d}{dx}\left\{x(1-x)^{-\frac{2}{3}}\right\}+\frac{d}{dx}\left\{\cos^2(2x+1)\right\}$$

$$=5\left[(1-x)^{-\frac{2}{3}}\frac{d}{dx}(x)+x\frac{d}{dx}\left\{(1-x)^{-\frac{2}{3}}\right\}\right]+2\cos(2x+1)\times\frac{d}{dx}\{\cos(2x+1)\} \qquad \left[\because \frac{d}{dx}(u\cdot v)=u\cdot\frac{dv}{dx}+v\cdot\frac{du}{dx}\right]$$

$$=5\left[(1-x)^{-\frac{2}{3}}+x\left(-\frac{2}{3}\right)(1-x)^{-\frac{5}{3}}(-1)\right]+2\cos(2x+1)\times\{-\sin(2x+1)\}\times 2$$

$$=5\left[\frac{1}{(1-x)^{\frac{2}{3}}}+\frac{2x}{3(1-x)^{\frac{5}{3}}}\right]-2\sin(4x+2) = 5\left\{\frac{3(1-x)+2x}{3(1-x)^{\frac{5}{3}}}\right\}-2\sin(4x+2)=\frac{5(3-x)}{3(1-x)^{\frac{5}{3}}}-2\sin(4x+2) \quad \textbf{(Ans)}$$

**1 (x) Solution :** Given, $y=\tan^{-1}\left(\dfrac{a+bx}{a-bx}\right)$

or, $$\frac{dy}{dx}=\frac{d}{dx}\left[\tan^{-1}\left(\frac{a+bx}{a-bx}\right)\right]=\frac{1}{1+\left(\frac{a+bx}{a-bx}\right)^2}\cdot\frac{d}{dx}\left(\frac{a+bx}{a-bx}\right)$$

$$=\frac{(a-bx)^2}{(a-bx)^2+(a+bx)^2}\times\frac{(a-bx)\frac{d}{dx}(a+bx)-(a+bx)\frac{d}{dx}(a-bx)}{(a-bx)^2}$$

$$=\frac{1}{a^2+b^2x^2-2abx+a^2+b^2x^2+2abx}[(a-bx)b-(a+bx)(-b)] = \frac{ab-b^2x+ab+b^2x}{2\left(a^2+b^2x^2\right)}=\frac{ab}{a^2+b^2x^2} \quad \textbf{(Ans)}$$

**2 (i) Solution :**Given, $y=\sec^{-1}\dfrac{\sqrt{x}+1}{\sqrt{x}-1}+\sin^{-1}\dfrac{\sqrt{x}-1}{\sqrt{x}+1} = \cos^{-1}\dfrac{\sqrt{x}-1}{\sqrt{x}+1}+\sin^{-1}\dfrac{\sqrt{x}-1}{\sqrt{x}+1} \quad \left[\because \sec^{-1}x=\cos^{-1}\frac{1}{x}\right]$

or, $y=\dfrac{\pi}{2} \qquad \left[\because \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}\right]$

Differentiating both sides with respect to x we get,

$\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{\pi}{2}\right)=0$ **(Ans)**

**2 (ii) Solution :** Given $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)\right\} = \frac{1}{\sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}} \times \frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$= \frac{1}{\sqrt{\frac{(b+a\cos x)^2-(a+b\cos x)^2}{(b+a\cos x)^2}}} \times \frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2} \quad \left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}\right]$$

$$= \frac{b+a\cos x}{\sqrt{b^2+2ab\cos x+a^2\cos^2 x-a^2-2ab\cos x-b^2\cos^2 x}} \times \frac{\sin x(a^2+ab\cos x-b^2-ab\cos x)}{(b+a\cos x)^2}$$

$$= \frac{(a^2-b^2)\sin x}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x}} \times \frac{1}{(b+a\cos x)^2} = \frac{(a^2-b^2)\sin x}{\sqrt{1-\cos^2 x}\cdot\sqrt{(b^2-a^2)}\cdot(b+a\cos x)}$$

$$= -\frac{\sqrt{(b^2-a^2)}\sin x}{\sin x\cdot(b+a\cos x)} = -\frac{\sqrt{(b^2-a^2)}}{b+a\cos x} \quad \textbf{(Ans)}$$

**2 (iii) Solution :** Given $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$$= \tan^{-1}\left(\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right) = \tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right) = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\frac{a}{b} - x$$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = -1 \quad \textbf{(Ans)} \qquad \left[\frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) = 0\right]$$

**2 (v) Solution :** Given $y = \tan^{-1}\left(\frac{\alpha+\beta x}{\beta-\alpha x}\right) = \tan^{-1}\left(\frac{\frac{\alpha+\beta x}{\beta}}{\frac{\beta-\alpha x}{\beta}}\right) = \tan^{-1}\left(\frac{\frac{\alpha}{\beta}+x}{1-\frac{\alpha}{\beta}x}\right) = \tan^{-1}\frac{\alpha}{\beta} + \tan^{-1}x$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = 0 + \frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2} \quad \textbf{(Ans)} \qquad \left[\frac{d}{dx}\left(\tan^{-1}\frac{\alpha}{\beta}\right) = 0\right]$$

**2 (viii) Solution :** $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) = \tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$ or, $y = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \quad \textbf{(Ans)}$$

**2 (ix) Solution :** Given, $y = \tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$

or, $y = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\right] = \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right]$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}\right] = \tan^{-1}\left[\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}\tan\frac{\pi}{4}}\right] = \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{\pi}{4} - \frac{x}{2}$$

Differentiating both sides with respect to x we get,

or, $\frac{dy}{dx} = -\frac{1}{2}$ **(Ans)**

**3 (i) Solution :** $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $x = \tan\theta$

Then, $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$ $\quad\left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\therefore\ y = 2\tan^{-1}x$

Differentiating both sides with respect to x we get,

$\frac{dy}{dx} = 2\frac{d}{dx}\left(\tan^{-1}x\right) = 2\times\frac{1}{1+x^2} = \frac{2}{1+x^2}$ **(Ans)**

**3 (iii) Solution :** $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let $x = \cos 2\theta$

$\therefore\ y = \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan^{-1}(\tan\theta) = \theta$

or, $y = \frac{1}{2}\cos^{-1}x$ $\quad\left[\because\ x = \cos 2\theta,\ \therefore\ 2\theta = \cos^{-1}x,\ \theta = \frac{1}{2}\cos^{-1}x\right]$

Differentiating both sides with respect to x we get,

$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{1}{2\sqrt{1-x^2}}$ **(Ans)**

**3 (v) Solution :** $y = \cot^{-1}\left[\sqrt{1+x^2} - x\right]$

Let $x = \tan\theta$

$\therefore\ y = \cot^{-1}\left[\sqrt{1+\tan^2\theta} - \tan\theta\right] = \cot^{-1}(\sec\theta - \tan\theta)$

$$= \cot^{-1}\left[\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right] = \cot^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right) = \cot^{-1}\left\{\frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}\right\} = \cot^{-1}\left\{\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right\} = \cot^{-1}\left\{\frac{\cot\frac{\theta}{2}\cdot\cot\frac{\pi}{2} - 1}{\cos\frac{\theta}{2} + \cot\frac{\pi}{2}}\right\}$$

$$= \cot^{-1}\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\pi}{4} + \frac{\theta}{2} = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}x$$

Differentiating both sides with respect to x we get,

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + \frac{1}{2}\tan^{-1}x\right) = 0 + \frac{1}{2}\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{2\left(x^2+1\right)}$ **(Ans)**

**3 (vi) Solution :** $y = \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$

Let $x = \cos\theta$

$$\therefore\ y = \sin\left[2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right] = \sin\left[2\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right]$$

$$= \sin\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right] = \sin\left[2\times\frac{\theta}{2}\right] = \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}$$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{1}{2\sqrt{1-x^2}}\times\frac{d}{dx}\left(-x^2\right) = -\frac{2x}{2\sqrt{1-x^2}}$$

$$\therefore\ \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$$ **(Ans)**

**4 (iii) Solution :** $x^3 + y^3 = 3axy$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(y^3\right) = 3a\frac{d}{dx}(xy) \quad \text{or,}\ 3x^2+3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx}+y\right) \quad \text{or,}\ x^2+y^2\frac{dy}{dx} = ax\frac{dy}{dx}+ay$$

$$\text{or,}\ \left(y^2-ax\right)\frac{dy}{dx} = ay-x^2 \quad \text{or,}\ \frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$$ **(Ans)**

**4 (v) Solution :** $e^{xy} - 4xy = 4$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}\left(e^{xy}\right)-4\frac{d}{dx}(xy) = 0 \quad \text{or,}\ e^{xy}\frac{d}{dx}(xy)-4\frac{d}{dx}(xy) = 0 \quad \text{or,}\ \frac{d}{dx}(xy)\left(e^{xy}-4\right) = 0$$

or, $\frac{d}{dx}(xy) = 0$ [assuming $e^{xy} \neq 4$] or, $x\frac{dy}{dx}+y = 0$ or, $\frac{dy}{dx} = -\frac{y}{x}$ **(Ans)**

**4 (vi) Solution :** $xy = \tan(x + y)$

Differentiating both sides with resptect to x we get,

$$\frac{d}{dx}(xy) = \frac{d}{dx}\{\tan(x+y)\} \quad \text{or,}\ x\frac{dy}{dx}+y = \sec^2(x+y)\times\frac{d}{dx}(x+y)$$

$$\text{or,}\ x\frac{dy}{dx}+y = \sec^2(x+y)\times\left(1+\frac{dy}{dx}\right) = \sec^2(x+y)+\sec^2(x+y)\frac{dy}{dx}$$

$$\text{or,}\ \left\{x-\sec^2(x+y)\right\}\frac{dy}{dx} = \sec^2(x+y)-y$$

$$\text{or,}\ \frac{dy}{dx} = \frac{\sec^2(x+y)-y}{x-\sec^2(x+y)} = \frac{1+\tan^2(x+y)-y}{x-1-\tan^2(x+y)} \quad \text{or,}\ \frac{dy}{dx} = \frac{1+x^2y^2-y}{x-1-x^2y^2}$$ **(Ans)**

**4 (viii) Solution :** $\log(x + y) = x^2 + y^2$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}\{\log(x+y)\} = \frac{d}{dx}\left(x^2+y^2\right) \quad \text{or,}\ \frac{1}{x+y}\times\frac{d}{dx}(x+y) = 2x+2y\frac{dy}{dx}$$

or, $\frac{1}{x+y}\times\left(1+\frac{dy}{dx}\right)=2x+2y\frac{dy}{dx}$ or, $\left(2y-\frac{1}{x+y}\right)\frac{dy}{dx}=\frac{1}{x+y}-2$

$\therefore$ at $x = 1$, $y = 1$, $\left(2-\frac{1}{2}\right)\frac{dy}{dx}=\frac{1}{2}-2$ or, $\frac{3}{2}\frac{dy}{dx}=-\frac{3}{2}$ or, $\frac{dy}{dx}=-1$ **(Ans)**

**5 (iv) Solution :** $y = (\tan x)^{\sin x}$ ..... (1)

Taking logarithm of both sides we get,

$\log y = \log(\tan x)^{\sin x} = \sin x \log(\tan x)$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y)=\frac{d}{dx}\{\sin x\times\log(\tan x)\}$$

or, $\frac{1}{y}\frac{dy}{dx}=\sin x\times\frac{d}{dx}\{\log(\tan x)\}+\log(\tan x)\times\frac{d}{dx}(\sin x)=\sin x\times\frac{1}{\tan x}\times\sec^2 x+\log(\tan x)\times\cos x$

$$\frac{dy}{dx}=y\left\{\sin x\times\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x}+\log(\tan x)\times\cos x\right\}$$

$$= y\{\sec x+\cos x\cdot\log(\tan x)\} = (\tan x)^{\sin x}\{\sec x+\cos x\cdot\log(\tan x)\}$$

$\therefore$ at $x=\frac{\pi}{4}$, $\frac{dy}{dx}=1^{\frac{1}{\sqrt{2}}}\left\{\sec\frac{\pi}{4}+\cos\frac{\pi}{4}\cdot\log\left(\tan\frac{\pi}{4}\right)\right\} = \left\{\sqrt{2}+\frac{1}{\sqrt{2}}\cdot\log(1)\right\}=\sqrt{2}$ **(Ans)**

**5 (vi) Solution :** $y=x^{1+\frac{1}{x}}+\left(1+\frac{1}{x}\right)^x$

or, $y = u + v$ ...................................... (1)

where $u=x^{1+\frac{1}{x}}$ and $v=\left(1+\frac{1}{x}\right)^x$

Taking logarithm of both sides we get,

$\log u=\left(1+\frac{1}{x}\right)\log x$ or, $\frac{d}{dx}(\log u)=\frac{d}{dx}\left\{\left(1+\frac{1}{x}\right)\log x\right\}$ [differentiating both sides with respect to x]

or, $\frac{1}{u}\frac{du}{dx}=\left(1+\frac{1}{x}\right)\frac{d}{dx}(\log x)+\log x\frac{d}{dx}\left(1+\frac{1}{x}\right)$

or, $\frac{1}{u}\frac{du}{dx}=\left(1+\frac{1}{x}\right)\frac{1}{x}+\log x\left(0-\frac{1}{x^2}\right)$ or, $\frac{du}{dx}=u\left\{\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^2}\log x\right\}$ ............................ (2)

and $\log v=x\log\left(1+\frac{1}{x}\right)$ or, $\frac{d}{dx}(\log v)=\frac{d}{dx}\left\{x\log\left(1+\frac{1}{x}\right)\right\}$

or, $\frac{1}{v}\frac{dv}{dx}=x\frac{1}{1+\frac{1}{x}}\left(0-\frac{1}{x^2}\right)+\left(1+\frac{1}{x}\right) = \frac{x^2}{x+1}\left(-\frac{1}{x^2}\right)+\log\left(1+\frac{1}{x}\right)$ or, $\frac{dv}{dx}=v\left\{-\frac{1}{x+1}+\log\left(1+\frac{1}{x}\right)\right\}$ ..... (3)

Differentiating both sides of (1) with respect to x we get,

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

or, $\frac{dy}{dx}=u\left\{\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^2}\log x\right\}+v\left\{-\frac{1}{x+1}+\log\left(1+\frac{1}{x}\right)\right\}$ [from (2) and (3)]

or $\frac{dy}{dx}=x^{1+\frac{1}{x}}\left\{\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^2}\log x\right\}+\left(1+\frac{1}{x}\right)^x\left\{\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right\}$ **(Ans)**

**5 (vii) Solution :** $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

or, $y = u + v$ ..... (1)

where, $u = (\sin x)^{\cos x}$, $v = (\cos x)^{\sin x}$

$\log u = \log(\sin x)^{\cos x} = \cos x \log(\sin x)$ [taking log]

Differentiating both sides with respect to x we get,

$\frac{d}{dx}(\log u) = \frac{d}{dx}(\cos x \log \sin x)$ or, $\frac{1}{u}\frac{du}{dx} = \cos x \frac{d}{dx}(\log \sin x) + \log \sin x \frac{d}{dx}(\cos x)$

$= \frac{1}{u}\frac{du}{dx} = \cos x \times \frac{1}{\sin x} \times \cos x + \log \sin x \times (-\sin x) = \cos x.\cot x - \sin x.\log(\sin x)$

$= \cos x \cdot \cot x - \sin x \cdot \log(\sin x)$

$\therefore \frac{du}{dx} = u\{\cos x \cdot \cot x - \sin x \cdot \log(\sin x)\}$

$= (\sin x)^{\cos x} \{\cos x.\cot x - \sin x.\log(\sin x)\}$ ..... (2)

and $\log v = \sin x.\log(\cos x)$

Differentiating both sides with respect to x we get,

$\frac{d}{dx}(\log v) = \frac{d}{dx}(\sin x \log \cos x)$ or, $\frac{1}{v}\frac{dv}{dx} = \sin x \frac{d}{dx}(\log \cos x) + \log \cos x \frac{d}{dx}(\sin x)$

$= \frac{1}{v}\frac{dv}{dx} = \sin x \times \frac{1}{\cos x} \times (-\sin x) + \log \cos x \times (\cos x) = -\tan x \cdot \sin x + \cos x \cdot \log(\cos x)$

$= (\cos x)^{\sin x}\{\cos x \log(\cos x) - \tan x \sin x\}$ ..... (3)

Differentiating both sides of (1) with respect to x we get,

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

or, $\frac{dv}{dx} = (\sin x)^{\cos x}\{\cos x.\cot x - \sin x.\log(\sin x)\} + (\cos x)^{\sin x}\{\cos x.\log(\cos x) - \tan x.\sin x\}$ **(Ans)**

**6 (i) Solution :** $x^y = y^x$

$y \log x = x \log y$ [taking logarithm of both sides]

Differentiating both sides with respect to x we get,

$\frac{d}{dx}(y \log x) = \frac{d}{dx}(x \log y)$ or, $y\frac{d}{dx}(\log x) + \log x \frac{dy}{dx} = x\frac{d}{dx}(\log y) + \log y \frac{d}{dx}(x)$

or, $y \times \frac{1}{x} + \log x \frac{dy}{dx} = x \times \frac{1}{y}\frac{dy}{dx} + \log y \times 1$ or, $\frac{y}{x} + \log x \frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$

or, $\left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \log y - \frac{y}{x}$ or, $\frac{dy}{dx} = \frac{\log y - \frac{y}{x}}{\log x - \frac{x}{y}}$ or, $\frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}$ **(Ans)**

**6 (ii) Solution :** $x^m y^n = (x + y)^{m+n}$

$\log(x^m y^n) = \log(x + y)^{m+n}$ [taking logarithm of both sides]

or, $m \log x + n \log y = (m + n) \log(x + y)$

Differentiating both sides with respect to x we get,

$$m\frac{d}{dx}(\log x)+n\frac{d}{dx}(\log y)=(m+n)\frac{d}{dx}\{\log(x+y)\}$$

or, $m\times\frac{1}{x}+n\times\frac{1}{y}\frac{dy}{dx}=(m+n)\times\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$ or, $\frac{m}{x}+\frac{n}{y}\times\frac{dy}{dx}=\frac{m+n}{x+y}+\frac{m+n}{x+y}\times\frac{dy}{dx}$

or, $\left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx}=\frac{m+n}{x+y}-\frac{m}{x}$ or, $\frac{nx+ny-my-ny}{y(x+y)}\times\frac{dy}{dx}=\frac{mx+nx-mx-my}{x(x+y)}$

or, $\frac{nx-my}{y(x+y)}\times\frac{dy}{dx}=\frac{nx-my}{x(x+y)}$ or, $\frac{1}{y}\times\frac{dy}{dx}=\frac{1}{x}$ or, $\frac{dy}{dx}=\frac{y}{x}$ **(Ans)**

**6 (iv) Solution :** $x^y = e^{x-y}$

Taking logarithm of both sides we get,

$y\log x = (x-y)\log e = x - y$ $[\because \log e = 1]$ or, $x = y + y\log x = y(1+\log x)$

or, $y=\frac{x}{1+\log x}$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{1+\log x}\right)=\frac{(1+\log x)\times\frac{d}{dx}(x)-x\times\frac{d}{dx}(1+\log x)}{(1+\log x)^2}=\frac{dy}{dx}=\frac{1+\log x-x\times\frac{1}{x}}{(1+\log x)^2}=\frac{1+\log x-1}{(\log e+\log x)^2}$$

$\therefore \frac{dy}{dx}=\frac{\log x}{\{\log(ex)\}^2}$ **(Ans)**

**6 (vi) Solution :** $x^y + y^x = 1$

or, $u + v = 1$

$\therefore \frac{du}{dx}+\frac{dv}{dx}=0$ [differentiating both sides with respect to x] ..... (1)

where, $u = x^y$ or, $\log u = y\log x$ or, $\frac{1}{u}\frac{du}{dx}=y\times\frac{1}{x}+\log x\frac{dy}{dx}$ [differentiating both sides with respect to x]

or, $\frac{du}{dx}=u\left(y\times\frac{1}{x}+\log x\frac{dy}{dx}\right)$ ..... (2)

and $v = y^x$ or, $\log v = x\log y$ or, $\frac{1}{v}\frac{dv}{dx}=x\times\frac{1}{y}\frac{dy}{dx}+\log y\times 1$ or, $\frac{dv}{dx}=v\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)$ ..... (3)

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in (1) we get,

$u\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+v\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)=0$ or, $x^y\times\frac{y}{x}+x^y\times\log x\frac{dy}{dx}+y^x\times\frac{x}{y}\frac{dy}{dx}+y^x\times\log y=0$

or, $\left(x^y\log x+y^{x-1}x\right)\frac{dy}{dx}=-\left(y^x\log y+x^{y-1}y\right)$ or, $\frac{dy}{dx}=-\frac{y^x\log y+x^{y-1}y}{x^y\log x+y^{x-1}x}$ **(Ans)**

**6 (vii) Solution :** $x^y.y^x = x + y$ or, $\log(x^y.y^x) = \log(x+y)$

or, $y\log x + x\log y = \log(x+y)$

or, $\frac{d}{dx}(y\log x)+\frac{d}{dx}(x\log y)=\frac{d}{dx}\{\log(x+y)\}$

[Differentiating both sides w. r. t. x ]

or, $y\frac{d}{dx}(\log x)+\log x\frac{dy}{dx}+x\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)=\frac{1}{x+y}\cdot\frac{d}{dx}(x+y)$

or, $y\cdot\frac{1}{x}+\log x\frac{dy}{dx}+x\cdot\frac{1}{y}\frac{dy}{dx}+\log y=\frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right)=\frac{1}{x+y}+\frac{1}{x+y}\cdot\frac{dy}{dx}$

or, $\left(\log x+\frac{x}{y}-\frac{1}{x+y}\right)\frac{dy}{dx}=\frac{1}{x+y}-\log y-\frac{y}{x}$ or, $\frac{dy}{dx}=\frac{\frac{1}{x+y}-\log y-\frac{y}{x}}{\log x+\frac{x}{y}-\frac{1}{x+y}}$ or, $\frac{dy}{dx}=\frac{y\{x-y(x+y)-x(x+y)\log y\}}{x\{y(x+y)\log x+x(x+y)-y\}}$ **(Ans)**

**7 (iii) Solution :** $y=x\sqrt{\frac{x^2+4}{x^2+3}}$

Taking logarithm of both sides we get,

$$\log y=\log x+\log\left(\frac{x^2+4}{x^2+3}\right)^{\frac{1}{2}}=\log x+\frac{1}{2}\log\left(x^2+4\right)-\frac{1}{2}\log\left(x^2+3\right)$$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(\log x)+\frac{1}{2}\frac{d}{dx}\left\{\log\left(x^2+4\right)\right\}-\frac{1}{2}\frac{d}{dx}\left\{\log\left(x^2+3\right)\right\}$$

or, $\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+\frac{1}{2}\times\frac{1}{x^2+4}\times 2x-\frac{1}{2}\times\frac{1}{x^2+3}\times 2x$

or, $\frac{dy}{dx}=y\left(\frac{1}{x}+\frac{x}{x^2+4}-\frac{x}{x^2+3}\right)=y\left\{\frac{1}{x}+\frac{x^3+3x-x^3-4x}{\left(x^2+4\right)\left(x^2+3\right)}\right\}=x\sqrt{\frac{x^2+4}{x^2+3}}\left\{\frac{1}{x}-\frac{x}{\left(x^2+4\right)\left(x^2+3\right)}\right\}$ **(Ans)**

**7 (v) Solution :** $y=\sqrt{\frac{(x-a)(x-b)}{(x-c)(x-d)}}$

or, $\log y=\log\left\{\frac{(x-a)(x-b)}{(x-c)(x-d)}\right\}^{\frac{1}{2}}$ [taking logarithm of both sides]

$=\frac{1}{2}\log(x-a)+\frac{1}{2}\log(x-b)-\frac{1}{2}\log(x-c)-\frac{1}{2}\log(x-d)$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y)=\frac{1}{2}\frac{d}{dx}\{\log(x-a)\}+\frac{1}{2}\frac{d}{dx}\{\log(x-b)\}-\frac{1}{2}\frac{d}{dx}\{\log(x-c)\}-\frac{1}{2}\frac{d}{dx}\{\log(x-d)\}$$

or, $\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-a}\times 1+\frac{1}{x-b}\times 1-\frac{1}{x-c}\times 1-\frac{1}{x-d}\times 1\right]$

or, $\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-a)(x-b)}{(x-c)(x-d)}}\times\left[\frac{1}{x-a}+\frac{1}{x-b}-\frac{1}{x-c}-\frac{1}{x-d}\right]$ **(Ans)**

**8 (ii) Solution :** $x=a\sec^2\theta,\ y=a\tan^3\theta$

Differentiating both sides with respect to $\theta$ we get,

$$\frac{dx}{d\theta}=a\frac{d}{d\theta}\left(\sec^2\theta\right)=a\times 2\sec\theta\times\frac{d}{d\theta}(\sec\theta)=2a\sec\theta\times\sec\theta.\tan\theta=2a\sec^2\theta.\tan\theta$$

$$\frac{dy}{d\theta}=a\frac{d}{d\theta}\left(\tan^3\theta\right)=3a\tan^2\theta\frac{d}{d\theta}(\tan\theta)=3a\tan^2\theta\times\sec^2\theta=3a\tan^2\theta.\sec^2\theta$$

$$\therefore\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3a\tan^2\theta\sec^2\theta}{2a\sec^2\theta\tan\theta}=\frac{3}{2}\tan\theta$$ **(Ans)**

**8 (iv) Solution :** $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

Differentiating both sides with respect to $\theta$ we get,

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta) = a(1-\cos\theta),\quad \frac{dy}{d\theta} = a\frac{d}{d\theta}(1+\cos\theta) = -a\sin\theta$$

$$\therefore\ \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} \qquad \therefore\ \frac{dy}{dx} = -\cot\frac{\theta}{2}\ \textbf{(Ans)}$$

**8 (vi) Solution :** $x = \sec^{-1}\frac{1}{2v^2-1}$ or, $x = \sec^{-1}\frac{1}{2\cos^2\theta - 1}$ (let $v = \cos\theta$)

$$= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1} v$$

$$\therefore\ \frac{dx}{dv} = 2\frac{d}{dv}\left(\cos^{-1} v\right) = -\frac{2}{\sqrt{1-v^2}}$$

Again, $y = \tan^{-1}\frac{v}{\sqrt{1-v^2}}$ or, $y = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$ [let $v$ = sinq]

$$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta$$

$$\therefore\ y = \sin^{-1} v\ \therefore\ \frac{dy}{dv} = \frac{d}{dv}\left(\sin^{-1} v\right) = \frac{1}{\sqrt{1-v^2}}$$

$$\therefore\ \frac{dy}{dx} = \frac{\frac{dy}{dv}}{\frac{dx}{dv}} = \frac{\frac{1}{\sqrt{1-v^2}}}{-\frac{2}{\sqrt{1-v^2}}} = -\frac{1}{2} \qquad \text{or, } \frac{dy}{dx} = -\frac{1}{2}\ \textbf{(Ans)}$$

**8 (viii) Solution :** Given $\tan y = \frac{2t}{1-t^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$ [let $t = \tan\theta$]

or, $y = 2\theta$ or, $\frac{dy}{d\theta} = 2$ [differentiating both sides with respect to $\theta$]

Again, $\sin x = \frac{2t}{1+t^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$ [let $t = \tan\theta$]

or, $x = 2\theta$ or, $\frac{dx}{d\theta} = 2$ [differentiating both sides with respect to $\theta$]

$$\therefore\ \frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = \frac{2}{2} = 1\ \textbf{(Ans)}$$

**8 (ix) Solution :** $x = \sin\theta\sqrt{\cos 2\theta},\ y = \cos\theta\sqrt{\sin 2\theta}$

Differentiating both sides with respect to $\theta$ we get,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(\sin\theta\sqrt{\cos 2\theta}\right) = \sin\theta\frac{d}{d\theta}\left(\sqrt{\cos 2\theta}\right) + \sqrt{\cos 2\theta}\frac{d}{d\theta}(\sin\theta)$$

$$= \sin\theta \times \frac{1}{2\sqrt{\cos 2\theta}}(-2\sin 2\theta) + \sqrt{\cos 2\theta} \times \cos\theta$$

and $\frac{dy}{d\theta} = \frac{d}{d\theta}\left(\cos\theta\sqrt{\sin 2\theta}\right) = \cos\theta \times \frac{1}{2\sqrt{\sin 2\theta}} \times 2\cos 2\theta + \sqrt{\sin 2\theta} \times (-\sin\theta)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{\cos\theta\cos2\theta}{\sqrt{\sin 2\theta}} - \sin\theta\sqrt{\sin2\theta}}{-\frac{\sin\theta\sin2\theta}{\sqrt{\cos2\theta}} + \cos\theta\sqrt{\cos2\theta}} = \frac{\cos\theta\cos2\theta - \sin\theta\sin2\theta}{\cos\theta\cos2\theta - \sin\theta\sin2\theta} \times \frac{\sqrt{\cos2\theta}}{\sqrt{\sin2\theta}} = \sqrt{\cot 2\theta}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \sqrt{\cot\frac{\pi}{2}} = 0 \quad \textbf{(Ans)}$$

**8 (x) Solution :** $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$

Differentiating both sides with respect to t we get,

$$\frac{dx}{dt} = a\left\{\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right\} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \times \sec^2\frac{t}{2} \times \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}\right) = a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{-\sin^2 t + 1}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$$

Again, $y = a\sin t \quad \therefore \frac{dy}{dt} = a\cos t$,

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\frac{\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t \qquad \therefore \frac{dy}{dx} = \tan t \quad \textbf{(Ans)}$$

**9 (ii) Solution :** Let $u = \sin^{-1}\frac{2x}{1+x^2}$

or, $u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$, [ let $x = \tan\theta$ ] $= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

$\therefore \frac{du}{dx} = 2\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{2}{1+x^2}$ [differentiating both sides with respect to x]

Again, $v = \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\left(\frac{2\tan\alpha}{1-\tan^2\alpha}\right)$ [ let $x = \tan\alpha$ ] $= \tan^{-1}(\tan 2\alpha) = 2\alpha$ $\left[\because \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\right]$

$\therefore v = 2\tan^{-1}x$

$\frac{dv}{dx} = 2\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{2}{1+x^2}$ [differentiating both sides with respectto x]

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1 \qquad \therefore \frac{du}{dv} = 1 \quad \textbf{(Ans)}$$

**9 (iv) Solution :** Let $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$, let $x = \tan\theta$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$\therefore u = \frac{1}{2}\tan^{-1}x = \frac{1}{2}v$ let $v = \tan^{-1}x$

$\therefore \frac{du}{dv} = \frac{1}{2}$ **(Ans)** [differentiating with respect to v]

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**9 (v) Solution :** Let $u = x^{\sin^{-1}x}$ and $v = \sin^{-1}x$

$\therefore\ u = x^v$ or, $\log u = v\log x$ [taking logarithm of both sides]

or, $\frac{d}{dv}(\log u) = \frac{d}{dv}(v\log x)$ [Differentiating both sides with respect to v,]

$$\frac{1}{u}\frac{du}{dv} = v\frac{d}{dv}(\log x) + \log x \times 1 = v \times \frac{1}{x}\frac{dx}{dv} + \log x$$

$$= \frac{\sin^{-1}x}{x}\sqrt{1-x^2} + \log x \qquad \left[\because v = \sin^{-1}x \therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dx}{dv} = \sqrt{1-x^2}\right]$$

$$\therefore \frac{du}{dv} = u\left(\frac{\sin^{-1}x}{x}\sqrt{1-x^2} + \log x\right)$$

$$\frac{du}{dv} = x^{\sin^{-1}x}\left(\frac{\sqrt{1-x^2}}{x}\sin^{-1}x + \log x\right) \text{ (Ans)}$$

**9 (vii) Solution :** Let $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right) = \sin^{-1}(2\sin\theta\cos\theta)$, let $x = \sin\theta$

or, $u = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$

Differentiating both sides with respect to x we get

$$\frac{du}{dx} = 2\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{2}{\sqrt{1-x^2}}$$

Again, $v = \tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$

or, $v = \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right]$ [let $x^2 = \cos\theta$]

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right] = \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right] = \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}}\right] = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{\pi}{4}-\frac{\theta}{2} = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(x^2\right)$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(x^2\right)\right\} = -\frac{1}{2}\frac{d}{dx}\left\{\cos^{-1}\left(x^2\right)\right\} = -\frac{1}{2}\left\{-\frac{1}{\sqrt{1-x^4}}\times 2x\right\} = \frac{x}{\sqrt{1-x^4}}$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{\sqrt{1-x^2}}}{\frac{x}{\sqrt{1-x^4}}} = \frac{2}{\sqrt{1-x^2}} \times \frac{\sqrt{\left(1-x^2\right)\left(1+x^2\right)}}{x} = \frac{2\sqrt{1+x^2}}{x} \text{ (Ans)}$$

**9 (xi) Solution :** Given $y = x^6$ and let $v = x^3$

Then, $\frac{dy}{dx} = 6x^5$ and $\frac{dv}{dx} = 3x^2$

Therefore, $\frac{dy}{dv} = \frac{6x^5}{3x^2} = 2x^3$ **(Ans)**

# MISCELLANEOUS PROBLEM SET

[.....Problems with '*' marks are solved at the end of the problem set.....]

**10.** (i) If $e^y - \frac{a+b\tan x}{a-b\tan x} = 0$ find $\frac{dy}{dx}$.

*(ii) If $y = e^{\sin^{-1}x}$, $z = e^{-\cos^{-1}x}$ show that $\frac{dy}{dx}$ is independent of x. **[HS – 97]**

*(iii) If $x^y = e^{x-y} + \sin x$, find $\frac{dy}{dx}$ **[WBSC – 85]**

(iv) If $y = e^{\cos^{-1}x} + x^{\sqrt{x}}$ find $\frac{dy}{dx}$.

*(v) If $\sin y = x\sin(a+y)$, $a \neq n\pi$, $n = 0, \pm 1$, .....prove that, $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a} = \frac{\sin a}{1-2x\cos a + x^2}$ **[WBSC – 85]**

*(vi) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, show that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **[WBSC – 92, 03, 06, 07]**

(vii) If $\sqrt{1-x^4} + \sqrt{1-y^4} = k\left(x^2 - y^2\right)$, prove that, $\frac{dy}{dx} = \frac{x\sqrt{1-y^4}}{y\sqrt{1-x^4}}$

*(viii) If $\frac{x}{x-y} = \log\frac{a}{x-y}$, show that, $\frac{dy}{dx} = 2 - \frac{x}{y}$ **[WBSC – 09]**

(ix) If $f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$, prove that $f'(0) = \left[2\log\frac{a}{b} + \frac{b^2-a^2}{ab}\right]\left(\frac{a}{b}\right)^{a+b}$

*(x) If $f(x) = \left(\frac{a+x}{1+x}\right)^{a+1+2x}$, find the value of $f'(0)$ **[WBSC – 97, 17 HS – 96]**

*(xi) If $y = \frac{1}{4\sqrt{2}}\log\left(\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right) + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$, find the value of $\frac{dy}{dx}$ in the simplest form. **[WBSC – 94]**

(xii) If $y = \log\frac{1+x}{1-x} + \frac{1}{2}\log\frac{1+x+x^2}{1-x+x^2} + \sqrt{3}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}$ show that, $\frac{dy}{dx} = \frac{6}{1-x^6}$

*(xiii) If $y = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \log\sqrt{x+\sqrt{x^2+1}}$ prove that, $2y = x\frac{dy}{dx} + \log\left(\frac{dy}{dx}\right)$.

*(xiv) Express y explicitly as a function of x and then find $\frac{dy}{dx}$, when $e^y - e^{-y} = 2x$ **[WBSC – 91]**

*(xv) If $y = x^{x^{x^{\cdots\infty}}}$ find $\frac{dy}{dx}$

*(xvi) If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots}}}$, find $\frac{dy}{dx}$.

(xvii) If $y = e^{x + e^{x + e^{x + \cdots \text{to } \infty}}}$ prove that $\frac{dy}{dx} = \frac{y}{1-y}$

**11.** *(i) If $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$, then show that, $\left(x^2+4\right)\left(\frac{dy}{dx}\right)^2 = n^2\left(y^2+4\right)$ **[IIT – 89]**

(ii) If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$ prove that $x^3 y \frac{dy}{dx} = -1$

(iii) If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$ prove that $x^3 y \frac{dy}{dx} = 1$

*(iv) If $x = \tan\frac{y}{2} - \log\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}$ show that, $\frac{dy}{dx} = \frac{1}{2}\sin y(1+\cos y + \sin y)$

(v) If $\tan^{-1}\frac{y}{x} = \log\sqrt{x^2+y^2}$ prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$

## ANSWERS

10. (i) $\frac{2ab}{a^2\cos^2 x - b^2\sin^2 x}$ (iii) $\frac{e^{x-y} - x^{y-1}.y + \cos x}{x^y \log x + e^{x-y}}$ (iv) $-\frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}} + \frac{x^{\sqrt{x}}}{\sqrt{x}}\left(1+\frac{1}{2}\log x\right)$ (x) $\left[2\log a + \frac{1-a^2}{a}\right]a^{a+1}$

(xi) $\frac{1}{1+x^4}$ (xiv) $\pm\frac{1}{\sqrt{x^2+1}}$ (xv) $\frac{y^2}{x(1-y\log x)}$ (xvi) $\frac{\cos x}{2y-1}$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

10 (ii) **Solution :** $y = e^{\sin^{-1}x}$, $z = e^{-\cos^{-1}x}$

$\therefore \frac{y}{z} = \frac{e^{\sin^{-1}x}}{e^{-\cos^{-1}x}} = e^{\sin^{-1}x + \cos^{-1}x} = e^{\frac{\pi}{2}}$ $\quad \therefore y = e^{\frac{\pi}{2}}.z$

Differentiating both sides with respect to z we get,

$\frac{dy}{dz} = e^{\frac{\pi}{2}}$, which is independent of x. **(Proved)**

10 (iii) **Solution :** $x^y = e^{x-y} + \sin x$ or, $u = e^{x-y} + \sin x$

or, $\frac{du}{dx} = e^{x-y} \times \left(1 - \frac{dy}{dx}\right) + \cos x$ ..... (1)

where $u = x^y$ or, $\log u = y\log x$

or, $\frac{1}{u}\frac{du}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx}$

or, $\frac{du}{dx} = u\left\{y \times \frac{1}{x} + \log x\frac{dy}{dx}\right\} = x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$

From (1) we get,

$x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) = e^{x-y} \times \left(1 - \frac{dy}{dx}\right) + \cos x$

or, $x^{y-1}.y + x^y\log x\frac{dy}{dx} = e^{x-y} - e^{x-y}\frac{dy}{dx} + \cos x$

or, $\left(x^y.\log x + e^{x-y}\right)\frac{dy}{dx} = e^{x-y} - x^{y-1}.y + \cos x$

$\therefore \frac{dy}{dx} = \frac{e^{x-y} - x^{y-1}.y + \cos x}{x^y.\log x + e^{x-y}}$ **(Ans)**

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10. (v) **Solution :** $\sin y = x\sin(a+y)$ ..... (1)

or, $x = \dfrac{\sin y}{\sin(a+y)}$

Differentiating both sides with respect to y we get,

or, $\dfrac{dx}{dy} = \dfrac{\sin(a+y)\times\frac{d}{dy}(\sin y)-\sin y\,\frac{d}{dy}\{\sin(a+y)\}}{\sin^2(a+y)} = \dfrac{\sin(a+y)\cos y-\sin y.\cos(a+y)}{\sin^2(a+y)} = \dfrac{\sin(a+y-y)}{\sin^2(a+y)} = \dfrac{\sin a}{\sin^2(a+y)}$

$\therefore \dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$, $a \neq n\pi$, $n = 0, \pm1$, ........................ ..... (2)

Now, $\sin y = x\sin(a+y)$.

or, $\sin y = x(\sin a\cos y + \cos a\sin y)$ or, $(1 - x\cos a)\sin y = x\sin a\cos y$

or, $\dfrac{\sin y}{x\sin a} = \dfrac{\cos y}{1-x\cos a} = \dfrac{\sqrt{\sin^2 y+\cos^2 y}}{\sqrt{x^2\sin^2 a+(1-x\cos a)^2}} = \dfrac{1}{\sqrt{x^2\sin^2 a+1+x^2\cos^2 a-2x\cos a}} = \dfrac{1}{\sqrt{1-2x\cos a+x^2}}$

$\therefore \sin y = \dfrac{x\sin a}{\sqrt{1-2x\cos a+x^2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{\sin^2 y}{x^2\sin a} = \dfrac{x^2\sin^2 a}{x^2\sin a(1-2x\cos a+x^2)} = \dfrac{\sin a}{1-2x\cos a+x^2}$ [from (1) & (2)]

$\therefore \dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a} = \dfrac{\sin a}{1-2x\cos a+x^2}$ $a \neq n\pi$, $n = 0, \pm1$, .... **(Proved)**

10 (vi) **Solution :** $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$

Let $x = \cos\theta$, $y = \cos\varphi$

$\therefore \sqrt{1-\cos^2\theta}+\sqrt{1-\cos^2\varphi} = a(\cos\theta-\cos\varphi)$ or, $\sin\theta + \sin\varphi = a(\cos\theta - \cos\varphi)$

or, $2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} = 2a\sin\frac{\theta+\varphi}{2}\sin\frac{\varphi-\theta}{2}$ or, $\cos\frac{\theta-\varphi}{2} = -a\sin\frac{\theta-\varphi}{2}$ or, $\dfrac{\cos\frac{\theta-\varphi}{2}}{\sin\frac{\theta-\varphi}{2}} = -a$

or, $\cot\frac{\theta-\varphi}{2} = -a$ or, $\frac{\theta-\varphi}{2} = \cot^{-1}(-a)$ or, $\theta - \varphi = 2\cot^{-1}(-a)$

or, $\cos^{-1}x - \cos^{-1}y = 2\cot^{-1}(-a)$

Differentiating both sides with respect to x we get,

$-\dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-y^2}}\cdot\dfrac{dy}{dx} = 0$ or, $\dfrac{dy}{dx} = \dfrac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$ **(Proved)**

10 (viii) **Solution :** $\dfrac{x}{x-y} = \log\dfrac{a}{x-y}$ or, $\dfrac{x}{x-y} = \log a - \log(x-y)$

Differentiating both sides with respect to x we get,

$\dfrac{(x-y)\times1-x\left(1-\frac{dy}{dx}\right)}{(x-y)^2} = 0 - \dfrac{1}{x-y}\left(1-\dfrac{dy}{dx}\right)$ or, $x - y - x\left(1-\dfrac{dy}{dx}\right) = -(x-y)\left(1-\dfrac{dy}{dx}\right)$

or, $x - y = (x - x + y)\left(1-\dfrac{dy}{dx}\right) = y - y\dfrac{dy}{dx}$ or, $y\dfrac{dy}{dx} = 2y - x$ or, $\dfrac{dy}{dx} = 2 - \dfrac{x}{y}$ **(Proved)**

10 (x) **Solution :** $f(x) = \left(\frac{a+x}{1+x}\right)^{a+1+2x}$

or, $\log f(x) = (a+1+2x)\log\left(\frac{a+x}{1+x}\right)$ [taking log of both sides]

or, $\log f(x) = (a+1+2x)\{\log(a+x) - \log(1+x)\}$

Differentiating both sides with respect to x we get,

$$\frac{1}{f(x)} \cdot f'(x) = (a+1+2x)\left(\frac{1}{a+x} - \frac{1}{1+x}\right) + (0 + 2\times 1)\log\left(\frac{a+x}{1+x}\right)$$

Putting $x = 0$, we get,

$$\frac{1}{f(0)} \cdot f'(0) = (a+1)\left(\frac{1}{a} - 1\right) + 2\log a = \frac{(1+a)(1-a)}{a} + 2\log a$$

or, $f'(0) = \left[\frac{1-a^2}{a} + 2\log a\right]f(0) = \left[2\log a + \frac{1-a^2}{a}\right]a^{a+1}$ $\left[\because f(x) = \left(\frac{a+x}{1+x}\right)^{a+1+2x} \therefore f(0) = a^{a+1}\right]$ **(Ans)**

10 (xi) **Solution :** $y = \frac{1}{4\sqrt{2}}\log\left(\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right) + \frac{1}{2\sqrt{2}} \cdot \tan^{-1}\frac{x\sqrt{2}}{1-x^2}$

$$= \frac{1}{4\sqrt{2}}\log\left(1 + x\sqrt{2} + x^2\right) - \frac{1}{4\sqrt{2}}\log\left(1 - x\sqrt{2} + x^2\right) + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \frac{1}{4\sqrt{2}} \times \frac{1}{1+x\sqrt{2}+x^2} \times \left(\sqrt{2} + 2x\right) - \frac{1}{4\sqrt{2}} \times \frac{1}{1-x\sqrt{2}+x^2} \times \left(-\sqrt{2} + 2x\right)$$

$$+ \frac{1}{2\sqrt{2}}\frac{1}{1+\frac{2x^2}{(1-x^2)^2}} \times \left\{\frac{\sqrt{2}\left(1-x^2\right) - x\sqrt{2}(-2x)}{\left(1-x^2\right)^2}\right\}$$

$$= \frac{\sqrt{2}x+1}{4\left(x^2+x\sqrt{2}+1\right)} - \frac{\sqrt{2}x-1}{4\left(x^2-\sqrt{2}x+1\right)} + \frac{\left(1-x^2\right)^2}{2\sqrt{2}\left\{\left(1-x^2\right)^2+2x^2\right\}} \times \frac{\sqrt{2}-\sqrt{2}x^2+2\sqrt{2}x^2}{\left(1-x^2\right)^2}$$

$$= \frac{\left(\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right) - \left(\sqrt{2}x-1\right)\left(x^2+x\sqrt{2}+1\right)}{4\left(x^2+1-x\sqrt{2}\right)\left(x^2+1+x\sqrt{2}\right)} + \frac{\sqrt{2}\left(1+x^2\right)}{2\sqrt{2}\left(1+x^4\right)}$$

$$= \frac{\sqrt{2}x\left(x^2-\sqrt{2}x+1-x^2-x\sqrt{2}-1\right)+1\left(x^2-\sqrt{2}x+1+x^2+x\sqrt{2}+1\right)}{4\left\{\left(x^2+1\right)^2-2x^2\right\}} + \frac{1+x^2}{2\left(1+x^4\right)}$$

$$= \frac{-4x^2+2x^2+2}{4\left(1+x^4\right)} + \frac{1+x^2}{2\left(1+x^4\right)} = \frac{1-x^2}{2\left(1+x^4\right)} + \frac{1+x^2}{2\left(1+x^4\right)} = \frac{1-x^2+1+x^2}{2\left(1+x^4\right)}$$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^4}$ **(Ans)**

**10 (xiii) Solution :** $y = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \log\sqrt{x+\sqrt{x^2+1}}$ ................ (1)

or, $y = \frac{x^2}{2} + \frac{x\sqrt{x^2+1}}{2} + \frac{1}{2}\log\left(x+\sqrt{x^2+1}\right)$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = x + \frac{1}{2}\sqrt{x^2+1} + \frac{x}{2}\times\frac{1}{2\sqrt{x^2+1}}\times 2x + \frac{1}{2}\times\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\sqrt{x^2+1}}\times 2x\right)$$

$$= x + \frac{1}{2}\sqrt{x^2+1} + \frac{x^2}{2\sqrt{x^2+1}} + \frac{1}{2}\frac{1}{x+\sqrt{x^2+1}}\times\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} = x + \frac{1}{2}\sqrt{x^2+1} + \frac{x^2}{2\sqrt{x^2+1}} + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{1}{2}\sqrt{x^2+1} + \frac{x^2+1}{2\sqrt{x^2+1}} = x + \frac{1}{2}\sqrt{x^2+1} + \frac{1}{2}\sqrt{x^2+1} \quad \text{or,} \quad \frac{dy}{dx} = x + \sqrt{x^2+1} \quad \text{..... (2)}$$

From (1) we get,

$$2y = x\left(x+\sqrt{x^2+1}\right) + \log\left(x+\sqrt{x^2+1}\right) \quad \text{or,} \quad 2y = x\frac{dy}{dx} + \log\left(\frac{dy}{dx}\right) \text{[from (2)]}$$ **(Proved)**

10 (xiv) **Solution :** $e^y - e^{-y} = 2x$ or, $(e^y)^2 - 1 = 2xe^y$ or, $(e^y)^2 - 2xe^y - 1 = 0$

$$\therefore e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4\times 1(-1)}}{2\times 1} = \frac{2x\pm\sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2+1}\right)$$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\log\left(x\pm\sqrt{x^2+1}\right)\right\} = \frac{1}{x\pm\sqrt{x^2+1}}\left(1 \pm \frac{1}{2\sqrt{x^2+1}}\times 2x\right)$$

$$= \frac{1}{x\pm\sqrt{x^2+1}}\left(1\pm\frac{x}{\sqrt{x^2+1}}\right) = \pm\frac{1}{x\pm\sqrt{x^2+1}}\left(\frac{x\pm\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) = \pm\frac{1}{\sqrt{x^2+1}}$$ **(Ans)**

10. (xv) **Solution :** $y = x^{x^{x\cdots\cdots\infty}}$

Taking log of both sides we get,

$$\log y = x^{x^{x\cdots\cdots\infty}}.\log x = y\log x$$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y\log x) \quad \text{or,} \quad \frac{1}{y}\frac{dy}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx} \quad \text{or,} \quad \frac{1}{y}\frac{dy}{dx} = y\times\frac{1}{x} + \log x\frac{dy}{dx}$$

$$\text{or,} \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x} \quad \text{or,} \quad \frac{dy}{dx} = \frac{y}{x\left(\frac{1}{y}-\log x\right)} = \frac{y^2}{x(1-y\log x)}$$ **(Ans)**

10 (xvi) **Solution :** $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots \text{to } \infty}}}$

Squaring we get,

$$y^2 = \sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots \infty}}} = \sin x + y$$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\sin x) + \frac{dy}{dx} \quad \text{or, } 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

or, $(2y-1)\dfrac{dy}{dx} = \cos x \quad \therefore \dfrac{dy}{dx} = \dfrac{\cos x}{2y-1}$ **(Ans)**

11 (i) **Solution :** Given, $x = \sec\theta - \cos\theta$

$$\therefore \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \tan\theta(\sec\theta + \cos\theta)$$

Again, $y = \sec^n\theta - \cos^n\theta$

$$\therefore \frac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta + n\cos^{n-1}\theta\sin\theta = n\tan\theta(\sec^n\theta + \cos^n\theta)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

$$\text{or, } \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2} = \frac{n^2\left\{(\sec^n\theta - \cos^n\theta)^2 + 4\right\}}{(\sec\theta + \cos\theta)^2 + 4} = \frac{n^2(y^2+4)}{x^2+4}$$

or, $(x^2+4)\left(\dfrac{dy}{dx}\right)^2 = n^2(y^2+4)$ **(Proved)**

11 (iv) **Solution :** Given, $x = \tan\frac{y}{2} - \log\dfrac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}$

$x = \tan\frac{y}{2} + \log\tan\frac{y}{2} - 2\log\left(1+\tan\frac{y}{2}\right)$ or, $x = t + \log t - 2\log(1+t)$ ............................(1)

where $\tan\frac{y}{2} = t$ or, $y = 2\tan^{-1}t$ or, $\dfrac{dy}{dt} = \dfrac{2}{1+t^2}$

and $\dfrac{dx}{dt} = 1 - \dfrac{2}{1+t} + \dfrac{1}{t} = \dfrac{t+t^2-2t+1+t}{t(1+t)} = \dfrac{1+t^2}{t(1+t)}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2}{1+t^2} \times \frac{t(1+t)}{1+t^2} = \frac{2}{\sec^4\frac{y}{2}} \times \tan\frac{y}{2}\left(1+\tan\frac{y}{2}\right)$$

$$= 2\cos^2\frac{y}{2}\left(\sin\frac{y}{2}\cos\frac{y}{2} + \sin^2\frac{y}{2}\right) = \frac{1}{2}(1+\cos y)(\sin y + 1 - \cos y)$$

$$= \frac{1}{2}\left\{\sin y(1+\cos y) + \sin^2 y\right\} = \frac{1}{2}\sin y(1+\sin y+\cos y)$$

or, $\dfrac{dy}{dx} = \dfrac{1}{2}\sin y(1+\cos y+\sin y)$ **(Proved)**

# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. $\frac{d}{dx}\left(\sin^2 x\right)$ is (a) 2sinx (b) $2\cos^2 x$ (c) sin2x (d) none of these. [WBSC – 12]

2. If y = logtanx, then $\frac{dy}{dx}$ is equal to : (a) 2sec2x, (b) 2cosec2x,, (c) $2\sec^3 x$, (d) $2\operatorname{cosec}^3 x$. [WBSC – 05]

3. If $y = \log\tan x^2$ then $\frac{dy}{dx}$ is – (a) $\cot x^2$ (b) $2x\cot x^2$ (c) $2x\operatorname{cosec}x^2$ (d) none of these. [WBSC – 12]

4. If f(x) = logsinax then $f'(x)$ is – (a) $\frac{a}{\sin ax}$, (b) cotx, (c) a cotax, (d) tanax. [WBSC – 10]

5. If $y = \tan^{-1}\left[\frac{\sin x}{1+\cos x}\right]$, then the value of $\frac{dy}{dx}$ is: (a) $-\frac{1}{2}$; (b) $\frac{1}{2}$; (c) –1; (d) None of these. [WBSC – 03]

6. If $y = \tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$, then the value of $\frac{dy}{dx}$ is: (a) 1; (b) $\frac{1}{2}$; (c) $-\frac{1}{2}$; (d) 2 [WBSC – 06]

7. If $f(x) = x + \log_e x^x$, $f'(x)$ is – (a) 2, (b) $\log_e x$, (c) $2 + \log_e x$, (d) none is true. [WBSC – 08, 09]

8. If $f(x) = \log_e e^x + e^{\log_e x}$ then $f'(x)$ is – (a) 2 (b) $e^x + 1$ (c) $e^x + x$ (d) none of these [WBSC – 11]

9. If $f(x) = e^{\tan^{-1}x}$, $f'(x)$ is – (a) $\frac{e^{\tan^{-1}x}}{1+x^2}$, (b) $e^{\tan^{-1}x}\cdot\tan^{-1}x$, (c) $\frac{1}{1+x^2}$; (d) $1 + x^2$ [WBSC – 08, 09]

10. Find $\frac{d}{dx}\left(e^{\tan^{-1}x}\right)$ [WBSC – 11]

11. If $y = \tan^{-1}\left(\frac{a+bx}{a-bx}\right)$, then $\frac{dy}{dx}$ is equal to : (a) $\frac{1}{1+x^2}$ (b) $-\frac{1}{1+x^2}$, (c) $1 + x^2$, (d) none [WBSC – 04]

12. If $y = \sin^{-1}\frac{2x}{1+x^2}$, then $\frac{dy}{dx}$ is – (a) 2, (b) $\frac{1}{1+x^2}$ (c) $\frac{2}{1+x^2}$ (d) none of these. [WBSC – 07]

13. If $y = \sqrt{1+x^2}$, $\frac{dy}{dx}$ is – (a) $2\sqrt{1+x^2}$ (b) $\frac{x}{\sqrt{1+x^2}}$ (c) $\frac{2x}{\sqrt{1+x^2}}$ (d) none of these. [WBSC – 09]

14. $\sqrt{x}+\sqrt{y}=4$ then $\frac{dy}{dx}$. at y = 1 is : (a) 3 (b) $\frac{1}{3}$ (c) $-\frac{1}{3}$ (d) –3 [WBSC – 08]

15. If $y = x^6$, then differential coefficient of y with respect to $x^3$ is – (a) $6x^6$ (b) $3x^2$ (c) $2x^3$ (d) $x^3$. [WBSC – 05]

16. Differential coefficients of $x^4$ with respect to $x^2$ is – (a) $4x^2$, (b) $4x^3$, (c) $2x^2$, (d) 4x. [WBSC – 10, 12]

17. The differential coefficient of $x^6$ with respect to $x^2$ is $3x^4$ – (a) True (b) False [WBSC – 08]

**18.** Differential coefficient of $x^6$ with respect to $x^2$ is – (a) $6x^4$ (b) $3x^4$ (c) $6x^5$ (d) none of these. **[WBSC – 11]**

**19.** If $y = \log_a x$, then $\frac{dy}{dx}$ is – (a) $\frac{x}{\log_e a}$ (b) $\frac{\log_e a}{x}$ (c) $x \log_e a$ (d) $\frac{1}{x \log_e a}$ **[WBSC – 09]**

**20.** If $y = \tan^{-1}x$, then $\frac{dy}{dx}$ is – (a) $\cos^{-1}x$ (b) $\sec^{-1}x$ (c) $\sec^2 x$ (d) none of these **[WBSC – 09]**

**21.** If $y = \sin(\sin nx)$ then $\frac{dy}{dx}$ is – (a) $\cos(\cos nx)$ (b) $\cos(\sin nx)$ (c) $n\cos(\sin nx)$ (d) none of these. **[WBSC – 09]**

## SUBJECTIVE TYPE

**1.** If $x^p . y^q = (x + y)^{p+q}$, prove that $\frac{dy}{dx} = \frac{y}{x}$ **[WBSC – 05, 08, 14]**

**2.** Find $\frac{dy}{dx}$: $x^3y^4 = (x+y)^7$ **[WBSC – 18]**

**3.** If $y = \tan\sqrt{\frac{x+1}{x-1}}$ find $\frac{dy}{dx}$ **[WBSC – 07, 08]**

**4.** If $y = \tan^{-1}\sqrt{\frac{1+x}{1-x}}$ find $\frac{dy}{dx}$ **[WBSC – 10]**

**5.** Find $\frac{dy}{dx}$; if $y = \cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ **[WBSC – 15]**

**6.** Find $\frac{dy}{dx}$ when $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ **[WBSC – 17]**

**7.** If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, show that : $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **[WBSC – 03, 06, 07, 09, 10, 16]**

**8.** If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then find the value of $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ **[WBSC – 09, 17]**

**9.** Find $\frac{dy}{dx}$: $e^{xy} - 4xy = 4$ **[WBSC – 18]**

**10.** If $y = \cos^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, find $\frac{dy}{dx}$. **[WBSC – 08]**

**11.** If $y = \sin^{-1}\frac{1-x^2}{1+x^2}$, find $\frac{dy}{dx}$ **[WBSC – 11]**

**12.** Find $\frac{dy}{dx}$ if $y = (\tan x)^{\sin x}$ **[WBSC – 16]**

**13.** If $x^x . y^y = x + y$, find $\frac{dy}{dx}$ **[WBSC – 08]**

**14.** If $x^y . y^x = x + y$, find $\frac{dy}{dx}$ **[WBSC – 04]**

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15. If $x^y = (x + y).y^{-x}$, find $\frac{dy}{dx}$ [WBSC – 06]

16. If $x^y = e^{x-y}$, then prove that : $\frac{dy}{dx} = \frac{\log x}{(\log ex)^2}$ [WBSC – 03, 12]

17. If $f(x) = \left(\frac{a+x}{1+x}\right)^x$, find the value of $f'(0)$ [WBSC – 17]

18. If $\frac{x}{x-y} = \log_e \frac{a}{x-y}$, show that $\frac{dy}{dx} = 2 - \frac{x}{y}$ [WBSC – 09]

19. If $\tan y = \frac{2t}{1-t^2}$ and $\sin x = \frac{2t}{1+t^2}$, prove that $\frac{dy}{dx} = 1$ [WBSC – 05]

20. If $y = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ and $x = a\sin\theta$ find $\frac{dy}{dx}$ [WBSC – 11]

21. Find the derivative of sinx with respect to $\log_e x$. [WBSC – 17]

22. Differentiate $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$ [WBSC – 11]

23. Differentiate $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$. [WBSC – 09, 12, 16]

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Chapter 5

# Second Order Derivative

## 5.1 INTRODUCTION :

For a differentiable function y = f(x), the derivative $\frac{dy}{dx}$ or $f'(x)$ or $y_1$ is called the first order derivative. We have seen that $\frac{dy}{dx}$ or $f'(x)$ or $y_1$ is in general a function of x. This new function $f'(x)$ or $\frac{dy}{dx}$ may have a derivative, which is called the **second order derivative.**

Thus the derivative of first order derivative is called the second order derivative or second order differential coefficient of the function y = f(x) and is denoted by $f''(x)$ or, $\frac{d^2y}{dx^2}$, or $y_2$.

That is, $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$ or, $\frac{d}{dx}\{f'(x)\}=f''(x)$

Now, since $f''(x)=\frac{d}{dx}\{f'(x)\}$, from definition of derivative we get,

$\frac{d^2y}{dx^2}$ or, $f''(x)=\underset{h\to 0}{\text{Lt}}\left[\frac{f'(x+h)-f'(x)}{h}\right]$, provided limit exist.

That is, $\underset{h\to 0-}{\text{Lt}}\left[\frac{f'(x+h)-f'(x)}{h}\right]=\underset{h\to 0+}{\text{Lt}}\left[\frac{f'(x+h)-f'(x)}{h}\right]$

That is, $Lf'(x)=Rf'(x)$

In general, second order derivative i.e., $\frac{d^2y}{dx^2}$ or, $f''(x)$ or, $y_2$ of y = f(x) is a function of x.

The derivative of the second order derivative is called the **third order derivative** of y = f(x) and is denoted by $\frac{d^3y}{dx^3}$ or, $f'''(x)$ or, $y_3$.

Similarly, if we differentiate y = f(x) successively n times with respect to x then the result obtained is called **nth order derivative** and is denoted by $\frac{d^ny}{dx^n}$ or, $f^n(x)$ or $y_n$.

For example, let $y = x^4$

Differentiating both sides with respect to x we get,

$\frac{dy}{dx}=f'(x)=\frac{d}{dx}\left(x^4\right)=4x^3$ [ is called First Order Derivative]

Again differentiating both sides with respective to x we get,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\{f'(x)\}=\frac{d}{dx}\left(4x^3\right)=12x^2 \text{ or, } \frac{d^2y}{dx^2}=f''(x)=12x^2$$ [ is called Second Order Derivative]

Again differentiating both sides with respective to x we get,

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\{f''(x)\}=\frac{d}{dx}\left(12x^2\right)=24x \text{ or, } \frac{d^3y}{dx^3}=f'''(x)=24x$$ [ is called Third Order Derivative] etc.

In this chapter we will discuss only about up to second order derivative.

## PROBLEM SET

[.....Problems with '*' marks are solved at the end of the problem set.....]

1. Find $\frac{d^2y}{dx^2}$ when,

(a) $y = \cos^2 x$ *(b) $y = x^2 \log x^2$, at $x = 1$ [HS – 96]

*(c) $y = e^{ax^4}$ *(d) $y = \log x^2$ [WBSC – 06]

*(e) $y = \log_e \tan\frac{x}{2}$ [WBSC – 07]

2. Find $\frac{d^2y}{dx^2}$ in the following cases :

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ii) $y^2 = 4ax$,

*(iii) $x^3 + y^3 = 3axy$, *(iv) $x^m y^n = (x + y)^{m-n}$ [JEE – 92]

*(v) $x\sqrt{1+y}+y\sqrt{1+x}=0$ (vi) $\sin x + \cos y = 1$

(vii) $y^4 + 5xy + y = 2$, at $x = 0$, $y = 1$. (viii) $x + y = e^{x-y}$ [JEE – 90]

*(ix) $y=\frac{1}{1+x+x^2+x^3}$, at $x = 0$ (x) $y=\frac{\log x}{x}$, at $x = 1$.

3. *(i) If $ax^2 + 2hxy + by^2 = 1$, prove that, $\frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^3}$ [WBSC – 91, 96, 15]

*(ii) If $y^2 = mx^2 + 2lx + n$, show that, $\frac{d^2x}{dy^2}=\frac{l^2-mn}{(mx+l)^3}$ [WBSC – 92]

4. Find $\frac{d^2y}{dx^2}$ in each of the following cases :

(i) $x = a\cos^3 t$, $y = b\sin^3 t$ *(ii) $x = \cos\theta(1 + \cos\theta)$, $y = \sin\theta(1 + \cos\theta)$ [WBSC – 88]

(iii) $x = a(k\sin t + \sin kt)$, $y = a(k\cos t + \cos kt)$. (iv) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, at $\theta=\frac{\pi}{2}$

[WBSC – 14]

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*(v) $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$, at $\theta = \frac{\pi}{3}$.

(vi) $x = \frac{3at}{1+t^2}$, $y = \frac{3at^2}{1+t^2}$

5. *(i) If $y = A.e^{mx} + B.e^{-mx}$, prove that, $\frac{d^2y}{dx^2} - m^2y = 0$ [WBSC – 83]

(ii) If $y = A\sin mx + B\cos mx$, prove that $\frac{d^2y}{dx^2} + m^2y = 0$

*(iii) If $y = Ae^{mx} + Be^{nx}$, prove that, $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

6. (i) If $y = (\cos^{-1}x)^2$, then prove that, $(1 - x^2)y_2 - xy_1 = 2$ [HS – 95]

*(ii) If $y = (\sin^{-1}x)^2$, then prove that, $(1 - x^2)y_2 - xy_1 = 2$ [WBSC – 08, 12, 16]

(iii) If $y = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$, then, prove that $(1 - x^2)y_2 - xy_1 = 4$.

7. *(i) If $y = \left(x + \sqrt{1+x^2}\right)^m$, then prove that, $(1 + x^2)y_2 + xy_1 = m^2y$ [WBSC – 10, 15]

(ii) If $2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$, show that $(x^2 - 1)y_2 + xy_1 = m^2y$.

*(iii) If $y^{\frac{1}{2}} + y^{-\frac{1}{2}} = 2x$, prove that, $(x^2 - 1)y_2 + xy_1 = 4y$ [WBSC – 88]

8. *(i) If $y = e^{ax}\cos bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$ [WBSC – 87, 99, 00]

(ii) If $y = e^{ax}\sin bx$, prove that, $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

*(iii) If $y = e^x(a\cos x + b\sin x)$, show that $y_2 - 2y_1 + 2y = 0$ [WBSC – 89, 90]

(iv) If $y = e^x(\cos x + \sin x)$, show that, $y_2 - 2y_1 + 2y = 0$

*(v) If $x = Ae^{-\frac{kt}{2}}\cos(pt + c)$, show that, $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + n^2x = 0$ where $n^2 = p^2 + \frac{k^2}{4}$

(vi) If $x = (a + bt)e^{-nt}$, show that, $\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + n^2x = 0$

9. (i) If $y = e^{m\sin^{-1}x}$, prove that, $(1 - x^2)y_2 - xy_1 = m^2y$ [WBSC – 11, 17]

*(ii) If $x = \sin\left(\frac{1}{a}\log_e y\right)$, show that, $(1 - x^2)y_2 - xy_1 - a^2y = 0$ [WBSC – 06, 07]

(iii) If $x = \cos(\log_e y)$, prove that, $(1 - x^2)y_2 - xy_1 - y = 0$

10. *(i) If $x = f(t)$, $y = \phi(t)$, show that, $\frac{d^2y}{dx^2} = \frac{x_1y_2 - x_2y_1}{x_1^3}$, where suffixes denote differentiation with respect to t.

*(ii) If $x = \sin t$, $y = \sin pt$, prove that, $(1 - x^2)y_2 - xy_1 + p^2y = 0$ [WBSC – 04]

(iii) If $x = \sin t$, $y = \cos pt$, prove that, $(1 - x^2)y_2 - xy_1 + p^2y = 0$ **[WBSC – 93]**

(iv) If $x = \tan t$, $y = \tan pt$, prove that, $(1 + x^2)y_2 + 2(x - py)y_1 = 0$.

(v) If $x = e^t \sin t$, $y = e^t \cos t$, show that, $(x+y)^2 \frac{d^2y}{dx^2} = 2\left(x\frac{dy}{dx} - y\right)$

(vi) If $x = \sin t$, $y = \sin 2t$, prove that, (a) $(1 - x^2){y_1}^2 = 4(1 - y^2)$ (b) $(1 - x^2)y_2 - xy_1 + 4y = 0$.

*(vii) If $x = \cos t$, $y = \log t$, prove that at $t = \frac{\pi}{2}$, $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

11. (i) If $y = \sin(2\sin^{-1}x)$, prove that, $(1 - x^2)y_2 - xy_1 + 4y = 0$ **[WBSC – 98]**

*(ii) If $y = \sin(m\sin^{-1}x)$, prove that, $(1 - x^2)y_2 - xy_1 + m^2y = 0$ **[WBSC – 03, 07, 14, 18, 19]**

(iii) If $y = \cos(m\sin^{-1}x)$ prove that, $(1 - x^2)y_2 - xy_1 + m^2y = 0$

(iv) If $y = \sin(4\cos^{-1}x)$, prove that (a) $(1 - x^2){y_1}^2 = 16(1 - y^2)$ (b) $(1 - x^2)y_2 - xy_1 + 16y = 0$

## ANSWERS

1 (a) $-2\cos 2x$ (b) $6 + 4.\log x$ (c) $4ax^2e^{ax^4}\left(3 + 4ax^4\right)$ (d) $-\frac{2}{x^2}$ (e) cosecx

2 (i) $-\frac{b^4}{a^2y^3}$ (ii) $-\frac{4a^2}{y^3}$ (iii) $-\frac{2a^3xy}{\left(y^2 - ax\right)^3}$ (iv) 0 (v) $\frac{2}{(1+x)^3}$ (vi) $-\frac{\sin^2x + \cos y}{\sin^3 y}$ (vii) $-\frac{2}{5}$ (viii) $\frac{4(x+y)}{(x+y+1)^3}$ (ix) 0, (x) –3.

4.(i) $\frac{b}{3a\sin\theta\cos^4\theta}$ (ii) $-\frac{3}{4}\operatorname{cosec}^3\frac{3\theta}{2}\sec\frac{\theta}{2}$ (iii) $-\frac{k+1}{4ak}\cdot\sec^3\frac{k+1}{2}t\cdot\sec\frac{k-1}{2}t$ (iv) $\frac{1}{a}$ (v) $-\frac{3}{2}$ (vi) $\frac{2\left(1+t^2\right)^3}{3a\left(1-t^2\right)^3}$.

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

1 (b) **Solution :** Given $y = x^2\log x^2$ or, $y = 2x^2\log x$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = 2\frac{d}{dx}\left(x^2\log x\right)$$

$$= 2\left\{x^2\frac{d}{dx}(\log x) + \log x\frac{d}{dx}\left(x^2\right)\right\} = 2\left\{x^2 \times \frac{1}{x} + \log x \times 2x\right\}$$

$$= 2(x + 2x\log x) = 2x\,(1 + 2\log x)$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = 2\frac{d}{dx}\{x(1 + 2\log x)\}$$

$$= 2\left\{x\frac{d}{dx}(1 + 2\log x) + (1 + 2\log x)\frac{d}{dx}(x)\right\} = 2\left\{x \times \frac{2}{x} + (1 + 2\log x) \times 1\right\}$$

$$= 2(2 + 1 + 2\log x) = 6 + 4\log x$$ **(Ans)**

1(c) **Solution :** $y = e^{ax^4}$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{ax^4}\right) = e^{ax^4} \times \frac{d}{dx}\left(ax^4\right) = ae^{ax^4} \times 4x^3 = 4a\left(x^3 e^{ax^4}\right)$$

Again differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = 4a\frac{d}{dx}\left(x^3 . e^{ax^4}\right) = 4a\left\{x^3\frac{d}{dx}\left(e^{ax^4}\right) + e^{ax^4} \times \frac{d}{dx}\left(x^3\right)\right\}$$

$$= 4a\left[x^3 \times e^{ax^4} \times 4ax^3 + e^{ax^4} \times 3x^2\right] = 4a\left(4ax^6 + 3x^2\right)e^{ax^4}$$ **(Ans)**

1(d) **Solution :** Given, $y = \log x^2$ or, $y = 2\log x$

Therefore, $\frac{dy}{dx} = \frac{2}{x}$ or, $\frac{d^2y}{dx^2} = -\frac{2}{x^2}$ **(Ans)**

1(e) **Solution :** $y = \log_e\left(\tan\frac{x}{2}\right)$

Therefore, $\frac{dy}{dx} = \frac{1}{\tan\frac{x}{2}} \times \sec^2\frac{x}{2} \times \frac{1}{2} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \operatorname{cosec} x$ **(Ans)**

2 (iii) **Solution :** $x^3 + y^3 = 3axy$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) = 3a\frac{d}{dx}(xy) \quad \text{or, } 3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

$$\text{or, } x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay \quad \text{or, } \left(y^2 - ax\right)\frac{dy}{dx} = ay - x^2$$

$$\text{or, } \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{ay - x^2}{y^2 - ax}\right) = \frac{\left(y^2 - ax\right)\frac{d}{dx}\left(ay - x^2\right) - \left(ay - x^2\right)\frac{d}{dx}\left(y^2 - ax\right)}{\left(y^2 - ax\right)^2}$$

$$= \frac{\left(y^2 - ax\right)\left(a\frac{dy}{dx} - 2x\right) - \left(ay - x^2\right)\left(2y\frac{dy}{dx} - a\right)}{\left(y^2 - ax\right)^2} = \frac{\left(ay^2 - a^2x - 2ay^2 + 2x^2y\right)\frac{dy}{dx} + \left(2ax^2 - 2xy^2 + a^2y - ax^2\right)}{\left(y^2 - ax\right)^2}$$

$$\text{or, } \frac{d^2y}{dx^2} = \frac{\left(2x^2y - ay^2 - a^2x\right)\frac{dy}{dx} + \left(ax^2 - 2xy^2 + a^2y\right)}{\left(y^2 - ax\right)^2} = \frac{\left(2x^2y - ay^2 - a^2x\right)\left(\frac{ay - x^2}{y^2 - ax}\right) + \left(ax^2 - 2xy^2 + a^2y\right)}{\left(y^2 - ax\right)^2}$$

$$= \frac{\left(ay - x^2\right)\left(2x^2y - ay^2 - a^2x\right) + \left(y^2 - ax\right)\left(ax^2 - 2xy^2 + a^2y\right)}{\left(y^2 - ax\right)^3}$$

$$= \frac{6ax^2y^2 - 2xy\left(x^3+y^3+a^3\right)}{\left(y^2-ax\right)^3} = \frac{6ax^2y^2 - 2xy\left(3axy+a^3\right)}{\left(y^2-ax\right)^3} \quad [\because x^3 + y^3 = 3axy]$$

$$= \frac{6ax^2y^2 - 6ax^2y^2 - 2a^3xy}{\left(y^2-ax\right)^3} = \frac{-2a^3xy}{\left(y^2-ax\right)^3}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{2a^3xy}{\left(y^2-ax\right)^3} \quad \textbf{(Ans)}$$

2 (iv) **Solution :** $x^my^n = (x + y)^{m+n}$

Taking log of both sides we get,

$m \log x + n \log y = (m + n) \log (x + y)$.

Differentiating both sides with respect to x we get,

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right) \text{ or, } \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\cdot\frac{dy}{dx}$$

$$\text{or, } \left(\frac{n}{y} - \frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x} \text{ or, } \left\{\frac{nx+ny-my-ny}{y(x+y)}\right\}\frac{dy}{dx} = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\text{or, } \frac{nx-my}{y}\frac{dy}{dx} = \frac{nx-my}{x} \text{ or, } \frac{dy}{dx} = \frac{y}{x}$$

Differentiating again both side with respect to x we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x\frac{dy}{dx} - y.1}{x^2} = \frac{x\left(\frac{y}{x}\right) - y}{x^2} = \frac{y-y}{x^2} = 0 \quad \frac{d^2y}{dx^2} = 0 \quad \textbf{(Ans)}$$

2 (v) **Solution :** $x\sqrt{1+y} + y\sqrt{1+x} = 0$ or, $x\sqrt{1+y} = -y\sqrt{1+x}$

or, $x^2(1 + y) = y^2(1 + x)$ [squaring both sides]

or, $x^2 + x^2y - y^2 - y^2x = 0$ or, $(x^2 - y^2) + (x^2y - xy^2) = 0$ or, $(x + y)(x - y) + xy (x - y) = 0$

or, $(x - y)(x + y + xy) = 0$ or, $x + y + xy = 0$ [$\because x \neq y$]

or, $y(1 + x) = -x$ or, $y = -\frac{x}{1+x} = -\frac{x+1-1}{x+1} = -1 + \frac{1}{x+1}$

$\frac{dy}{dx} = -\frac{1}{(x+1)^2}$ [Differentiating both sides with respect to x]

Again differentiating with respect to x we get,

$$\frac{d^2y}{dx^2} = \frac{2}{(1+x)^3} \quad \textbf{(Ans)}$$

2. (ix) **Solution :** $y = \frac{1}{1+x+x^2+x^3}$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = -\frac{1}{\left(1+x+x^2+x^3\right)^2} \times \left(1 + 2x + 3x^2\right) = -\frac{3x^2+2x+1}{\left(1+x+x^2+x^3\right)^2}$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left\{\frac{3x^2+2x+1}{\left(1+x+x^2+x^3\right)^2}\right\} = -\frac{\left(1+x+x^2+x^3\right)^2(6x+2)-\left(3x^2+2x+1\right)^2\times 2\left(1+x+x^2+x^3\right)}{\left(1+x+x^2+x^3\right)^4}$$

$\therefore$ at $x = 0$, $\frac{d^2y}{dx^2} = -\frac{1\times 2-1\times 2}{1^4} = 0$ **(Ans)**

3.(i) **Solution :** $ax^2 + 2hxy + by^2 = 1$ ..... (1)

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}\left(ax^2\right) + 2h\frac{d}{dx}(xy) + \frac{d}{dx}\left(by^2\right) = \frac{d}{dx}(1) \quad \text{or,} \quad 2ax + 2h\left(y.1 + x\frac{dy}{dx}\right) + 2by\frac{dy}{dx} = 0$$

$$\text{or,} \quad ax + hy + hx\frac{dy}{dx} + by\frac{dy}{dx} = 0 \quad \text{or,} \quad (hx + by)\frac{dy}{dx} = -(ax + hy)$$

$$\text{or,} \quad \frac{dy}{dx} = -\frac{ax+hy}{hx+by} \qquad ..... (1)$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{ax+hy}{hx+by}\right) = -\frac{(hx+by)\frac{d}{dx}(ax+hy)-(ax+hy)\frac{d}{dx}(hx+by)}{(hx+by)^2}$$

$$= -\frac{(hx+by)\left(a+h\frac{dy}{dx}\right)-(ax+hy)\left(h+b\frac{dy}{dx}\right)}{(hx+by)^2} = -\frac{\frac{dy}{dx}\left(h^2x+bhy-abx-bhy\right)+\left(ahx+aby-ahx-h^2y\right)}{(hx+by)^2}$$

$$= -\frac{\left(h^2-ab\right)x\frac{dy}{dx}-\left(h^2-ab\right)y}{(hx+by)^2} = \frac{\left(h^2-ab\right)\left(y-x\frac{dy}{dx}\right)}{(hx+by)^2} = \frac{\left(h^2-ab\right)\left(y+x\cdot\frac{ax+hy}{hx+by}\right)}{(hx+by)^2}.$$

$$= \frac{\left(h^2-ab\right)\left(hxy+by^2+ax^2+hxy\right)}{(hx+by)^3} = \frac{\left(h^2-ab\right)\left(ax^2+2hxy+by^2\right)}{(hx+by)^3} = \frac{\left(h^2-ab\right)\cdot 1}{(hx+by)^3}$$

$\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$ **(Proved)** $[\because ax^2 + 2hxy + by^2 = 1]$

3.(ii) **Solution :** $y^2 = mx^2 + 2lx + n$ ..... (1)

Differentiating both sides with respect to y we get,

$$2y = 2mx\frac{dx}{dy} + 2l\frac{dx}{dy} \quad \text{or,} \quad y = (mx+l)\frac{dx}{dy}$$

$$\text{or,} \quad \frac{dx}{dy} = \frac{y}{mx+l} \qquad ..... (2)$$

Again differentiating both sides with respect to y we get,

$$\frac{d^2x}{dy^2} = \frac{(mx+l)\cdot 1-y\cdot m\frac{dx}{dy}}{(mx+l)^2} = \frac{mx+l-my\left(\frac{y}{mx+l}\right)}{(mx+l)^2} = \frac{(mx+l)^2-my^2}{(mx+l)^3}$$

$$= \frac{m^2x^2+2lmx+l^2-my^2}{(mx+l)^3} = \frac{m^2x^2+2lmx+l^2-m\left(mx^2+2lx+n\right)}{(mx+l)^3} \quad \text{or,} \quad \frac{d^2x}{dy^2} = \frac{l^2-mn}{(mx+l)^3}$$ **(Proved)**

**4.(ii) Solution :** Given, $x = \cos\theta(1 + \cos\theta)$, $y = \sin\theta(1 + \cos\theta)$

Differentiating with respect to $\theta$ we get,

$$\frac{dx}{d\theta} = \cos\theta \frac{d}{d\theta}(1 + \cos\theta) + (1 + \cos\theta)\frac{d}{d\theta}(\cos\theta)$$

$$= \cos\theta\ (-\sin\theta) + (1 + \cos\theta)(-\sin\theta) = -\sin\theta(\cos\theta + 1 + \cos\theta) = -\sin\theta(1 + 2\cos\theta) = -(\sin\theta + \sin 2\theta)$$

$$\frac{dy}{d\theta} = \sin\theta \frac{d}{d\theta}(1 + \cos\theta) + (1 + \cos\theta)\frac{d}{d\theta}(\sin\theta)$$

$$= \sin\theta(-\sin\theta) + (1 + \cos\theta)\cos\theta = \cos\theta + \cos^2\theta - \sin^2\theta = \cos\theta + \cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta} = -\frac{2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}}{2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{3\theta}{2}$$

Again, differentiating both sides with respect to x we get,

$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{d\theta}\left(\cot\frac{3\theta}{2}\right)\times\frac{d\theta}{dx} = -\frac{3}{2}\operatorname{cosec}^2\frac{3\theta}{2}\times\frac{1}{\sin\theta + \sin 2\theta}$$

$$= -\frac{3}{2}\operatorname{cosec}^2\frac{3\theta}{2}\times\frac{1}{2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}} = -\frac{3}{4}\operatorname{cosec}^3\frac{3\theta}{2}\sec\frac{\theta}{2} \quad \textbf{(Ans)}$$

4.(v) **Solution :** $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$

Differentiating both sides with respect to $\theta$ we get,

$$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta, \qquad \frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\cos\theta - 2\cos 2\theta}{-2\sin\theta + 2\sin 2\theta} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta} = \frac{2\sin\frac{3\theta}{2}.\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{3\theta}{2}}$$

$$\therefore \frac{dy}{dx} = \tan\frac{3\theta}{2}$$

Again, differentiating with respect to x we get,

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\tan\frac{3\theta}{2}\right)\frac{d\theta}{dx} = \frac{3}{2}\sec^2\frac{3\theta}{2}\times\frac{1}{2(\sin 2\theta - \sin\theta)} = \frac{3}{4}\sec^2\frac{3\theta}{2}\times\frac{1}{2\sin\frac{\theta}{2}\cdot\cos\frac{3\theta}{2}} = \frac{3}{8}\sec^3\frac{3\theta}{2}\times\operatorname{cosec}\frac{\theta}{2}$$

$\therefore$ at $\theta = \frac{\pi}{2}$

$$\frac{d^2y}{dx^2} = \frac{3}{8}\times\sec^3\frac{3\pi}{4}\cdot\operatorname{cosec}\frac{\pi}{4} = \frac{3}{8}\left(-\sqrt{2}\right)^3\times\sqrt{2} = -\frac{3}{8}\times 2\times 2 = -\frac{3}{2} \quad \textbf{(Ans)}$$

5.(i) **Solution :** $y = Ae^{mx} + Be^{-mx}$ .................................... (1)

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = A\frac{d}{dx}\left(e^{mx}\right) + B\frac{d}{dx}\left(e^{-mx}\right) = Ame^{mx} + B(-m)e^{-mx}$$

Again, differentiating with respect to x we get,

$$\frac{d^2y}{dx^2} = Am^2e^{mx} + B(-m)^2e^{-mx} = m^2(Ae^{mx} + Be^{-mx}) = m^2y \text{ ............ [from (1)]}$$

$$\frac{d^2y}{dx^2} - m^2y = 0 \quad \textbf{(Proved)}$$

**5.(iii) Solution :** $y = Ae^{mx} + Be^{nx}$ ............................... (1)

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = A\frac{d}{dx}\left(e^{mx}\right) + B\frac{d}{dx}\left(e^{nx}\right) = Ame^{mx} + Bne^{nx}$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = Am\frac{d}{dx}\left(e^{mx}\right) + Bn\frac{d}{dx}\left(e^{nx}\right) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \text{ L.H.S} = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Ae^{mx}(m^2 - m^2 - mn + mn) + Be^{nx}(n^2 - mn - n^2 + mn) = 0 + 0 = 0 =$ R.H.S **(Proved)**

**6.(ii) Solution :** Given, $y = (\sin^{-1}x)^2$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = 2\left(\sin^{-1}x\right) \times \frac{d}{dx}\left(\sin^{-1}x\right) = 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$$

or, $\sqrt{1-x^2}\,\frac{dy}{dx} = 2\sin^{-1}x$ or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = 4\left(\sin^{-1}x\right)^2$ [squaring both sides]

or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = 4y$

Differentiating both sides with respect to x we get,

$\frac{d}{dx}\left\{\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2\right\} = 4\frac{dy}{dx}$ or, $\left(1-x^2\right) \times 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \times (-2x) = 4\frac{dy}{dx}$

or, $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$ or, $(1 - x^2)y_2 - xy_1 = 2$ **(Proved)**

**7.(i) Solution :** $y = \left(x + \sqrt{1+x^2}\right)^m$

$\log y = m\log\left(x + \sqrt{1+x^2}\right)$ [taking logarithm of both sides]

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y) = m\frac{d}{dx}\left\{\log\left(x + \sqrt{1+x^2}\right)\right\}$$

or, $\frac{1}{y}\frac{dy}{dx} = m \times \frac{1}{x+\sqrt{x^2+1}} \times \left(1 + \frac{1}{2\sqrt{x^2+1}} \times 2x\right) = \frac{m}{x+\sqrt{x^2+1}} \times \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} = \frac{m}{\sqrt{x^2+1}}$

or, $\sqrt{x^2+1}\,\frac{dy}{dx} = my$ or, $\left(x^2+1\right)\left(\frac{dy}{dx}\right)^2 = m^2y^2$ or, $\left(x^2+1\right) \times 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 2m^2y\frac{dy}{dx}$

or, $\left(x^2+1\right) \cdot \frac{d^2y}{dx^2} + x\frac{dy}{dx} = m^2y$ or, $(1 + x^2)\,y_2 + xy_1 = m^2y$. **(Proved)**

**7.(iii) Solution :** Given $y^{\frac{1}{2}} + y^{-\frac{1}{2}} = 2x$

or, $a + \frac{1}{a} = 2x$ or, $a^2 - 2ax + 1 = 0$ $\left[\text{let } a = y^{\frac{1}{2}}\right]$

or, $a = \frac{2x \pm \sqrt{4x^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = x \pm \sqrt{x^2 - 1}$ or, $y^{\frac{1}{2}} = x \pm \sqrt{x^2 - 1}$

or, $y = \left(x \pm \sqrt{x^2 - 1}\right)^2$ or, $y = \left(x + \sqrt{x^2 - 1}\right)^2$ [taking +ve sign]

$\log y = 2\log\left(x + \sqrt{x^2 - 1}\right)$ [taking logarithm on both sides we get]

Differentiating both sides with respect to x we get,

$$\frac{1}{y}\frac{dy}{dx} = 2 \times \frac{1}{x + \sqrt{x^2 - 1}} \times \frac{d}{dx}\left(x + \sqrt{x^2 - 1}\right) = \frac{2}{x + \sqrt{x^2 - 1}}\left(1 + \frac{1}{2\sqrt{x^2 - 1}} \times 2x\right)$$

$$= \frac{2}{x + \sqrt{x^2 - 1}}\left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{2}{x + \sqrt{x^2 - 1}}\left(\frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}\right)$$

or, $\sqrt{x^2 - 1}\,\frac{dy}{dx} = 2y$ or, $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2$

Again, differentiating both sides with respect to x we get,

$$\left(x^2 - 1\right) \times 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} + 2x \cdot \left(\frac{dy}{dx}\right)^2 = 8y\frac{dy}{dx}$$

$\left(x^2 - 1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 4y$ or, $(x^2 - 1)\, y_2 + xy_1 = 4y$ **(Proved)**

**8.(i) Solution :** $y = e^{ax}.\cos bx$

or, $e^{-ax}.y = \cos bx$

Differentiating both sides with respect to x we get,

$e^{-ax}\frac{dy}{dx} + y.e^{-ax}(-a) = -b\sin bx$ or, $e^{-ax}\left(\frac{dy}{dx} - ay\right) = -b\sin bx$

Again, differentiating both sides with respect to x we get,

$$e^{-ax}\frac{d}{dx}\left(\frac{dy}{dx} - ay\right) + \left(\frac{dy}{dx} - ay\right)\frac{d}{dx}\left(e^{-ax}\right) = -b\frac{d}{dx}(\sin bx)$$

or, $e^{-ax}\left(\frac{d^2y}{dx^2} - a\frac{dy}{dx}\right) + \left(\frac{dy}{dx} - ay\right)\left(-ae^{-ax}\right) = -b^2\cos bx$

or, $e^{-ax}\left\{\frac{d^2y}{dx^2} - a\frac{dy}{dx} - a\frac{dy}{dx} + a^2y\right\} = -b^2\left(e^{-ax}y\right)$ $[\because e^{-ax}y = \cos bx]$

or, $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = -b^2y$ or, $y_2 - 2ay_1 + (a^2 + b^2)y = 0$ **(Proved)**

**8.(iii) Solution :** $y = e^x(a\cos x + b\sin x)$ or, $e^{-x}y = a\cos x + b\sin x$ ..... (1)

Differentiating both sides with respect to x we get,

$e^{-x}\frac{dy}{dx} + \left(-e^{-x}\right).y = -a\sin x + b\cos x$ or, $e^{-x}\left(\frac{dy}{dx} - y\right) = -a\sin x + b\cos x$

Again, differentiating both sides with respect to x we get,

$e^{-x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) - e^{-x}\left(\frac{dy}{dx} - y\right) = -a\cos x - b\sin x$ or, $e^{-x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{dy}{dx} + y\right) = -(a\cos x + b\sin x) = -e^{-x}y$

or, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = -y$ or, $y_2 - 2y_1 + 2y = 0$ **(Proved)**

**8.(v) Solution :** $x = A.e^{-\frac{kt}{2}}\cos(pt + c)$

or, $e^{\frac{kt}{2}}x = A.\cos(pt + c)$ ..... (1)

Differentiating both sides with respect to t we get,

$e^{\frac{kt}{2}}\frac{dx}{dt} + \frac{k}{2}e^{\frac{kt}{2}}x = -Ap\sin(pt + c)$ or, $e^{\frac{kt}{2}}\left(\frac{dx}{dt} + \frac{kx}{2}\right) = -Ap\sin(pt + c)$

Again, differentiating both sides with respect to t we get,

$e^{\frac{kt}{2}}\left(\frac{d^2x}{dt^2} + \frac{k}{2}\frac{dx}{dt}\right) + \frac{k}{2}e^{\frac{kt}{2}}\left(\frac{dx}{dt} + \frac{kx}{2}\right) = -Ap^2\cos(pt + c) = -p^2.e^{\frac{kt}{2}}x$ [by (1)]

or, $\frac{d^2x}{dt^2} + \frac{k}{2}\frac{dx}{dt} + \frac{k}{2}\frac{dx}{dt} + \frac{k^2}{4}x = -p^2x$ or, $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \left(p^2 + \frac{k^2}{4}\right)x = 0$

$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + n^2x = 0$ where, $n^2 = p^2 + \frac{k^2}{4}$ **(Proved)**

**9.(ii) Solution :** $x = \sin\left(\frac{1}{a}\log_e y\right)$ or, $\frac{1}{a}\log_e y = \sin^{-1}x$

or, $\log_e y = a\sin^{-1}x$ or, $y = e^{a\sin^{-1}x}$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = e^{a\sin^{-1}x} \times \frac{d}{dx}\left(a\sin^{-1}x\right) = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

Again, differentiating both sides with respect to x we get,

$\left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = 2a^2y\frac{dy}{dx}$ or, $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$

or, $\left(1 - x^2\right)y_2 - xy_1 - a^2y = 0$ **(Proved)**

**10.(i) Solution :** $x = f(t),\ y = \phi(t)$

Differentiating both sides with respect to t we get,

$$\frac{dx}{dt} = f'(t) = x_1,\ \frac{dy}{dt} = \varphi'(t) = y_1$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y_1}{x_1}$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y_1}{x_1}\right) = \frac{d}{dt}\left(\frac{y_1}{x_1}\right) \times \frac{dt}{dx} = \frac{x_1\frac{d}{dt}(y_1) - y_1\frac{d}{dt}(x_1)}{x_1^2} \times \frac{1}{x_1} = \frac{x_1y_2 - y_1x_2}{x_1^3} \textbf{ (Proved)}$$

**10.(ii) Solution :** $x = \sin t,\ y = \sin pt$

Differentiating both sides with respect to t we get,

$$\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = p\cos pt$$

$$\therefore\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t} \quad \text{or,}\ \left(\frac{dy}{dx}\right)^2 = \frac{p^2\cos^2 pt}{\cos^2 t}$$

$$\text{or,}\ \left(\frac{dy}{dt}\right)^2 = \frac{p^2(1-\sin^2 pt)}{1-\sin^2 t} = \frac{p^2(1-y^2)}{1-x^2} \quad \text{or,}\ (1-x^2)\left(\frac{dy}{dt}\right)^2 = p^2 - p^2y^2$$

Again, differentiating both sides with respect to x we get,

$$(1-x^2)2\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + (-2x)\cdot\left(\frac{dy}{dx}\right)^2 = -2p^2y\cdot\frac{dy}{dx}$$

$$\text{or,}\ (1-x^2)\cdot\frac{d^2y}{dx^2} - x\cdot\frac{dy}{dx} = -p^2y \quad \text{or,}\ (1-x^2)y_2 - xy_1 + p^2y = 0 \textbf{ (Proved)}$$

**10.(vii) Solution :** $x = \cos t,\ y = \log t$

Differentiating both sides with respect to t we get,

$$\frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \frac{1}{t}$$

$$\therefore\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t}}{-\sin t} = -\frac{1}{t\sin t}$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = -\frac{d}{dt}\frac{1}{(t\sin t)} \times \frac{dt}{dx} = \frac{1}{t^2\sin^2 t} \times \frac{d}{dt}(t\sin t) \times \left(\frac{1}{-\sin t}\right) = -\frac{1}{t^2\sin^3 t}(t\cos t + \sin t)$$

$$\therefore\ \text{at } t = \frac{\pi}{2},\ \frac{dy}{dx} = -\frac{1}{\frac{\pi}{2}.\sin\frac{\pi}{2}} = -\frac{2}{\pi},\ \frac{d^2y}{dx^2} = \frac{0+1}{-\frac{\pi^2}{4}\times 1^3} = -\frac{4}{\pi^2}$$

$$\therefore\ \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{4}{\pi^2} + \left(-\frac{2}{\pi}\right)^2 = -\frac{4}{\pi^2} + \frac{4}{\pi^2} = 0 \quad \therefore \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \textbf{ (Proved)}$$

**11.(ii) Solution :** $y = \sin(m\sin^{-1}x)$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \cos\left(m\sin^{-1}x\right)\times\frac{d}{dx}\left(m\sin^{-1}x\right) = \cos\left(m\sin^{-1}x\right)\times\frac{m}{\sqrt{1-x^2}}$$

or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = m^2\cos^2\left(m\sin^{-1}x\right) = m^2\left\{1-\sin^2\left(m\sin^{-1}x\right)\right\} = m^2 - m^2y^2$

Differentiating again both sides with respect to x we get,

$$\left(1-x^2\right)\times 2\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + (-2x\times\left(\frac{dy}{dx}\right)^2 = 0 - 2m^2y\frac{dy}{dx}$$

or, $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$ or, $\left(1-x^2\right)y_2 - xy_1 + m^2y = 0$ **(Proved)**

## MISCELLANEOUS

***1.** A particle is moving along a straight line, on which O is a point. After time t, the distance of the particle from O is given by a cosnt + b sinnt (a, b, n are constant). Prove that the acceleration of the particle is proportional to its distance from O. **[WBSC – 04]**

***2.** If $x^2 + xy + 3y^2 = 1$, prove that $(x + 6y)^3.y_2 + 22 = 0$ **[WBSC – 94]**

***3.** If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, show that, $p+\frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$

**4.** If $y = x\log\left(\frac{x}{a+bx}\right)$, prove that, $x^3y_2 = (y - xy_1)^2$.

***5.** If $2y = x\left(1+\frac{dy}{dx}\right)$, prove that, $\frac{d^2y}{dx^2}$ = constant.

**6.** If $\cos^{-1}\frac{y}{b} = \log\left(\frac{x}{n}\right)^n$, prove that, $x^2y_2 + xy_1 + n^2y = 0$

***7.** If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that $\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**2. Solution :** $x^2 + xy + 3y^2 = 1$ ..... (1)

Differentiating both sides with respect to x we get,

$$2x + y + x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0 \text{ or, } (x+6y)\frac{dy}{dx} + y + 2x = 0 \text{ or, } \frac{dy}{dx} = -\frac{y+2x}{x+6y} \quad \text{..... (2)}$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = -\frac{(x+6y)\left(\frac{dy}{dx}+2\right)-(y+2x)\left(1+6\frac{dy}{dx}\right)}{(x+6y)^2} = -\frac{\frac{dy}{dx}(x+6y-6y-12x)+(2x+12y-y-2x)}{(x+6y)^2}$$

$$= -\frac{11x\times\left(\frac{y+2x}{x+6y}\right)+11y}{(x+6y)^2} = -\frac{11x(y+2x)+11y(x+6y)}{(x+6y)^3} = -\frac{11(xy+2x^2+xy+6y^2)}{(x+6y)^3}$$

$$= -\frac{11(2x^2+2xy+6y^2)}{(x+6y)^3} = -\frac{22(x^2+xy+3y^2)}{(x+6y)^3} \quad \text{or,}\ (x+6y)^3 y_2 + 22 = 0 \ \textbf{(Proved)}$$

**3. Solution :** Given $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$ ..... (1)

$$\therefore 2p\frac{dp}{d\theta} = 2(b^2-a^2)\sin\theta\cos\theta \qquad \therefore p\frac{dp}{d\theta} = (b^2-a^2)\sin\theta\cos\theta \qquad \text{..... (2)}$$

$$\text{or,}\ p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (b^2-a^2)(\cos^2\theta-\sin^2\theta)$$

$$\text{or,}\ p^3\frac{d^2p}{d\theta^2} = p^2(b^2-a^2)(\cos^2\theta-\sin^2\theta) - (b^2-a^2)^2\sin^2\theta\cos^2\theta$$

$$\text{or,}\ p^4 + p^3\frac{d^2p}{d\theta^2} = (a^2\cos^2\theta + b^2\sin^2\theta)^2$$

$$+(b^2-a^2)(\cos^2\theta-\sin^2\theta)(a^2\cos^2\theta+b^2\sin^2\theta) - (b^2-a^2)^2\sin^2\theta\cos^2\theta$$

$$= (a^2\cos^2\theta+b^2\sin^2\theta)(a^2\cancel{\cos^2\theta}+b^2\cancel{\sin^2\theta}+b^2\cos^2\theta-a^2\cancel{\cos^2\theta}$$

$$-b^2\cancel{\sin^2\theta}+a^2\sin^2\theta) - (a^4+b^4-2a^2b^2)\sin^2\theta\cos^2\theta$$

$$= (a^2\cos^2\theta + b^2\sin^2\theta)(a^2\sin^2\theta + b^2\cos^2\theta) - (a^4+b^4-2a^2b^2)\sin^2\theta\cos^2\theta$$

$$= a^4\cancel{\sin^2}\theta\cos^2\theta + a^2b^2\sin^4\theta + a^2b^2\cos^4\theta + b^4\cancel{\sin^2}\theta\cos^2\theta$$

$$-a^4\cancel{\sin^2}\theta\cos^2\theta - b^4\cancel{\sin^2}\theta\cos^2\theta + 2a^2b^2\sin^2 q\cos^2 q.$$

$$= a^2b^2(\sin^2\theta + \cos^2\theta)^2$$

$$\therefore p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3} \ \textbf{(Proved)}$$

**5. Solution :** $2y = x\left(1+\frac{dy}{dx}\right)$

Differentiating both sides with respect to x we get,

$$2\frac{dy}{dx} = 1 + \frac{dy}{dx} + x\frac{d^2y}{dx^2} \quad \text{or,}\ \frac{dy}{dx} = 1 + x\frac{d^2y}{dx^2}$$

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dx^2} + x\frac{d^3y}{dx^3} \qquad \text{or,}\ x\frac{d^3y}{dx^3} = 0 \qquad \text{or,}\ \frac{d^3y}{dx^3} = 0$$

Hence, $\frac{d^2y}{dx^2}$ = constant. **(Proved)**

**7. Solution :** $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ..... (1)

Differentiating both sides with respect to x we get,

$$2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx}(hx + by + f) = -(ax + hy + g)$ $\therefore \frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}$ ..... (2)

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = \frac{-(hx+by+f)\left(a+h\frac{dy}{dx}\right)+(ax+hy+g)\left(h+b\frac{dy}{dx}\right)}{(hx+by+f)^2}$$

$$= \frac{\frac{dy}{dx}\left(-h^2x - b\cancel{h}y - fh + abx + b\cancel{h}y + bg\right)+\left(-a\cancel{h}x - aby - af + a\cancel{h}x + h^2y + gh\right)}{(hx+by+f)^2}$$

$$= \frac{\left\{(ab-h^2)x + bg - hf\right\}\frac{dy}{dx} - \left\{(ab-h^2)y + (af-gh)\right\}}{(hx+by+f)^2} = \frac{(ab-h^2)\left(x\frac{dy}{dx}-y\right)+\left\{(bg-hf)\frac{dy}{dx}-(af-gh)\right\}}{(hx+by+f)^2}$$

$$= \frac{(h^2-ab)\left(y+\frac{ax^2+hxy+gx}{hx+by+f}\right)+\left\{(gh-af)+(hf-bg)\left(\frac{ax+hy+g}{hx+by+f}\right)\right\}}{(hx+by+f)^2}$$

$$= \frac{(h^2-ab)(ax^2+2hxy+by^2+gx+fy)+\left\{(gh-af)\cdot(hx+by+f)+(hf-bg)(ax+hy+g)\right\}}{(hx+by+f)^3}$$

$$= \frac{(h^2-ab)(-gx-fy-c)+gh^2x-afhx+bghy-abfy+fgh-af^2+ahfx-abgx+h^2fy-bghy+fgh-bg^2}{(hx+by+f)^3}$$

$$= \frac{abc-ch^2-g\cancel{h}^2x-f\cancel{h}^2y+abgx+abfy+g\cancel{h}^2x-af\cancel{h}x+bg\cancel{h}y-abfy+2fgh-af^2+ahfx-abgx+h^2fy-bg\cancel{h}y-bg^2}{(hx+by+f)^3}$$

or, $\frac{d^2y}{dx^2} = \frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}$ **(Proved)**

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# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. If $y = \tan^{-1}x$ find $\frac{d^2y}{dx^2}$ [WBSC – 12]

2. If $v = \frac{A}{r} + B$ then $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$, – (a) True, (b) False. [WBSC – 05]

3. If $y = \log\sin x$ then $\frac{d^2y}{dx^2}$ is (a) $\cot x$ (b) $\log\cos x$ (c) $-\operatorname{cosec}^2 x$ (d) none is true. [WBSC – 12]

4. If $y = \log_e\left(\tan\frac{x}{2}\right)$ then $\frac{d^2y}{dx^2}$ is – (a) $-\operatorname{cosec}^2 x$, (b) $-\operatorname{cosec}x\cot x$ (c) $\cos^2 x$ (d) none of these [WBSC – 07]

5. If $y = \log x^2$, then the value of $\frac{d^2y}{dx^2}$ is :(a) $\frac{2}{x^2}$, (b) $-\frac{2}{x^2}$, (c) $\frac{2}{x}$, (d) none of these.

   [WBSC – 04, 06, 09, 10]

6. If $y = \log x^2$, the value of $\frac{d^2y}{dx^2}$ is – (a) $\frac{2}{x^2}$ (b) $-\frac{2}{x^2}$ (c) $\frac{2}{x^3}$ (d) $\frac{2}{x^4}$ [WBSC – 07]

## SUBJECTIVE TYPE

1. $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$ [WBSC – 14]

2. If $ax^2 + 2hxy + by^2 = 1$, prove that $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$ [WBSC – 15]

3. If $y = \sin(m\sin^{-1}x)$, prove that: $(1 - x^2)y_2 - xy_1 + m^2y = 0$. [WBSC – 03, 07, 14, 18, 19]

4. If $x = \sin t$ and $y = \sin pt$, show that $(1 - x^2)y_2 - xy_1 + p^2y = 0$. [WBSC – 04]

5. If $x = \sin\left(\frac{\log y}{a}\right)$, prove that $(1 - x^2)y_1 - xy_1 - a^2y = 0$ [WBSC – 06, 07]

6. If $y = e^{a\sin^{-1}x}$, show that $(1 - x^2)y_2 - xy_1 - a^2y = 0$ [WBSC – 11, 17]

7. If $y = (\sin^{-1}x)^2$, prove that $(1 - x^2)y_2 - xy_1 = 2$ [WBSC – 08, 12, 16]

8. If $y = \left(x + \sqrt{1 + x^2}\right)^m$ prove that $(1 + x^2)y_2 + xy_1 - m^2y = 0$ where $y_1$ and $y_2$ are the differential coefficient of y with respect to 'x' once and twice respectively. [WBSC – 10, 15]

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Chapter – 6

# SIGNIFICANCE OF DERIVATIVE

## 6.1 DIFFERENTIAL COEFFICIENT AS RATE MEASURE.

Let $y = f(x)$ be a single-valued function of x. Now, if the value of the independent variable x changes from x to $x + \Delta x$, then the increment of $x = x + \Delta x - x = \Delta x$; the corresponding increment in the function

$$= \Delta y = (y + \Delta y) - y = f(x + \Delta x) - f(x).$$

Then the ratio $\frac{\Delta y}{\Delta x}$ is called the **average rate of change** of y with respect to x in the interval $(x, x + \Delta x)$. Again, as $\Delta x \to 0$, the limiting value of the ratio $\frac{\Delta y}{\Delta x}$ (if it exists) is called the **instantaneous rate of change** of y with respect to x at the point x. But by definition of derivative, $\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$.

Therefore, the derivative $\left[f'(x) \text{ or}, \frac{dy}{dx}\right]$ of the function $y = f(x)$ at the point x represents the **rate measure** of y with respect to x at the point x.

### EXERCISE : SET – I

1. *(i) Find the **average** rate of change of the function $y = 16 - x^2$ between $x = 3$ and $x = 4$; also find the rate of change of the function at $x = 4$.

   (ii) Find the **average** rate of change of the function $y = x^2$ between $x = 2$ and $x = 5$; also find the rate of change of the function at $x = 2$.

2. *(i) If the area of a circle changes uniformely with respect to time, show that the rate of change of its circumference varies inversely as its radius. [HS – 90, 94]

   (ii) The radius of a circle increases at a rate $\frac{1}{\pi}$. Find the rate of change of its (a) circumference and (b) area when its radius is 2 unit.

   *(iii) The time rate of change of the radius of a sphere is $\frac{1}{2\pi}$. When its radius is 5 cm, find the rate of change of the area of the surface of the sphere with time. [HS – 98, 00]

(iv) If $\gamma$ be the increase in volume of the cube of unit volume and $\beta$ be the increase in area of each surface of the cube, show that $2\gamma = 3\beta$. [HS – 91]

*(v) A solid cube changes its volume such that its shape remains unchanged. For such a cube of unit volume show that, rate of change of volume = $\frac{3}{2}\times$ rate of change of area of any face of the cube. [HS – 99]

(vi) A spherical balloon is being filled with air at the rate of 25 cu. cm./sec. How fast is the radius increasing when the balloon is 20 cm. in diameter?

*(vii) A spherical balloon is being inflated so that its volume increases uniformly at the rate of 40 $cm^3$/min. How fast is its surface area increasing, when the radius is 8 cm. Find approximately how much the radius will increase during the next $\frac{1}{2}$ minute. [HS – 99]

(viii) Air is expelled from a spherical balloon by decreasing the radius at the rate of $\frac{1}{8}$ cm./sec. At what rate the air escaping when the radius is 10 cm.?

(ix) A particle moves along the curve $y^2 = 8x$. At what point on the curve the abscissa and ordinate increase at the same rate ?

*(x) Find the co-ordinate of the position of a particle moving along the parabola $y^2 = 4x$ at which the rate of increase of the abscissa is twice the rate of increase of the ordinate. [HS – 94]

(xi) A man 5 ft. long walks away from the foot of a lamp-post $12\frac{1}{2}$ ft. high at the rate of 3 m. p. h. Find the rate at which his shadow is increasing.

*(xii) A man 6 ft. tall walks away from the foot of a lamp-post 15 ft. high at the rate of 3 m. p. h. Find the rate at which his shadow is increasing.

(xiii) Water is flowing into a right circular conical vessel, 45 cm. deep and 27 cm. in diameter at the rate of 11 c. c. per minute. How fast is the water-level rising when the water is 30 cm. deep?

*(xiv) Water is flowing into a right circular conical vessel, 24 inches deep and 12 inches in diameter at the rate of 100 cu. inches per minute. How fast is the water-level rising when the water is 10 inches deep?

(xv) A ladder 20 ft. long leans against a vertical wall. If the top end slides downward at the rate of 2 ft. per second, find the rate at which the lower end moves on a horizontal floor when it is 12 ft. from the wall. Find also the rate at which the slope of the ladder changes.

*(xvi) A ladder 26 ft. long leans against a vertical wall. If the top end slides downward at the rate of 10 inches per second, find the rate at which the lower end moves on a horizontal floor when it is 10 ft. from the wall. Find also the rate at which the slope of the ladder changes.

*(xvii) The top of a ladder 30 ft. long leans against a vertical wall and the lower end rests on the level pavement. The ladder begins to slide outwards so that the lower end is being moved away from the wall at a rate (per second) equal to twice its distance from the wall. How fast is the top sliding downwards at the instant when the lower end is 10 ft. away from the wall? How far is the lower end from the wall when it and the top are moving at the same rate ?

3. *(i) A Parachutist falls through a distance $x = \log_e(6 - 5.e^{-t})$ cm. in the t-th second of its motion. Find $\frac{dx}{dt}$ at $t = 0$.

(ii) The Bending Moment M of a simply supported beam is given by $M = \frac{W}{2}\left(\frac{l}{2} - x\right)$, where W, l are constants. Find the shearing force $\frac{dM}{dx}$ at $x = \frac{l}{2}$

*(iii) The deflection y at a distance x from the centre of a simply supported beam is given by $y = \frac{W}{2EI}\left(\frac{lx^2}{4} - \frac{x^3}{6}\right)$, where W, E, I are constants, Find $\frac{dy}{dx}$.

*(iv) The pressure P and the volume V of a gas obeys the law $PV^{1.4} = C$, where C is a constant. When pressure is 10 kg/m$^2$ and volume is 3 m$^3$ and is changing 0.3 m$^3$/sec., how fast pressure P is changing at that instant (explain the negative sign) [WBSC – 07]

(v) A spherical ice-ball melts and its radius decreases at the rate of 0.1 cm/sec. Taking $\pi = \frac{22}{7}$ and specific gravity of ice 0.9, find the amount of water formed when the radius of the sphere is 7 cm. [WBSC – 07]

## ANSWERS

3.(i) 5 (ii) $-\frac{W}{2}$ (iii) $\frac{W}{4EL}\left(lx - x^2\right)$

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1.(i) Solution :** Let, $y = f(x) = 16 - x^2$

Since x changes from 3 to 4 then $Dx = 4 - 3 = 1$

And $\Delta y = f(4) - f(3) = (16 - 4^2) - (16 - 3^2) = -7$

Therefore, the required average rate of change of the given function is,

$\frac{\Delta y}{\Delta x} = \frac{-7}{1} = -7$ **(Ans)**

Again, the rate of change of the function at $x = 4$ is, $\left(\frac{dy}{dx}\right)_{x=4} = (-2x)_{x=4} = -8$ **(Ans)**

**2.(i) Solution :** Let r be the radius of the circle.

Then by the problem, $\frac{d}{dt}\left(\pi r^2\right) = k$, constant

or, $2\pi r \frac{dr}{dt} = k$ or, $\frac{dr}{dt} = \frac{k}{2\pi r}$ ..................... (1)

Now circumference of the circle, $C = 2\pi r$

$\therefore \frac{dC}{dt} = 2\pi \frac{dr}{dt} = 2\pi\left(\frac{k}{2\pi r}\right) = \frac{k}{r}$

$\therefore \frac{dC}{dt} \propto \frac{1}{r}$, which shows that the rate of change of its circumference varies inversely as its radius. **(Proved)**

**2.(iii) Solution :** Let r and s be respectively the radius and surface area of the sphere at time t.

Then by the problem, $\frac{dr}{dt} = \frac{1}{2\pi}$

Now, $s = 4\pi r^2$

or, $\frac{ds}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt} = 8\pi r \frac{dr}{dt}$ [differentiating both sides with respect to t]

When r = 5 and $\frac{dr}{dt} = \frac{1}{2\pi}$ then,

$\frac{ds}{dt} = 8\pi \cdot 5 \cdot \frac{1}{2\pi} = 20$

Hence, the required rate of change of the area of the surface of the sphere is 20 sq. cm. **(Ans)**

**2.(v) Solution :** Let x and v be respectively the length of a side and volume of the cube at time t. Again let s be the area of any face of the cube.

Then we know, $v = x^3$ ..................... (1) and $s = x^2$ ..................... (2)

From (1) and (2) we get,

$v^2 = x^6 = (x^2)^3 = s^3$ or $v^2 = s^3$

Differentiating both sides with respect to t we get,

$2v\frac{dv}{dt} = 3s\frac{ds}{dt}$ or, $v\frac{dv}{dt} = \frac{3}{2}s\frac{ds}{dt}$ ..................... (3)

Now for a cube of unit volume we have, v = 1 which gives,

$x^3 = 1$ or, x = 1 [since x is real] and hence s = 1 [from (2)]

Therefore, from (3) we get, $\frac{dv}{dt} = \frac{3}{2}\frac{ds}{dt}$

Hence, rate of change of volume = $\frac{3}{2}\times$ rate of change of area of any face of the cube. **(Ans)**

**2.(vii) Solution :** Let r, s and v be respectively the radius, surface area and volume of the spherical balloon at time t.

Then we know, $v = \frac{4}{3}\pi r^3$ ..................... (1) and $s = 4\pi r^2$ .................. (2)

Differentiating both sides of (1) and (2) with respect to t we get,

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$ ..................... (3) and $\frac{ds}{dt} = 8\pi r \frac{dr}{dt}$ ..................... (4)

Now if $\frac{dv}{dt} = 40$ and $r = 8$, then from (3) we get,

$$40 = 4\pi \cdot 8^2 \cdot \frac{dr}{dt} \quad \text{or,} \quad \frac{dr}{dt} = \frac{5}{32\pi} \qquad \ldots\ldots\ldots\ldots (5)$$

Putting $r = 8$ and $\frac{dr}{dt} = \frac{5}{32\pi}$ in (4) we get,

$$\frac{ds}{dt} = 8\pi \cdot 8 \cdot \frac{5}{32\pi} = 10$$

Therefore, the surface area of the spherical balloon increases at the rate of 10 cm$^2$/min. **(Ans)**

Let the radius r increases by dr when time t increases by dt.

Then, $dr = \frac{dr}{dt} dt$

By the problem, $\frac{dr}{dt} = \frac{5}{32\pi}$ and $dt = \frac{1}{2}$

Then, $dr = \frac{5}{32\pi} \cdot \frac{1}{2} = \frac{5 \times 7}{32\,22\,2} = 0.0248$ cm. (approximately). **(Ans)**

**2.(x) Solution :** We have, $y^2 = 4x$ $\qquad \ldots\ldots\ldots\ldots (1)$

Let (x, y) be the position of the particle on the parabola (1) at time t.

Now, differentiating both sides of (1) with respect to t we get,

$$2y\frac{dy}{dt} = 4\frac{dx}{dt} \quad \text{or,} \quad y\frac{dy}{dt} = 2\frac{dx}{dt} \qquad \ldots\ldots\ldots\ldots (2)$$

By the problem, $\frac{dx}{dt} = 2\frac{dy}{dt}$ $\qquad \ldots\ldots\ldots\ldots (3)$

From (2) and (3) we get, $y\frac{dy}{dt} = 4\frac{dy}{dt}$ or, $y = 4$ $\left[\because \frac{dy}{dt} \neq 0\right]$

From (1) we get, $4^2 = 4x$ or, $x = 4$

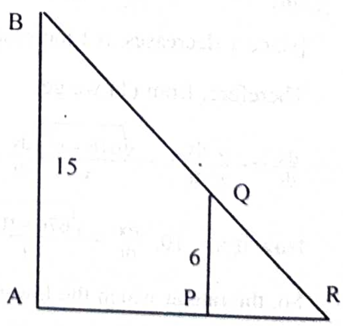
Hence, the required co-ordinates are (4, 4). **(Ans)**

**2.(x) Solution :** Let AB be the lamp-post whose foot is A and B is the source of light; given $\overline{AB} = 15$ ft.

Let PQ denote the position of the man at time t where $\overline{PQ} = 6$ ft.Join BQ and produce it to meet AP (pro duced) at R [as shown in the figure]. Then the length of the man's shadow = $\overline{RP}$.

Assume $\overline{AP} = x$ and $\overline{PR} = y$.

Then, $\overline{RA} = \overline{AP} + \overline{PR} = x + y$ and $\frac{dx}{dt}$ = velocity of the man = 3




Clearly triangles ARB and PRQ are similar

$$\therefore \frac{\overline{PR}}{\overline{PQ}} = \frac{\overline{RA}}{\overline{AB}} \quad \text{or,} \quad \frac{y}{6} = \frac{x+y}{15} \quad \text{or,} \quad 3y = 2x \quad \text{or,} \quad y = \frac{2}{3}x$$

$$\therefore \frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3} \times 3 = 2$$

So, the length of the man's shadow increases at the rate of 2 m.p.h. **(Ans)**

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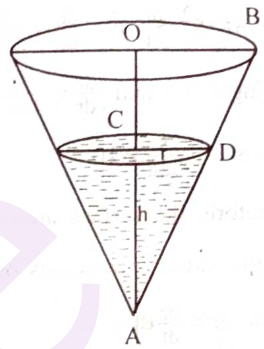
**2.(xiv) Solution :** Let at time t, r and h be respectively the radius and the height of the water level. Then the volume of water V at time t is given by,




$V = \frac{1}{3}\pi r^2 h$ ........................ (1)

Given, the radius of the base of the cone

$= \overline{OB} = \frac{12}{2} = 6$ and its height, $\overline{OA} = 24$ inches.

Again, at time t, the radius of the water-level

$= r = \overline{CD}$ and its height $= h = \overline{CA}$

Clearly, the triangles OAB and CAD are similar.

$\therefore \frac{\overline{CD}}{\overline{CA}} = \frac{\overline{OB}}{\overline{OA}}$ or, $\frac{r}{h} = \frac{6}{24} = \frac{1}{4}$ or, $r = \frac{h}{4}$

From (1) we get, $V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3$

Differentiating both sides with respect to t we get,

$\frac{dV}{dt} = \frac{\pi}{48}\cdot 3h^2\frac{dh}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}$

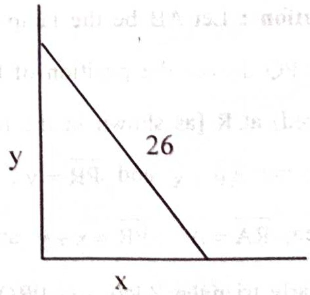
When the height of water-level is 10 inches i.e., h = 10 inches, then

$100 = \frac{\pi}{16}\cdot 10^2\frac{dh}{dt}$ [since for all values of t we have, $\frac{dV}{dt} = 100$]

$\frac{dh}{dt} = \frac{16}{\pi} = \frac{16\times 7}{22} = \frac{8\times 7}{11} = 5.1$

Hence, the water-level is rising at the rate 0f 5.1 inches per minute when the water is 10 inches deep. **(Ans)**

**2.(xvi) Solution :** Let x ft. be the distance of the lower end of the ladder from the vertical wall and y be the height of the top from the floor at any instant t.




Then $x^2 + y^2 = 676$ or, $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

............... (1)

By the problem,

$\frac{dy}{dt} = -10$ inches $= -\frac{10}{12}$ ft.

[since y decreases as t increases]

Therefore, from (1) we get,

$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = -\frac{\sqrt{676-x^2}}{x}\frac{dy}{dt} = \frac{\sqrt{676-x^2}}{x}\times\frac{10}{12}$

Now if x = 10, $\frac{dx}{dt} = \frac{\sqrt{676-10^2}}{10}\times\frac{10}{12} = \frac{24}{10}\times\frac{10}{12} = 2$ ft./sec.

So, the rate at which the lower end moves on a horizontal floor when it is 10 ft. from the wall is 2 ft./sec. **(Ans)**

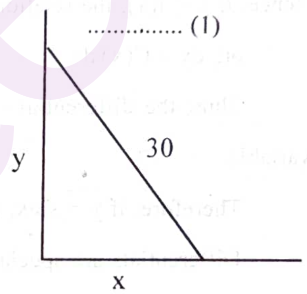
**Second part :** If $\theta$ be the angle makes with the horizontal floor at time t, then the slope of the ladder at time t is

$$\tan\theta = \frac{y}{x}$$

Therefore, the rate at which the slope of the ladder changes when it is 10 ft. from the wall is

$$\frac{d}{dt}\left(\frac{y}{x}\right) = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2} = \frac{10\times\left(-\frac{10}{12}\right) - 24\times 2}{10^2} = \frac{5\times\left(-\frac{5}{3}\right) - 24\times 2}{10^2} = -\frac{169}{300} \text{ per sec. } \textbf{(Ans)}$$

**2.(xvii) Hints :** Let x ft. be the distance of the lower end of the ladder from the vertical wall and y be the height of the top from the pavement at any instant t. Then $x^2 + y^2 = 900$. or, $x\frac{dy}{dt} + y\frac{dy}{dt} = 0$ . ............... (1)




By the problem, $\frac{dx}{dt} = 2x$

Therefore, $\frac{dy}{dt} = -\frac{2x^2}{y} = -\frac{2x^2}{\sqrt{900 - x^2}}$

Now if $x = 10$, $\frac{dy}{dt} = -5\sqrt{2}$ ft./sec.

Negative sign indicates that y decreases as t increases i.e., the top is sliding downwards at the rate of $5\sqrt{2}$ ft./sec.

Second part : If $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are equal numerically, then from (1) we get $x = y$, numerically.

Hence, $x^2 + x^2 = 900$ or, $x = 15\sqrt{2}$ ft. which gives the required distance. **(Ans)**

**3.(i) Solution :** Given, $x = \log_e(6 - 5e^{-t})$

Differentiating both sides with respect to t we get,

$$\frac{dx}{dt} = \frac{d}{dt}\left\{\log\left(6 - 5e^{-t}\right)\right\} = \frac{1}{6-5e^{-t}} \times \left\{0 + 5e^{-t}\right\} = \frac{5e^{-t}}{6-5e^{-t}}$$

$$\therefore \text{ at } t = 0,\quad \frac{dx}{dt} = \frac{5\times e^0}{6-5\times e^0} = 5 \ \textbf{(Ans)}$$

**3.(iii) Solution :** Given, $y = \frac{W}{2EI}\left(\frac{lx^2}{4} - \frac{x^3}{6}\right)$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \frac{W}{2EI}\left\{\frac{l}{4}\frac{d}{dx}\left(x^2\right) - \frac{1}{6}\frac{d}{dx}\left(x^3\right)\right\} = \frac{W}{2EI}\left[\frac{l}{4}\times 2x - \frac{1}{6}\times 3x^2\right] = \frac{W}{2EI}\left[\frac{l}{2}x - \frac{1}{2}x^2\right] = \frac{W}{4EI}\left[lx - x^2\right] \ \textbf{(Ans)}$$

**3.(iv) Solution :** Given, $PV^{1.4} = C$

Taking log of both sides we get,

$\log P + 1.4 \log V = \log C$

Taking differentials we get, $\frac{1}{P}dP + \frac{1.4}{V}dV = 0$

Now by the problem, $P = 10$, $V = 3$ and $dV = 0.3$

Therefore, $\frac{1}{10}dP + \frac{1.4}{3}\times 0.3 = 0$

or, $dP = -\frac{1.4}{3}\times 0.3\times 10 = -1.4$ **(Ans)** [Negative sign indicates the volume is decreasing.]

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## 6.2 Differentials :

Let $f'(x)$ is the derivative of f(x), and $\Delta x$ is the increment of x. Then the differential of f(x), denoted by the symbol df(x), is defined by the relation $df(x) = f'(x)\,\Delta x$ ......................... (1)

If f(x) = x, then $f'(x) = 1$, and (1) reduces to $dx = \Delta x$ .

Thus, when x is the independent variable, the differential of x ( = dx) is identical with $\Delta x$ .

Hence, if y = f(x), the relation (1) becomes $df(x) = f'(x)\,dx$ .

or, $dy = f'(x)\,dx$ .............................. (2)

Thus, the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

Therefore, if y = sinx, then dy = cosx dx, If y = tanx, then dy = $\sec^2 x$ dx.

Differentials are specially useful in applications of integral calculus.

**Note :** In general $\Delta y \neq dy$ , where y is the dependent variable for a given function y = f(x).

(ii) The relation (2) can be written as $\frac{dy}{dx} = f'(x)$; thus, the quotient of the differentials of y and x is equal to the derivative of y with respect to x.

### EXERCISE : SET – II

3. Find the differential of the following functions:

*(i) $\sqrt[3]{x^3+8}$ (ii) $\log(x^2 + 1)$

4. Find the differential of the following functions:

(i) $y = x^2 - 3x + 2$ (ii) $y = \frac{x}{x^2+a^2}$

*(iii) $y = \tan\sqrt{x}$ (iv) $y = \log(x^2+6x+12)$

(v) $y = e^x(\sin x + \cos x)$

5. Find the following differentials:

(i) $d(x^2y^2)$ (ii) $d(x^2 + y^2)$

*(iii) $d\left(\frac{x}{y}\right)$ (iv) $d(x^2 \sin y)$

6. *(i) Find the differential of the function $f(x) = 2x^2 - 2x + 2$ when x changes from 3 to 3.01

(ii) Find the differential of the function $y = x^2 + 2x + 3$, when x changes from 3 to 2.98

*(iii) Find the differential of the function $y = x^2 - 2x + 3$, when x changes from 3 to 2.97

(iv) Find the differential of the function $y = 3x^2 + 2x - 3$, when x changes from 3 to 2.98

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**3.(i) Solution :** Let $y = f(x) = \sqrt[3]{x^3+8}$

$$\therefore \quad f'(x) = \frac{d}{dx}\left(\sqrt[3]{x^3+8}\right) = \frac{1}{3}\left(x^3+8\right)^{-\frac{2}{3}} \frac{d}{dx}\left(x^3+8\right) = \frac{x^2}{\sqrt[3]{\left(x^3+8\right)^2}}$$

Hence, the required differential is $dy = f'(x)\,dx = \frac{x^2}{\sqrt[3]{\left(x^3+8\right)^2}}dx$ **(Ans)**

**4.(iii) Solution :** Let $y = f(x) = \sin\sqrt{x}$

$$\therefore \quad f'(x) = \frac{d}{dx}\left(\sin\sqrt{x}\right) = \cos\sqrt{x}\,\frac{d}{dx}\left(\sqrt{x}\right) = \cos\sqrt{x}\,\frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

Hence, the required differential is $dy = f'(x)\,dx = \frac{\cos\sqrt{x}}{2\sqrt{x}}dx$ **(Ans)**

**5.(iii) Solution :** Let $u = \frac{y}{x}$

$$\therefore \quad \frac{du}{dx} = \frac{y - x\frac{dy}{dx}}{y^2} \quad \text{or,} \quad du = \frac{ydx - xdy}{y^2} \quad \therefore \quad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$ **(Ans)**

**6.(i) Solution :** Differential of $y = f(x) = x^2 - 2x + 3$ is $dy = f'(3)\,\Delta x$ .......... (1)

Here, $\Delta x = 3.01 - 3 = .01$

and $f'(x) = 4x - 2$ which gives $f'(3) = 12 - 2 = 10$

Therefore, from (1) we get, $dy = 10 \times 0.01 = 0.1$ **(Ans)**

**6.(iii) Solution :** Differential of $y = f(x) = 2x^2 - 2x + 2$ is $dy = f'(2)\,\Delta x$ ...... (1)

Here, $\Delta x = 2.97 - 3 = -0.03$

and $f'(x) = 2x - 2$ which gives $f'(2) = 4 - 2 = 2$

Therefore, from (1) we get, $dy = 2 \times (-0.03) = -0.06$ **(Ans)**

## 6.3 Approximate calculations and small errors :

Let $y = f(x)$, since $\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = f'(x)$, $\Delta y$ is approximately $= f'(x)\,\Delta x$ for small values of $\Delta x$. Thus, dy and $\Delta y$ may be taken as approximately equal, when $\Delta x$ (dx) is small.

Hence, when only an approximate value of the change of a function is desired, it is usually convenient to calculate the value of the corresponding differential and use this value.

Small errors arising in the value of a function due to an assumed small error in the independent variable may also be calculated on the same principle.

## EXERCISE : SET – III

7. (i) If $\log_{10}3 = 0.4771$ and $\log_{10}e = 0.4343$, using calculus find the value of $\log_{10}30.5$, correct to 3 places of decimal.

(ii) Using the method of differentials find the approximate value of $\cos 62^0$, given $1^0 = 0.01745$.

(iii) Determine the approximate values of the following using the method of differentials: *(a) $\sqrt{50}$ (b) $\sqrt[4]{627}$

8. (i) The length of a side of a cube is 10 cm.; if an error of 0.05 cm. Is made in measuring the side find the approximate error, relative error and percentage rror made in calculating its volume.

*(ii) Estimate the error made in calculating the area of the triangle ABC in which the side a and b measured accurately as 25 cm. And 16 cm. While the angle C is measured as $60^0$ but $\left(\frac{1}{2}\right)^0$ in error.

(iii) If in a triangle ABC, the side c and the angle C remain unchanged while the other sides and angles are changed slightly, show that, $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$

*(iv) If a triangle ABC inscribed in a fixed circle be slightly varied in such a way that its vertices are always on the circle, show that, $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$

(v) In a triangle ABC, if the sides a, b remain constant but the base angles A and B vary, then show that, $\frac{dA}{\sqrt{a^2 - b^2\sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2\sin^2 B}}$.

*(vi) If the radius of a spherical balloon increases by 0.1%, find approximately the percentage increase in volume.

*(vii) The distance x of a particle from a fixed point at time t is given by $x = 5 + A.\sin 2t + B.\cos 2t$, where A and B are given to be 3, 4 respectively. However, it is found, on measurement, that there is a 1% error in the maximum value of x and this is due to an error in A only. Find the percentage error in A. **[JEE – 91]**

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**6.(iii)(a) Solution :** Let $y = \sqrt{x}$ $\therefore$ $dy = \frac{1}{2\sqrt{x}}dx$ ................ (1)

Let us assume $x = 49$ and $dx = 1$. Then from (1) we get, $dy = \frac{1}{2\sqrt{49}} \times 1 = \frac{1}{14} = 0.07143$

Hence, for an increment of x, the increment in y is 0.07143.

Therefore, $\sqrt{50} = \sqrt{49} + 0.07143 = 7 + 0.07143 = 7.07143 = 7.07$

- If dx is the error in x, then the ratio (a) $\frac{dx}{x}$ is the **relative error** and (b) $\frac{dx}{x} \times 100$ is the **percentage error.**

**8.(ii) Solution :** Let A be the area of the triangle ABC.

Then we know, $A = \frac{1}{2}ab\sin C$, where a and b are constants.

Taking differentials we get, $dA = \frac{1}{2}ab\cos C\,dC$ ................. (1)

By the problem, a = 25, b = 16 , C = $60^0$ and $dC = \left(\frac{1}{2}\right)^0 = \frac{1}{2} \times \frac{\pi}{180}$ radian.

$\therefore$ from (1) we get, $dA = \frac{1}{2} \times 25 \times 16 \times \cos 60^0 \times \frac{1}{2} \times \frac{\pi}{180}$ sq.cm.

$= \frac{1}{2} \times 25 \times 16 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{180} \times \frac{22}{7} = \frac{55}{63}$ sq.cm. **(Ans)**

**8.(iv) Solution :** We know in triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = k$ (say) = constant .......................... (1)

From (1) we get, $\frac{a}{\sin A} = 2R = k$ or, $a = k \sin A$

Taking differentials we get, $da = k \cos A\, dA$ or, $\frac{da}{\cos A} = k \cdot dA$ ...... (2)

Similarly from, $\frac{b}{\sin B} = 2R$ we get, $\frac{db}{\cos B} = k \cdot dB$ ..... (3) and from $\frac{c}{\sin C} = 2R$ we get, $\frac{dc}{\cos C} = k \cdot dC$ ..... (4)

Adding (2), (3) and (4) we get,

$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = k(dA + dA + dC) = k\, d(A + B + C) = k\, d(\pi) = 0$ [since $A + B + C = \pi$]

or, $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$ **(Proved)**

**8.(vi) Solution :** Let v be the volume of a sphere of radius r. Then $v = \frac{4}{3}\pi r^3 \therefore \frac{dv}{dr} = 4\pi r^2$

By the given condition $\frac{\Delta r}{r} \times 100 = 0.1$ .................. (1)

Now, $dv = \frac{dv}{dr}\Delta r = 4\pi r^2 \times \Delta r = 4\pi r^2 \times \frac{0.1 \times r}{100} = \frac{4\pi r^3}{1000}$ [by (1)]

$\therefore \frac{dv}{v} = \frac{4\pi r^3 \times 3}{1000 \times 4\pi r^3} = \frac{3}{1000}$ $\therefore \frac{dv}{v} \times 100 = \frac{3}{1000} \times 100 = \frac{3}{10} = 0 \cdot 3$ **(Ans)**

**8.(vii) Solution :** Given, x = 5 + A.sin2t + B.cos2t

or, $x = 5 + r \sin\theta \sin 2t + r \cos\theta \cos 2t = 5 + r\cos(2t - \theta)$ ............... (1) where, $r\cos\theta = B$ and $r\sin\theta = A$

$\therefore A^2 + B^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2$ or, $r = \sqrt{A^2 + B^2}$ and $\frac{r\sin\theta}{r\cos\theta} = \frac{A}{B}$ or, $\tan\theta = \frac{A}{B}$

From (1) we get, $r = 5 + \sqrt{A^2 + B^2} \cos(2t - \theta)$

We know the maximum value of $\cos(2t - \theta)$ is 1 and therefore the maximum value of x is $5 + \sqrt{A^2 + B^2}$

Let $u = 5 + \sqrt{A^2 + B^2}$ ................. (2) Putting A = 3 and B = 4 in (2) we get, u = 5 + 5 = 10

By the problem, error in u is 1% i.e., $\frac{du}{u} \times 100 = 1$ or, $\frac{du}{10} \times 100 = 1$ or, $du = \frac{1}{10}$

Again, from (2) we get, $du = \frac{d}{dA}\left(5 + \sqrt{A^2 + B^2}\right) dA$ [since, error in u is due to an error in A only]

or, $du = \frac{1}{2} \cdot \frac{1}{\sqrt{A^2 + B^2}} 2A \cdot dA$ ............................... (3)

Putting $du = \frac{1}{10}$, A = 3 and B = 4 in (3) we get, $\frac{1}{10} = \frac{1}{2} \cdot \frac{3}{\sqrt{3^2 + 4^2}} dA$ or, $dA = \frac{1}{6}$

Therefore, the required percentage error in A is $\frac{dA}{A} \times 100 = \frac{1}{6} \times \frac{1}{3} \times 100 = \frac{50}{9}$ **(Ans)**

N.Tute

## 6.3 Velocity and acceleration :

Let a particle moving along the straight line OX from O to X and P be the position of the particle at any time t where OP = x.

Let Q be its position of the particle at time $(t + \delta t)$, such that $OQ = x + \delta x$


Therefore, $PQ = \delta x$ = the displacement in time $\delta t$.

Since, the velocity is the rate of displacement, the average velocity of the particle in timt $\delta t = \frac{\delta x}{\delta t}$ and at P the velocity v of the particle is given by

$v = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\delta x}{\delta t} = \frac{dx}{dt}$ in the direction from O to P. Here x increases with t.

Again let $v$, $v + dv$ be the velocity of the particle at P , Q respectively.

Then the average acceleration of the particle in time $dt = \frac{\delta v}{\delta t}$

Hence, at P the acceleration f of the particle is given by

$f = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\delta v}{\delta t} = \frac{dv}{dt}$, in terms of v and t.

Now, $f = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = v\frac{dv}{dx}$, in terms of v and x.

and $f = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$, in terms of x and t.

# EXERCISE SET – IV

**1.** The displacement in metre of a particle 'x' in time t is given by $x = \frac{1}{2}t^2 + \sqrt{t}$ . Find Its velocity at t = 4 sec.

***2.** The displacement of a particle in meter 's' in time 't' in second is given by $x = ut + \frac{1}{2}at^2$ , where u is its initial velocity and 'a' its constant acceleration. Find the velocity of the particle at t = 2 when u = 20 m/s and a = 50 m/s$^2$. **[WBSC – 08]**

**3.** A ball travels 8 cm in t sec so that $s = 8t + 15t^2 + 5$. Find the acceleration of the ball when t = 3 secs

**4.** A ball travels s ft. in t sec, where $s = 8t - 10t^2$, find the velocity of the ball when t = 2. **[WBSC – 03]**

**5.** A ball travels S cm. in t sec. so that $s = 8t + 15t^2$, find the acceleration of the ball when t = 2 sec.

**6.** The distance x cm moved by a particle in time t sec is given by $s = 2t^3 - 4t^2 + 3t$. Find the velocity and acceleraton when t = 3 sec.

***7.** A particle moves along a straight line according to the law $s^2 = 6t^2 + 4t + 3$.. Prove that the acceleration varies as $\frac{1}{s^3}$.

**8.** A particle moves along a straight line so that after t sec its dietance s from a fixed point O on the line is given by $s = (t - 1)^2(t - 2)$. Find the distance from O when its velocity is zero.

*9. A particle is moving along a straight line, on which O is a point. After time t, the distance of the particle from O is given by a.cos nt + b.sin nt (a, b, n are constant). Prove that the acceleration of the particle is proportional to its distance from O. **[WBSC – 03, 17]**

*10. A particle is moves according to the law $v^2 = 4(x\sin x + \cos x)$ where v is the velocity at a distance x. Find its acceleration.

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**2. Solution :** Given, $x = ut + \frac{1}{2}at^2$

$\frac{dx}{dt} = u + at$, is the velocity at time t. [ Differentiating both sides with respect to t ]

Now, given t = 2 when u = 20 m/s and a = 50 m/s².

Therefore, at t = 2, $\frac{dx}{dt} = 20 + 50 \cdot 2 = 20 + 100 = 120$ m/s. is the required velocity **(Ans)**

**7. Solution :** Given, $s^2 = 6t^2 + 4t + 3$.

Differentiating both sides with respect to t we get,

$$2s\frac{ds}{dt} = 12t + 4 \quad \text{or,} \quad \frac{ds}{dt} = \frac{6t+2}{s} \qquad \text{--------------------- (1)}$$

Again differentiating both sides with respect to t we get,

$$\frac{d^2s}{dt^2} = \frac{6s - (6t+2)\frac{ds}{dt}}{s^2} = \frac{6s - (6t+2)\cdot\frac{6t+2}{s}}{s^2} \quad \textbf{[ from (1) ]}$$

$$= \frac{6s^2 - (6t+2)^2}{s^3} = \frac{6(6t^2+4t+3) - (36t^2+24t+4)}{s^3} = \frac{14}{s^3}$$

**or,** $\frac{d^2s}{dt^2} \propto \frac{1}{s^3}$

Hence, the acceleration varies as $\frac{1}{s^3}$. **(Proved)**

**9. Solution :** Let, x = a.cos nt + b.sin nt, where x is the distance of the particle from O after time t.

$\frac{dx}{dt} = -$ an sinnt + bn cosnt [ Differentiating both sides with respect to t ]

or, $\frac{d^2x}{dt^2} = -an^2\cos nt - bn^2\sin nt = -n^2(a\cos nt + b\sin nt)$ [ Again differentiating both sides with respect to t ]

or, $\frac{d^2x}{dt^2} = -n^2x$ or, or, $\frac{d^2x}{dt^2} \propto x$ [Since, $-n^2$ is constant]

Hence, the acceleration of the particle is proportional to its distance from O. **(Proved)**

**10. Solution :** Given, $v^2 = 4(x\sin x + \cos x)$

$2v\frac{dv}{dx} = 4(x\cos x + \sin x - \sin x) = 4x\cos x$ [ Differentiating both sides with respect to x ]

or, $v\frac{dv}{dx} = 2x\cos x$ is the required acceleration. **(Ans)**



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# MISCELLANEOUS

## OBJECTIVE TYPE MULTIPLE CHOICE

1. If the radius of a circle increased by 100%, then its area is increased by:

   (a) 100% (b) 200% (c) 300% (d) 400% [WBSC – 03, 07]

2. The function $y = f(x)$ is increasing in the interval $a \le x \le b$ if:

   (a) $\frac{dy}{dx} = 0$; b) $\frac{dy}{dx} < 0$; (c) $\frac{dy}{dx} > 0$; (d) none of these. [WBSC – 03]

3. A ball travels 8 cm in t sec so that $s = 8t + 15t^2 + 5$, the acceleration of the ball when $t = 3$ secs is –

   (a) 30 cm/sec² (b) 45 cm/sec² (c) 60 cm/sec² (d) none of these [WBSC – 07]

4. The displacement in metre of a particle 'x' in time t is given by $x = \frac{1}{2}t^2 + \sqrt{t}$. Its velocity at t = 4 sec is –

   (a) $4\frac{1}{4}$ m/s (b) $8\frac{1}{2}$ m/s (c) $3\frac{3}{4}$ m/s (d) none of these. [WBSC – 07]

5. The displacement of a particle in meter 's' in time 't' in second is given by $x = ut + \frac{1}{2}at^2$, where u is its initial velocity and 'a' its constant acceleration. Find the velocity of the particle at t = 2 when u = 20 m/s and a = 50 m/s². [WBSC – 08]

6. A ball travels s ft. in t sec, where $s = 8t - 10t^2$, find the velocity of the ball when t = 2. [WBSC – 03]

7. A ball travels S cm. in t sec. so that $s = 8t + 15t^2$, the acceleration of the ball when t = 2 sec. is

   (a) 60 cm/sec², (b) 30 cm/sec², (c) 16 cm/sec², (d) 68 cm/sec². [WBSC – 06]

## SUBJECTIVE TYPE

1. A particle is moving along a straight line, on which O is a point. After time t, the distance of the particle from O is given by a.cos nt + b.sin nt (a, b, n are constant). Prove that the acceleration of the particle is proportional to its distance from O. [WBSC – 03, 17]

2. The pressure P and the volume V of a gas obeys the law $PV^{1.4} = C$, where C is a constant. When pressure is 10 kg/m² and volume is 3 m³ and is changing 0.3 m³/sec., how fast pressure P is changing at that instant (explain the negative sign) [WBSC – 07]

3. A spherical ice-ball melts and its radius decreases at the rate of 0.1 cm/sec. Taking $\pi = \frac{22}{7}$ and specific gravity of ice 0.9, find the amount of water formed when the radius of the sphere is 7 cm. [WBSC – 07]

4. If the area of a circle changes uniformly with respect to time, show that the rate of change of its circumference varies inversely as its radius. [WBSC – 14]

5. The area of a circular disc increases with uniform rate. Prove that the rate of change of its perimeter varies inversely as the radius of the disc. [WBSC – 18]

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# Chapter – 7

## MAXIMA AND MINIMA

### 7.1 Increasing and Decreasing function :

(a) If the value of a function $y = f(x)$ increases or decreases as x increases or decreases then the function $f(x)$ is said to be an increasing function of x.

If the value of a function $y = f(x)$ decreases or increases as x increases or desreases then the function $f(x)$ is said to be a decreasing function of x.

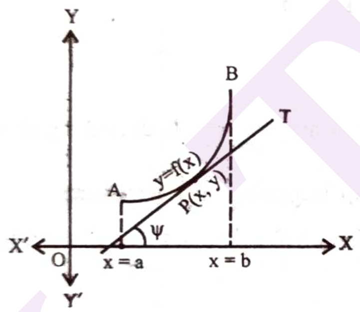



Fig. – 1

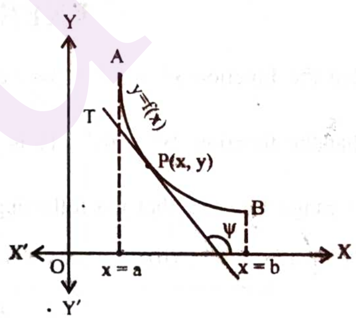



Fig. – 2

**For example :** $f(x) = x^2 + 5$ is an increasing function and $f(x) = \frac{1}{x}$ is a decreasing function for all values of x.

Suppose $P(x, y)$ be any point in $a \le x \le b$ of the function $y = f(x)$.

If the tangent PT to the curve at P makes an angle $\psi$ with the positive direction of x–axis, then we know $\frac{dy}{dx} = \tan\psi$.

Clearly, $\frac{dy}{dx} = \tan\psi > 0$ when $\psi$ is acute angle. From **Fig. – 1** it is clear that $y = f(x)$ increases or decreases as x increases or decreases. Therefore, if $\frac{dy}{dx} > 0$ in $a \le x \le b$, then the function $y = f(x)$ is increasing in $a \le x \le b$.

Again, $\frac{dy}{dx} = \tan\psi < 0$, when $\psi$ is obtuse. From **Fig. – 2** it is clear that $y = f(x)$ decreases or increases when x increases or decreases. Therefore, if $\frac{dy}{dx} < 0$ in $a \le x \le b$, then the function $y = f(x)$ is decreasing in $a \le x \le b$.

Hence, the function $y = f(x)$ is increasing or decreasing in the interval $a \le x \le b$ according as $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ in that interval.

N.Tute

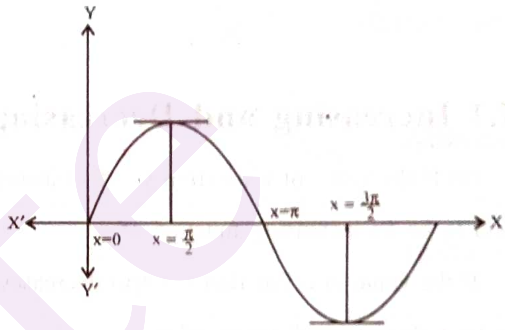
## 7.2 Turning point or Stationary point :

The point at which the function changes its nature is called the turning point.

At turning point, the increasing function may change to decreasing one and vice versa.

For example the function $y = \sin x$ is an increasing function between $x = 0$ and $x = \frac{\pi}{2}$ and decreasing between $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

At $x = \frac{\pi}{2}$ the function is neither increasing nor decreasing but stationary. Tangent at the turning point $x = \frac{\pi}{2}$ to the curve $y = \sin x$ is parallel to x-axis, i.e., $y = 0$.




$\therefore$ at the turning point, $\tan\psi = 0 \Rightarrow \frac{dy}{dx} = \tan\psi = 0$.

### EXERCISE : SET – I

1. (i) Show that the function $x^3 - 3x^2 + 3x$ increases with x.

*(ii) Show that the function $4x^2 - 6x - 11$ is increasing at $x = 4$ and $\frac{x^2}{x^2+16}$ is decreasing at $x = -2$.

2. Find the range for x so that the following functions are (a) increasing (b) decreasing.

(i) $2x^3 - 9x^2 + 12x + 6$ *(ii) $6 - 36x + 15x^2 - 2x^3$ (iii) $x^3 - 3x^2 + 15x + 3$ (iv) $3 - 12x + 3x^2 - x^3$

### ANSWERS

2 (i) (a) $x < 1$, $x > 2$, (b) $1 < x < 2$, (ii) (a) $2 < x < 3$, (b) $x < 2$, $x > 3$, (iii) (a) for all values of x (iv) (b) for all values of x.

### SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1.(ii) Solution :** Let $f(x) = 4x^2 - 6x - 11 \therefore f'(x) = 8x - 6$

At $x = 4$, $f'(x) = 8 \times 4 - 6 = 26 > 0$ Therefore, f(x) is increasing at $x = 4$

Let $g(x) = \frac{x^2}{x^2+16} \therefore g'(x) = \frac{(x^2+16)\times 2x - x^2 \times 2x}{(x^2+16)^2} = \frac{32x}{(x^2+16)^2}$

$\therefore$ at $x = -2$, $g'(x) = \frac{-64}{(4+16)^2} < 0$

Therefore, at $x = -2$, g(x) is decreasing. **(Proved)**

**2.(ii) Solution :** Let $f(x) = 6 - 36x + 15x^2 - 2x^3 \therefore f'(x) = -36 + 30x - 6x^2$

$= -6(6 - 5x + x^2) = -6(3 - x)(2 - x) = 6(x - 2)(3 - x)$

$\therefore f'(x) > 0$, when $(x - 2)(3 - x) > 0$ i.e., $2 < x < 3$ and $f'(x) < 0$, when $(x - 2)(3 - x) < 0$ i.e., $x < 2$, $x > 3$.

Hence, given function is increasing when $2 < x < 3$ and decreasing when $x < 2$, $x > 3$ **(Ans)**

## 7.3 Maxima and Minima of a Function.

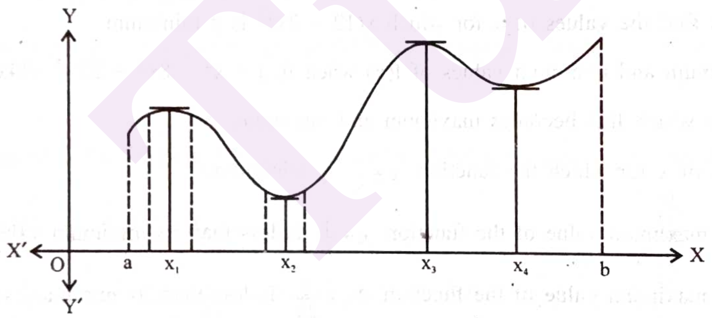
**Definition :** Let $y = f(x)$ be a continuous function of $x$ and $x = c$ be a point in the interval of definition of the function.

(a) $f(x)$ is said to have a **maximum** value at $x = c$, if there exist a suitable neighbourhood of c, wherein for every $x$, $f(c) > f(x)$ and

(b) $f(x)$ is said to have a **minimum** value at $x = c$, if there exist a suitable neighbourhood of c, wherein for every $x$, $f(c) < f(x)$.

The function $y = f(x)$ is said to have an **extreme** or **turning** value at $x = c$ if $y = f(x)$ has a maximum at $x = c$ or $y = f(x)$ has a minimum at $x = c$.

**Graphically :** At an extreme point the ordinate of the curve $y = f(x)$, is either greater than (maximum) or less than (minimum) the adjacent ordinates on either side in the immediate neighbourhood.




From the above figure : it is clear that, in $a \le x \le b$, $y = f(x)$ has maxima at $x_1$, $x_3$ and minima at $x_2$, $x_4$. Observe that maximum value of $f(x)$ at $x_1$ is less than the minimum value of $f(x)$ at $x_4$.

## 7.4 Test for Extreme values of a Function :

Let $x = c$ be a point in the interval of definition of the function $f(x)$ and $f'(c) = 0$.

Then

(i) If $f'(x)$ changes sign, from positive on the left to negative on the right of the point $x = c$, $f(x)$ has a maximum at $x = c$.

(ii) If $f'(x)$ changes sign from negative on the left to positive on the right of the point $x = c$, $f(x)$ has a minimum at $x = c$.

(iii) If $f'(x)$ does not change in sign on either side of the point $x = c$, $f(x)$ has neither maximum nor minimum at $x = c$.

Now, when $f'(x)$ changes sign, from positive on the left to negative on the right of $x = c$, then $f'(x)$ is a decreasing function at $x = c$. Therefore, the derivative of $f'(x)$ at $x = c$ is $< 0$ i.e., $f''(c) < 0$. [ In case (i) ]

Again, when $f'(x)$ changes sign from negative on the left to positive on the right of $x = c$, then $f'(x)$ is an increasing function at $x = c$. Therefore, the derivative of $f'(x)$ at $x = c$ is $> 0$ i.e., $f''(c) > 0$ [ In case (ii) ]

**Hence, a function $y = f(x)$ is continuous in $a \leq x \leq b$ has a maximum at $x = c$ on the curve in $a < x < b$ if (i) $f'(c) = 0$ and $f''(c) < 0$ and has a minimum at $x = d$ on the curve in $a < x < b$ if $f'(d) = 0$ and $f''(d) > 0$.**

## EXERCISE : SET – II

1. *(i) Find the maximum and minimum values of f(x) when $f(x) = \frac{x^2 - 7x + 6}{x - 10}$. Also obtain the values of x for which f(x) becomes maximum and minimum. [HS – 91]

(ii) Find the local maximum and local minimum of $x^3 - 9x^2 + 24x - 12$. [HS – 00]

(iii) For what value of x will $(x - 1)(3 - x)$ have its maximum ?

(iv) Using calculus find the values of x for which $x(12 - 2x)^2$ is a minimum.

(v) Find the maximum and minimum values of f(x) when $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 8$. Also obtain the values of x for which f(x) becomes maximum and minimum.

(vi) Find the value of x for which the function $x + \frac{1}{x}$ is minimum. [HS – 00]

2. *(i) Show that the maximum value of the function $x + \frac{1}{x}$ is less than its minimum value. [WBSC – 14]

(ii) Show that the maximum value of the function $2x + \frac{1}{2x}$ is less than its minimum value.

(iii) For what values of x will the function $y = x + \frac{1}{x}$ be a maximum or minimum ? [HS – 96]

**(iv) Find the maximum and minimum values of the function $x^3 + \frac{1}{x^3}$.** [WBSC – 15]

**(v) Prove that the maximum value of the function** $y = 4x + \frac{3}{x}$ is less than its minimum value. [HS – 04]

**(vi)** Show that the minimum value of $\frac{(2x - 1)(x - 8)}{(x - 1)(x - 4)}$ is greater than its maximum value. [HS – 07]

3. **(a) Show that the following functions has neither a maximum nor a minimum value.**

(i) $\frac{2}{3}x^3 - 6x^2 + 20x - 5$   *(ii) $x^3 - 3x^2 + 24x + 30$

(iii) $\frac{\sin(x + a)}{\sin(x + b)}$   (iv) $\frac{3x + 4}{5x + 6}$   (v) $x^3 - 6x^2 + 24x + 4$ [HS – 05]

**(b) Show that if t is real and finite, then the function** $x(t) = \frac{e^t}{1 + e^t}$ has no extreme value. [HS – 07]

4. (i) Show that the function $\sin^3 x \cdot \cos x$ has a maximum value at $x = \frac{\pi}{3}$.

**(ii) Show that the function $\sin x \cdot (1 + \cos x)$ has a maximum value at $x = \frac{\pi}{3}$.**

(iii) Show that sinx· + cosx will be maximum at $x = \frac{\pi}{4}$. [WBSC – 16]

(iv) Prove that the minimum value of $9e^x + 25e^{-x}$ is 30.

(v) Prove that $4^x - 8x \cdot \log 2$ is minimum at x = 1.

(vi) Show that $f(x) = x^x$ has a minimum value at $x = e^{-1}$. [HS – 03]

*(vii) Prove that $x^2 \log\left(\frac{1}{x}\right)$ is maximum at $x = \frac{1}{\sqrt{e}}$.

(viii) State whether the following statement is true or false :

$f(x) = (x - 1)(3 - x)$ has an extreme value at the point x = 2. [HS – 07]

(ix) What do you understand by the statement "In case of a function f of a real variable x, the function f has a maximum at x = c". Show that $y = \left(\frac{1}{x}\right)^x$ has the maximum value $e^{\frac{1}{e}}$. [HS – 01]

5. (i) Find the maximum value of sinx + cosx and the value of x for which it is maximum.

(ii) Find the maximum and minimum values of 4sinx + 3cosx.

(iii) Obtain the maximum and minimum values of the function $\cos^2 x + \cos x + 3$ in the interval $0 \le x \le \frac{\pi}{2}$.

[HS – 92]

*(iv) For what values of x the function y = 2·sinx + cos2x ($0 \le x \le 2\pi$) attains maximum and minimum values.

6. *(i) Find the maximum value of $f(x) = x^{\frac{1}{x}}$. [HS – 94, 99]

(ii) Show that the difference of the maximum and minimum values of $\left(a - \frac{1}{a} - x\right)\left(4 - 3x^2\right)$ is $\frac{4}{9}\left(a + \frac{1}{a}\right)^3$.

7. *(i) Find the maximum value of the product of the two numbers if their sum is 12. [JEE – 90]

(ii) Let x and y be two real variables such that x > 0 and xy = 1. Find the minimum value of x + y.

(iii) If x > 0, y > 0 and xy = 25, find the minimum value of x + y.

(iv) Divide 25 into two parts in such a way that the product of the square of one part and the cube of the other part is a maximum. [HS – 03]

*(v) Divide 20 into two parts such that the product of one part and the cube of the other is maximum.

[WBSC – 05, 18]

8. (i) Find the point on the straight line 2x + 3y = 6 which is closest to the origin.

*(ii) Find the co-ordinates of a point on the parabola $y = x^2 + 7x + 2$ which is closest to the straight line y = 3x – 3. JEE – 91]

*(iii) Find the shortest distance of the point (0, c) from the parabola $y = x^2$ where $0 \le c \le 5$.

9. *(i) Prove that the greatest rectangle inscribed in a circle is a square. [HS – 92, 98, 06]

(ii) Show that of all rectangles of a given area, the square has the least perimeter. [HS – 92, 97, 07]

(iii) Find the maximum area of a rectangle incribed in a circle $x^2 + y^2 = a^2$. [HS – 00]

(iv) The perimeter of a triangle is 8 cm. If one of the sides is 3 cm, find the length of the other sides so that the area of the triangle may be a maximum.

(v) The perimeter of a rectangle is 100 cm. If the area of a rectangle is maximum, find the lengths of its sides. [HS – 05]

(vi) Show that of all rectangles of given perimeter, the square has the greatest area. [HS – 99]

(vii) A wire of length 40 cm. is to form a rectangle; find the length and breadth of that rectangle which has the maximum area. [HS – 01]

*(viii) A wire of length $l$ is to be cut into two pieces, one being bent to form a square and the other to form a circle. How should the wire be cut if the sum of the areas enclosed by the two pieces to be a minimum.

10. *(i) Prove that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base. [JEE – 94]

*(ii) Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.

## ANSWERS

1. (i) Max. at x = 4; Max. value = 1, Min. at x = 16, Min value = 25. (ii) Max. at x = 2; Max. value = 8, Min. at x = 4, Min value = 4. (iii) 2 (iv) x = 6 (v) Max at x = 2; Max. value = 0, Min at x = 1, x = 3 Min value = –1.

2. (iii) Max at x = –1, Min at x = 1(iv) Min value = 2 (at x = 1), Max value = –2(at x = –1)

5. (i) $\sqrt{2}$ at $x=(4n+1)\frac{\pi}{4}$ where n = 0 or any integer. (ii) Max value = 5 and Min value = –5

(iii) Max. value = 5 at x = 0, no minimum in the given interval.

(iv) At $x=\frac{\pi}{2},\frac{3\pi}{2}$ y is minimum; at $x=\frac{\pi}{6},\frac{5\pi}{6}$ is maximum.

6. (i) $e^{\frac{1}{e}}$ 7. (i) 36 (ii) Min value = 2 (iii) 10 (v) 5, 15

8. (i) $\left(\frac{12}{13},\frac{18}{13}\right)$ (ii) (–2, –8) (iii) S.D = $\frac{1}{2}$ unit when c = $\frac{1}{2}$ and S.D = $\sqrt{c-\frac{1}{4}}$ when $\frac{1}{2}<c\le 5$.

9. (iii) $2a^2$ sq. unit (iv) Length of each side = 2.5 cm. (v) Length = Breadth = 25 cm.

(vii) Length = Breadth = 10 cm. (viii) $\frac{4l}{\pi+4},\frac{l\pi}{\pi+4}$.

## SOLUTION OF THE PROBLEMS WITH " * " MARKS

**1.(i) Solution :** $f(x)=\frac{x^2-7x+6}{x-10}$ $\therefore$ $f'(x)=\frac{(x-10)(2x-7)-(x^2-7x+6)\times 1}{(x-10)^2}$

or, $f'(x)=\frac{2x^2-20x-7x+70-x^2+7x-6}{(x-10)^2}$ or, $f'(x)=\frac{x^2-20x+64}{(x-10)^2}$

$\therefore$ $f''(x)=\frac{(x-10)^2(2x-20)-(x^2-20x+64)\times 2(x-10)}{(x-10)^4}$

Now, $f'(x)=0$ gives $x^2-20x+64=0$ or, $(x-4)(x-16)=0 \Rightarrow x=4, 16$

At $x=4$, $f''(x)=\frac{2(4-10)^3}{(4-10)^4}=-\frac{2}{6}=-\frac{1}{3}<0$ and at $x=16$, $f''(x)=\frac{2(16-10)^3}{(16-10)^4}=\frac{2}{6}=\frac{1}{3}>0$

$\therefore$ at $x=4$, $f(x)$ is maximum and the maximum value $=\frac{4^2-7\times 4+6}{4-10}=1$

and at $x=16$, $f(x)$ is minimum and the minimum value $=\frac{16^2-7\times 16+6}{16-10}=25$ **(Ans)**

**2.(i) Solution :** Let $f(x)=x+\frac{1}{x}$ $\therefore f'(x)=1-\frac{1}{x^2}$, $f''(x)=\frac{2}{x^3}$

$f'(x)=0$ gives, $1-\frac{1}{x^2}=0$ or, $x^2=1 \Rightarrow x=\pm 1$

At $x=1$, $f''(x)=2>0$; At $x=-1$, $f''(x)=-2<0$

$\therefore$ at $x=1$, $f(x)$ is minimum and at $x=-1$, $f(x)$ if maximum.

The maximum value $=-1+\frac{1}{-1}=-2$ and the minimum value $=1+\frac{1}{1}=2$.

Clearly, maximum value < minimum value. Hence, the result.

**3.(ii) Solution :** Let $f(x)=x^3-3x^2+24x+30$ $\therefore$ $f'(x)=3x^2-6x+24$, $f''(x)=6x-6$

$f'(x)=0$ gives, $3(x^2-2x+8)=0$ or, $x^2-2x+8=0$ $\therefore$ $x=\frac{2\pm\sqrt{4-32}}{2}=1\pm\sqrt{-7}$ which is not real.

Hence, the given function has neither a maximum nor a minimum. **(Ans)**

**4.(vii) Solution :** Let $f(x)=x^2\log\left(\frac{1}{x}\right)=-x^2\log x$ $[\because \log 1=0]$

$\therefore$ $f'(x)=-x^2\times\frac{1}{x}-2x\log x=-x(1+2\log x)$

$f''(x)=-1(1+2\log x)-x\times\frac{2}{x}=-1-2\log x-2=-3-2\log x$

At $x=\frac{1}{\sqrt{e}}$, $f'(x)=-\frac{1}{\sqrt{e}}\left(1+2\log\frac{1}{\sqrt{e}}\right)=-\frac{1}{\sqrt{e}}\left(1+2\log e^{-\frac{1}{2}}\right)=-\frac{1}{\sqrt{e}}\left(1+2\left(-\frac{1}{2}\right)\right)=0$ $[\because \log e=1]$

At $x = \frac{1}{\sqrt{e}}$, $f''(x) = -3 - 2\log\frac{1}{\sqrt{e}} = -3 - 2\log e^{-\frac{1}{2}} = -3 - 2\left(-\frac{1}{2}\right) = -2 < 0$

$\therefore$ $x = \frac{1}{\sqrt{e}}$, $x^2 \log\left(\frac{1}{x}\right)$ is maximum. **(Proved)**

**5.(iv) Solution :** We have , $y = 2\sin x + \cos 2x$ ............................(1)

Differentiating both sides with respect to x we get,

$\frac{dy}{dx} = 2\cos x - 2\sin 2x$ ........................... (2)

$\frac{d^2y}{dx^2} = -2\sin x - 4\cos 2x$ ........................... (3)

For maximum or minimum value of y we have, $\frac{dy}{dx} = 0$, which gives,

$2\cos x - 2\sin 2x = 0$ or, $\cos x - \sin 2x = 0$ [from (2)]

or, $\cos x - 2\sin x\cos x = 0$ or, $\cos x(1 - 2\sin x) = 0$

Therefore, either, $\cos x = 0$, i.e., $x = (2n+1)\frac{\pi}{2}$

or, $1 - 2\sin x = 0$ or, $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$ $\therefore$ $x = n\pi + (-1)^n\frac{\pi}{6}$

$\therefore$ $x = (2n+1)\frac{\pi}{2}$, $x = n\pi + (-1)^n\frac{\pi}{6}$, where n = 0, ±1, ±2, ........

For, n = 0, $x = \frac{\pi}{2}, \frac{\pi}{6}$ For n = 1, $x = \frac{3\pi}{2}, \pi - \frac{\pi}{6} = \frac{3\pi}{2}, \frac{5\pi}{6}$

For n = 2, $x = \frac{5\pi}{2}, 2\pi + \frac{\pi}{6}$

Since, $0 \le x \le 2\pi$, $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{6}$

$\therefore$ $\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{6}} = -2\sin\frac{\pi}{6} - 4\cos\frac{\pi}{3} = -2\times\frac{1}{2} - 4\times\frac{1}{2} = -3 < 0$

$\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{2}} = -2\sin\frac{\pi}{2} - 4\cos\pi = -2 + 4 = 2 > 0$

$\left(\frac{d^2y}{dx^2}\right)_{\frac{3\pi}{2}} = -2\sin\frac{3\pi}{2} - 4\cos 3\pi = 2 + 4 = 6 > 0$

$\left(\frac{d^2y}{dx^2}\right)_{\frac{5\pi}{6}} = -2\sin\frac{5\pi}{6} - 4\cos\frac{5\pi}{3} = -2\sin\left(\pi - \frac{\pi}{6}\right) - 4\cos\left(2\pi - \frac{\pi}{3}\right)$

$= -2\sin\frac{\pi}{6} - 4\cos\frac{\pi}{3} = -2\times\frac{1}{2} - 4\times\frac{1}{2} = -3 < 0$

Therefore, y attains minimum values at $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and maximum values at $x = \frac{\pi}{6}, \frac{5\pi}{6}$ **(Ans)**

**6.(i) Solution :** Let $f(x) = x^{\frac{1}{x}}$ $\therefore$ $\log f(x) = \frac{1}{x}\log x$

Differentiating both side with respect to x we get,

$$\frac{1}{f(x)}f'(x) = \frac{1}{x}\times\frac{1}{x} - \frac{1}{x^2}\times\log x \qquad \therefore\ f'(x) = f(x)\cdot\frac{1}{x^2}(1-\log x)$$

$$f''(x) = f'(x)\cdot\frac{1}{x^2}(1-\log x) + f(x)\left(-\frac{2}{x^3}\right)(1-\log x) + f(x)\cdot\frac{1}{x^2}\left(-\frac{1}{x}\right)$$

$f'(x) = 0$ gives, $1 - \log x = 0 \Rightarrow \log x = 1$ $\therefore$ $x = e^1 = e$

At $x = e$, $f''(x) = 0 + f(e)\left(-\frac{2}{e^3}\right)(1-\log e) + f(e)\cdot\frac{1}{e^2}\left(-\frac{1}{e}\right) = -\frac{1}{e^3}f(e) < 0$

$\therefore$ at $x = e$, $f(x)$ is maximum and the maximum value $= e^{\frac{1}{e}}$ **(Ans)**

**7.(i) Solution :** Let x and y be two numbers, such that $x + y = 12$ ..... (1)

Let $u = xy = x(12 - x) = 12x - x^2$

Differentiating two times successively with respect to x we get,

$$\frac{du}{dx} = 12 - 2x, \quad \frac{d^2u}{dx^2} = -2$$

For maximum or minimum values of u, $\frac{du}{dx} = 0$

$\therefore$ $12 - 2x = 0$ or, $2x = 12$ or, $x = 6$

And $\frac{d^2u}{dx^2} = -2 < 0$ $\therefore$ u is maximum when $x = 6$

From (1) $y = 12 - x = 6$

$\therefore$ the required maximum value of the product $= 6 \times 6 = 36$ **(Ans)**

**7.(v) Solution :** Let x and y be the two parts. Then $x + y = 20$ .............. (1)

We are to find the maximum value of $u = xy^3$ ................ (2)

From (1) and (2) we get, $u = y^3(20 - y)$ [eliminating x]

or, $u = 20y^3 - y^4$

or, $\frac{du}{dy} = 60y^2 - 4y^3$ and $\frac{d^2u}{dy^2} = 120y - 12y^2$

For maximum or minimum value of u, $\frac{du}{dy} = 0$ which gives,

$60y^2 - 4y^3 = 0$ or, $4y^2(15 - y) = 0$ or, $y = 0, 15$. But $y = 0$ is not possible.

Now, $\left[\frac{d^2u}{dy^2}\right]_{y=15} = 120\times15 - 12\times15^2 = -900 < 0$

Therefore, u is maximum when $y = 15$ and then $x = 20 - 15 = 5$

Hence the two parts are 5 and 15 **(Ans)**

**8.(ii) Solution :** Let $P(h, h^2 + 7h + 2)$ be any point on the parabola.

If p be the perpendicular distance of the straight line

$y = 3x - 3$, i.e., $y - 3x + 3 = 0$ from P, then

$$p = \frac{h^2 + 7h + 2 - 3h + 3}{\sqrt{1^2 + 3^2}} = \frac{h^2 + 4h + 5}{\sqrt{10}} = \frac{(h+2)^2 + 1}{\sqrt{10}} \quad \text{clearly, p is positive.}$$

$$\text{Now, } \frac{dp}{dh} = \frac{2(h+2)}{\sqrt{10}}, \quad \frac{d^2p}{dh^2} = \frac{2}{\sqrt{10}}$$

For maximum and minimum value of p, $\frac{dp}{dh} = 0$, which gives, $h = -2$

At $h = -2$, $\frac{d^2p}{dh^2} = \frac{2}{\sqrt{10}} > 0$. Hence, p is minimum at $h = -2$.

Therefore, the co-ordinates of the required point on the given parabola are

$[-2, (-2)^2 + 7(-2) + 2]$ i.e., $(-2, 4 - 14 + 2)$ i.e., $(-2, -8)$ **(Ans)**

**8.(iii) Solution :** The distance of the point $A(0, c)$ from any point $P(x, y)$ on the parabola $y = x^2$ is $AP = \sqrt{x^2 + (y-c)^2}$

$\therefore AP^2 = y + (y - c)^2 = y + y^2 - 2cy + c^2$

$$= y^2 - 2\left(c - \tfrac{1}{2}\right)y + c^2 = \left\{y - \left(c - \tfrac{1}{2}\right)\right\}^2 + c^2 - \left(c - \tfrac{1}{2}\right)^2 = \left\{y - \left(c - \tfrac{1}{2}\right)\right\}^2 + \left(c - \tfrac{1}{4}\right)$$

Thus the minimum value of AP is $\sqrt{c - \frac{1}{4}}$ when $y - \left(c - \frac{1}{2}\right) = 0$ i.e., $y = c - \frac{1}{2}$

For the given parabola, $y \geq 0 \quad \therefore c - \frac{1}{2} \geq 0$ or, $c \geq \frac{1}{2}$

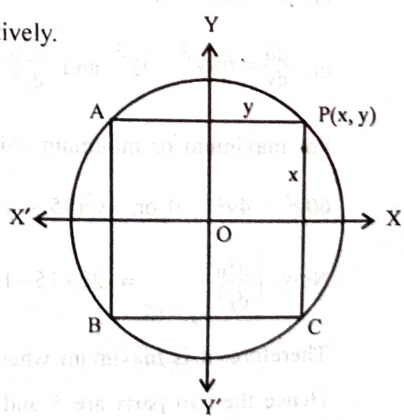
$\therefore$ shortest distance is $\sqrt{\frac{1}{2} - \frac{1}{4}} = \frac{1}{2}$ unit when $c = \frac{1}{2}$

and $\sqrt{c - \frac{1}{4}}$ for $\frac{1}{2} < c \leq 5$ **(Ans)**

**9.(i) Solution :** Let $x^2 + y^2 = a^2$ .................. (1) be the equation of the circle with centre at (0, 0) and radius a.

Let $P(x, y)$ be any point on the circle.

Then length and breadth of the rectangle PABC are 2y and 2x respectively.




Let A be the area of the rectangle, then $A = 2x \cdot 2y = 4xy$

or, $A^2 = 16x^2y^2 = 16x^2(a^2 - x^2)$ [from (1)] or, $A^2 = 16a^2x^2 - 16x^4$

$$\therefore \frac{dA^2}{dx} = 32a^2x - 64x^3 \text{ and } \frac{d^2A^2}{dx^2} = 32a^2 - 192x^2$$

$\frac{dA^2}{dx} = 0$ gives, $32a^2x - 64x^3 = 0$ or, $2x^2 = a^2$ [$\because x \neq 0$]

$\Rightarrow x = \frac{a}{\sqrt{2}}$ [$\because x > 0$]

$\therefore$ at $x = \frac{a}{\sqrt{2}}$,

$\frac{d^2A^2}{dx^2} = 32a^2 - 192 \times \frac{a^2}{2} = -64a^2 < 0$ Hence, A is maximum at $x = \frac{a}{\sqrt{2}}$

From (1), $y^2 = a^2 - x^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$ $\therefore$ $y = \frac{a}{\sqrt{2}}$, $[\because y > 0]$

Therefore, A is maximum when $x = \frac{a}{\sqrt{2}}$, $y = \frac{a}{\sqrt{2}}$ i.e., when x = y

or 2x = 2y i.e., when the rectangle is a square. Hence the result.

**9.(viii) Solution :** Let x and y be the length of two pieces.

Then $x + y = l$ ...................... (1)

Let a be the length of a side of the square and r be the radius of the circle. Then by the problem, 4a = x, or $a = \frac{x}{4}$ and $2\pi r = y$ or, $r = \frac{y}{2\pi}$

$\therefore$ area of the square = $a^2 = \frac{x^2}{16}$, area of the circle = $\pi r^2 = \pi \cdot \frac{y^2}{4\pi^2} = \frac{y^2}{4\pi}$

Let s be the sum of the areas of the square and circle.

Then, $s = \frac{x^2}{16} + \frac{y^2}{4\pi}$

$\frac{ds}{dx} = \frac{x}{8} + \frac{y}{2\pi} \cdot \frac{dy}{dx} = \frac{x}{8} - \frac{y}{2\pi}$ $\left[\because x + y = l \therefore 1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1\right]$

$\frac{d^2s}{dx^2} = \frac{1}{8} - \frac{1}{2\pi} \cdot \frac{dy}{dx} = \frac{1}{8} + \frac{1}{2\pi} = \frac{\pi + 4}{8\pi} > 0$

Now, for maximum or minimum value of s we have,

$\frac{ds}{dx} = 0$ which gives, $\frac{x}{8} - \frac{y}{2\pi} = 0$ or, $\frac{x}{8} = \frac{y}{2\pi}$ or, $x = \frac{4y}{\pi}$

From (1) we get, $x + y = l$ or, $\frac{4y}{\pi} + y = l$ *or,* $y\left(\frac{4+\pi}{\pi}\right) = l$

or, $y = \frac{l\pi}{4+\pi}$ $\therefore$ $x = \frac{4}{\pi} \times \frac{l\pi}{4+\pi} = \frac{4l}{4+\pi}$

Therefore, the required length of two pieces are $\frac{4l}{4+\pi}$ and $\frac{l\pi}{4+\pi}$ **(Ans)**

**10.(i) Solution :** Let r, h and l be respectively the radius of the base, height and the slant height of the conical tent.

Then volume of the conical tent, $v = \frac{1}{3}\pi r^2 h$, where v is constant. ....... (1)

and $l^2 = r^2 + h^2$ ................(2) and amount of the canvas $s = \pi r l$

or, $s^2 = \pi^2 r^2 l^2 = \pi^2 r^2 (r^2 + h^2) = \pi^2 (r^4 + r^2 h^2)$

Differentiating with respect to r we get

N.rute

$$\frac{ds^2}{dr} = \pi^2\left(4r^3 + 2rh^2 + 2r^2h\frac{dh}{dr}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From (1), $r^2h = \frac{3v}{\pi}$ = constant. $\therefore$ $2rh + r^2\frac{dh}{dr} = 0 \Rightarrow \frac{dh}{dr} = -\frac{2h}{r}$ ........... (4)

From (3) and (4) we get,

$$\frac{ds^2}{dr} = \pi^2\left[4r^3 + 2rh^2 + 2r^2h \times \left(-\frac{2h}{r}\right)\right] = \pi^2[4r^3 - 2rh^2]$$

Again differentiating with respect to r we get,

$$\frac{d^2s^2}{dr^2} = \pi^2\left(12r^2 - 2h^2 - 4rh\frac{dh}{dr}\right) = \pi^2\left[12r^2 - 2h^2 - 4rh \times \left(-\frac{2h}{r}\right)\right]$$

$$= \pi^2[12r^2 + 6h^2] = 6\pi^2[2r^2 + h^2]$$

For maximum and minimum value of s we have,

$\frac{ds^2}{dr} = 0$ which gives $4r^3 - 2rh^2 = 0$ or, $2r^2 - h^2 = 0$

or, $h^2 = 2r^2$ $\therefore$ $h = \sqrt{2}\,r$

and when $h = \sqrt{2}\,r\ [\because r \neq 0]$ $\frac{d^2s^2}{dr^2} = 6\pi^2(2r^2 + 2r^2) = 24\pi^2r^2 > 0$.

Hence, for least amount of canvas, $h = \sqrt{2}\,r$

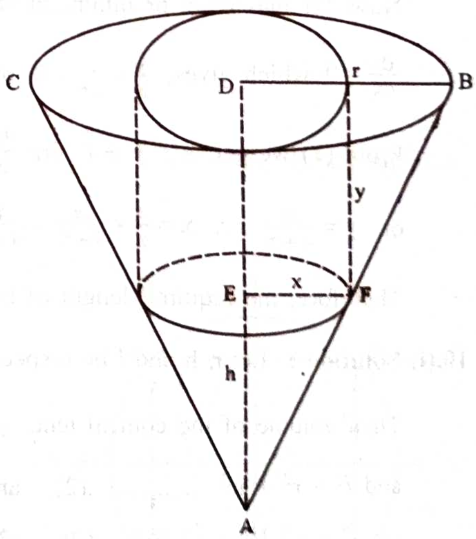
i.e., height is $\sqrt{2}$ times the radius of the base. **(Proved)**

**10.(ii) Solution :** Let r, h be respectively the radius of the base and height of the right circular cone and x, y are the radius of the base and height of the right circular cylinder which is inscribed in the given cone.

$\therefore$ BD = r, EF = x,

AD = h, DE = y.

Then from similar triangles ABD and AFE we get,




$$\frac{BD}{FE} = \frac{AD}{AE} = \frac{AD}{AD - DE}$$

or, $\frac{r}{x} = \frac{h}{h-y}$ or, $h - y = \frac{hx}{r}$

or, $y = h\left(1 - \frac{x}{r}\right) = \frac{h}{r}(r - x)$ ................. (1)

Let s be the area of the curved surface of the cylinder, then

$$s = 2\pi xy = 2\pi x \times \frac{h}{r}(r - x) = \frac{2\pi h}{r}\left(rx - x^2\right)$$

$$\therefore \frac{ds}{dx} = \frac{2\pi h}{r}(r - 2x) \text{ and } \frac{d^2s}{dx^2} = -\frac{4\pi h}{r}$$

Now, for maximum or minimum value of s we have, $\frac{ds}{dx} = 0$,

which gives, $\frac{2\pi h}{r}(r-2x)=0$ or, $r = 2x$ or, $x=\frac{r}{2}$

and $\left(\frac{d^2s}{dx^2}\right)_{x=\frac{r}{2}} = -\frac{4\pi h}{r} < 0$

$\therefore$ the curved surface of the cylinder s is maximum when $x=\frac{r}{2}$

i.e., when the radius of the right circular cylinder is half that of the cone. **(Proved)**

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# MISCELLANEOUS

## SUBJECTIVE TYPE

1. A particle is moving in a straight line such that its distance x from a fixed point O at any time t is given by the relation $x = t^4 - 10t^3 + 24t^2 + 36t + 12$. When is it moving slowly ? [WBSC – 06]

2. A particle is moving in a straight line such that its distance at any time t is given by $s = \frac{1}{4}t^4 - 2t^3 + 4t^2 - 7$. Find when its velocity is maximum and acceleration is minimum. [WBSC – 05, 08]

3. Divide 20 into two parts such that the product of one part and the cube of the other is maximum. [WBSC – 05, 18]

4. Find the minimum distance from the point (4, 2) to the parabola $y^2 = 8x$ [WBSC – 09]

5. Show that the maximum value of the function $x + \frac{1}{x}$ is less than its minimum value. [WBSC – 14]

6. Find the maximum and minimum value of $x^3 + \frac{1}{x^3}$ [WBSC – 15]

7. Show that the maximum value of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$ occurs at $x = 1$. Find the maximum value. [WBSC – 12]

8. If bending moment M of a beam supported at one end at a distance x from one end is given by $M = \frac{1}{2}wlx - \frac{1}{2}wx^2$, where ' l ' is the length of the beam and w is the uniform load per unit length (l, w are being constant). Find the point of maximum moment of the beam. [WBSC – 09]

9. Given that $W = 12i - 16i^3$, when W is the power in watt and ' i ' is the current in ampere. For what value of ' i ' the power will be maximum ? Find maximum power. [WBSC – 11]

10. Show that sinx + cosx will be maximum at $x = \frac{\pi}{4}$ [WBSC – 16]

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December, 2014

**MATHEMATICS**

Time Allowed: 3 Hours

Full Marks : 70

Answer to Question No.1 is compulsory and is to be answered first.

This answer is to be made in separate loose script(s) provided for the purpose.

Maximum time allowed is 45 minutes, after which the loose answer scripts will be collected and fresh scripts for answering the remaining part of the question will be provided.

On early submission of answer scripts of Question No.1, a student will get the remaining script earlier.

Answer any five Questions taking at least one from Group – A, B & C.

**1.** Answer **any twenty** questions with minimum justification: $20 \times 1 = 20$

(i) The equation $\log_e x + \log_e(1 + x) = 0$ may be expressed as –

(a) $x^2 + x + 1 = 0$ (b) $x^2 + x - 1 = 0$ (c) $x^2 + x + e = 0$ (d) $x^2 + x - e = 0$

**Solution :** Given equation, $\log_e x + \log_e(1 + x) = 0$

or, $\log_e x(1 + x) = 0$ or, $x(1 + x) = e^0$ or, $x^2 + x - 1 = 0$

**Ans.** (b)

(ii) If x, y are real and $x + 3i$ and $-2 + iy$ are conjugate to each other then x, y are –

(a) 2, –3 (b) 2, 3 (c) –2, 3 (d) –2, –3

**Solution :** By the problem, $x + 3i$ and $-2 + iy$ are conjugate to each other.

Therefore, $x - 3i = -2 + iy$ which gives, $x = -2$, $y = -3$

**Ans.** (d)

(iii) If the sum of the roots of $ax^2 + 2x + 3a = 0$ is equal to their product, value of a is –

(a) $-\frac{2}{3}$ (b) $-\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{2}$.

**Solution :** Given equation is $ax^2 + 2x + 3a = 0$

Sum of the roots $= -\frac{2}{a}$ and product of the roots $= \frac{3a}{a} = 3$

By the problem, $-\frac{2}{a} = 3$ or, $a = -\frac{2}{3}$

**Ans.** (a)

(iv) The term independant of x in the expansion of $\left(x - \frac{1}{x}\right)^{10}$ is (a) 4th (b) 5th (c) 6th (d) none of these.

**Solution :** Let, (r + 1)th term is independant of x in the expansion of $\left(x - \frac{1}{x}\right)^{10}$

Now, $t_{r+1} = {}^{10}C_r\, x^{10-r}\left(-\frac{1}{x}\right)^r = (-1)^r.{}^{10}C_r\, x^{10-r}.\frac{1}{x^r} = (-1)^r.{}^{10}C_r\, x^{10-2r}$

By the problem, $10 - 2r = 0$ or, $r = 5$

Therefore, (5 + 1)th = 6th term is the independant term.

**Ans.** (c)

(v) If $\log_x \log_2 \log_3 81 = 1$ then x = (a) 2 (b) 3 (c) 1 (d) none of these.

**Solution :** Given, $\log_x \log_2 \log_3 81 = 1$ or, $x = \log_2 \log_3 81$

or, $x = \log_2 \log_3 3^4 = \log_2(4\log_3 3) = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$

**Ans.** (a)

(vi) The values of p and q (which are both real) where $2+i\sqrt{3}$ is a root of $x^2 + px + q = 0$ are –

(a) p = 4, q = –7 (b) p = –4, q = 7 (c) p = –4, q = –7 (d) p = 4, q = 7

**Solution :** By the problem, the two roots are $2+i\sqrt{3}$ and $2-i\sqrt{3}$

$\therefore 2+i\sqrt{3}+2-i\sqrt{3} = -p$ or, p = –4 and $(2+i\sqrt{3})(2-i\sqrt{3}) = q$ or, q = 4 + 3 = 7

**Ans.** (b)

(vii) The middle term in the expansion of $\left(ax+\frac{1}{ax}\right)^6$ is – (a) 20 (b) –20 (c) 15a (d) $15a^2$

**Solution :** The middle term in the expansion of $\left(ax+\frac{1}{ax}\right)^6$ is

$$t_{\frac{6}{2}+1} = t_{3+1} = {}^6C_3(ax)^{6-3}\left(\frac{1}{ax}\right)^3 = \frac{6!}{3!3!}(ax)^3\frac{1}{(ax)^3} == \frac{6\cdot5\cdot4}{3\cdot2} = 20$$

**Ans.** (a)

(viii) Position vectors of two points A and B are respectively A(2, 3, 4) and B(2, 5, 6). The position vector of the middle point joining AB is – (a) 4i + 8j + 10k (b) –2j – 2k (c) 2ι + 4j + 5k (d) none of these.

**Solution :** Position vectors of the given points A and B are respectively

2i + 3j + 4k and 2i + 5j + 6k

Therefore, the position vector of the middle point joining AB is

$$\frac{(2i+3j+4k)+(2i+5j+6k)}{2} = \frac{4i+8j+10k}{2} = 2\imath + 4j + 5k$$

**Ans.** (c)

(ix) If the vectors $\bar{a}$ and $\vec{b}$ are collinear where $\bar{a} = 2i - 3j + k$ and $\vec{b} = 4i + mj + 2k$ then m is –

(a) 0 (b) 3 (c) –6 (d) none of these.

**Solution :** By the problem, $\vec{b} = x\bar{a}$ or, 4i + mj + 2k = x(2i – 3j + k)

Equating the coefficients of i, j, k from both sides we get,

2x = 4, m = –3x, x = 2 or, x = 2, m = –6

**Ans.** (c)

(x) If x = cosθ + i sinθ, then $x^m + \frac{1}{x^m}$ is – (a) 2 cosmθ (b) 2 sinmθ (c) 2i cosmθ (d) 2i sinmθ

**Solution :** Given, x = cosθ + i sinθ

Then $x^m + \frac{1}{x^m} = (\cos\theta + i\sin\theta)^m + (\cos\theta + i\sin\theta)^{-m} = \cos m\theta + i\sin m\theta + \cos m\theta - i\sin m\theta = 2\cos m\theta$

**Ans.** (a)

(xi) The unit vector in the direction of the vector $\bar{a} = 2i - 2j + k$ is –

(a) $i + j + k$ (b) $\pm\frac{1}{3}(2i - 2j + k)$ (c) $\frac{1}{3}(2i - 2j + k)$ (d) none of these.

**Solution :** Required unit vector $= \pm\frac{2i-2j+k}{\sqrt{2^2+(-2)^2+1^2}} = \pm\frac{2i-2j+k}{\sqrt{4+4+1}} = \pm\frac{1}{3}(2i - 2j + k)$

**Ans.** (b)

(xii) The vectors $\lambda\vec{i}+\lambda\vec{j}+3\vec{k}$ and $3\vec{i}+3\vec{j}-2\lambda\vec{k}$ are perpendicular to each other if $\lambda$ is equal to

(a) 0 (b) 3 (c) 1 (d) all real values.

**Solution :** Since, $\lambda\vec{i}+\lambda\vec{j}+3\vec{k}$ and $3\vec{i}+3\vec{j}-2\lambda\vec{k}$ are perpendicular to each other therefore, $(\lambda\vec{i}+\lambda\vec{j}+3\vec{k})\cdot(3\vec{i}+3\vec{j}-2\lambda\vec{k})=0$ or, $3\lambda + 3\lambda - 6\lambda = 0$ and hence true for all real values of $\lambda$.

**Ans.** (d)

(xiii) $\tan\theta = -\frac{1}{\sqrt{5}}$, $\sin\theta$ is negative then the value of $\cos\theta$ is – (a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{2}$ (c) $\sqrt{\frac{5}{6}}$ (d) $-\sqrt{\frac{5}{6}}$.

**Solution :** Given, $\tan\theta = -\frac{1}{\sqrt{5}}$ $\therefore \sec\theta = \pm\sqrt{1+\tan^2\theta} = \pm\sqrt{1+\frac{1}{5}} = \pm\sqrt{\frac{6}{5}}$ $\therefore \cos\theta = \pm\sqrt{\frac{5}{6}}$

Since, $\sin\theta$ is negative $\therefore \cos\theta = \sqrt{\frac{5}{6}}$

**Ans.** (c)

(xiv) The latus rectum of the parabola $y^2 = -48x$ is – (a) 48; (b) –12; (c) –48; (d) 12.

**Solution :** Given parabola $y^2 = -48x$ or, $y^2 = -4.12.x$ Here, $a = 12$

Therefore, length of latus rectum $= 4a = 48$

**Ans.** (a)

(xv) Find the coordinates of the centroid of the triangle whose vertices are (3, –5), (–7, 4) and (10, –2).

**Solution :** Coordinates of the centroid of the triangle whose vertices are (3, –5), (–7, 4) and (10, –2) are

$\left(\frac{3-7+10}{3}, \frac{-5+4-2}{3}\right) = (2, -1)$ **(Ans)**

(xvi) The least value of $\sin^2\theta + \mathrm{cosec}^2\theta$ is – (a) 1 (b) 2 (c) 0 (d) 2.5

**Solution :** Given, $\sin^2\theta + \mathrm{cosec}^2\theta = (\sin\theta - \mathrm{cosec}\theta)^2 + 2\sin\theta\,\mathrm{cosec}\theta = (\sin\theta - \mathrm{cosec}\theta)^2 + 2$

Since $(\sin\theta - \mathrm{cosec}\theta)^2$ is a perfect square, its least value = 0

Hence, least value of the given expression = 2

**Ans.** (b)

(xvii) The chord of a circle measures 24 cm and its distance from the centre is 4 cm. Then the radius of the circle is – (a) 16 cm (b) 12.65 cm (c) 14.33 cm (d) none of these.

**Solution :** Given, chord = 24 cm and distance from the centre = 4 cm

Radius of the circle is $= \sqrt{12^2+4^2} = \sqrt{144+16} = \sqrt{160} = 12.65$ cm (apprx)

**Ans.** (b)

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(xviii) If $g(x)=\frac{1-x}{1+x}$, then find $g\left(\frac{1}{x}\right)+g(x)$ – (a) –1 (b) 0 (c) 1 (d) $\frac{1}{2}$

**Solution :** Given, $g(x)=\frac{1-x}{1+x}$ $\therefore g\left(\frac{1}{x}\right)=\frac{1-\frac{1}{x}}{1+\frac{1}{x}}=\frac{x-1}{x+1}=-\frac{1-x}{1+x}=-g(x)$ $\therefore g\left(\frac{1}{x}\right)+g(x)=0$

**Ans.** (b)

(xix) The area of the lateral surface of a right prism with regular octagonal base of side 2 cm and height 6 cm is – (a) $12\sqrt{3}$ cm$^2$ (b) $36\sqrt{3}$ cm$^2$ (c) 96 cm$^2$ (d) 72 cm$^2$

**Solution :** Perimeter of the base = $8\times 2=16$ cm.

Hence the required lateral surface = perimeter of the base × height = 16 × 6 = 96 cm$^2$

**Ans.** (c)

(xx) The function y = f(x) is increasing in the interval $a\le x\le b$: (a) $\frac{dy}{dx}=0$ (b) $\frac{dy}{dx}<0$ (c) $\frac{dy}{dx}>0$ (d) none.

**Solution :** The function y = f(x) is increasing in the interval $a\le x\le b$ if $\frac{dy}{dx}>0$

**Ans.** (c)

(xxi) $\underset{x\to 0}{\text{Lt}}\frac{\tan 3x}{\tan x}$ is (a) $\frac{1}{3}$ (b) 1 (c) 3 (d) 0

**Solution :** Given, $\underset{x\to 0}{\text{Lt}}\frac{\tan 3x}{\tan x}=\underset{3x\to 0}{\text{Lt}}\frac{\tan 3x}{3x}\cdot\frac{3}{\underset{x\to 0}{\text{Lt}}\frac{\tan x}{x}}=1\cdot\frac{3}{1}=3$

**Ans.** (c)

(xxii) If f(x) = logsinax, f′(x) is (a) $\frac{a}{\sin ax}$ (b) –a cotax (c) a cotax (d) a tanax

**Solution :** Given, f(x) = logsinax or, $f'(x)=\frac{1}{\sin ax}\cdot\frac{d}{dx}(\sin ax)=\frac{a\cos ax}{\sin ax}=a\cot ax$

**Ans.** (c)

(xxiii) A ball travels s cm in t sec so that s = 8t + 10t$^2$, find the velocity of the ball when t = 2 sec.

**Solution :** Given, s = 8t + 10t$^2$ or, $\frac{ds}{dt}=8+20t$

Therefore, velocity of the ball when t = 2 sec is $8+20\times 2=48$ cm/sec. **(Ans)**

(xxiv) Find the domain of the function f(x) where $f(x)=\frac{1}{\sqrt{3-x}}$

**Solution :** Given $f(x)=\frac{1}{\sqrt{3-x}}$. For real and finite value of f(x), 3 – x > 0 or, x < 3

Hence the required domain of the function is $-\infty<x<3$ **(Ans)**

(xxv) A sphere and a right circular cylinder have same radius and equal volume. Ratio between radius of the sphere and height of the cylinder is – (a) 3 : 4 (b) 4 : 3 (c) 3 : 8 (d) 1 : 2

**Solution :** Let the same radius be r.

Then by the problem, $\frac{4}{3}\pi r^3=\pi r^2 h$, h is the height of the cylinder. or, 4r = 3h or, r : h = 3 : 4

**Ans.** (a)

## Group – A

**2.** (a) If $\sqrt[3]{x+iy} = a+ib$ show that $\frac{x}{a}+\frac{y}{b} = 4(a^2 - b^2)$

**Solution :** $\sqrt[3]{x+iy} = a+ib$ or $x + iy = (a + ib)^3$ [cubing both sides]

or, $x + iy = a^3 + 3a^2ib + 3ai^2b^2 + i^3b^3 = a^3 + 3a^2bi - 3ab^2 - ib^3 = (a^3 - 3ab^2) + i(3a^2b - b^3)$

Equating real and imarginary parts from both sides we get,

$x = a^3 - 3ab^2$, $y = 3a^2b - b^3$

$$\therefore \frac{x}{a}+\frac{y}{b} = \frac{a^3-3ab^2}{a}+\frac{3a^2b-b^3}{b} = a^2 - 3ab + 3ab - b^2 = 4(a^2 - b^2)$$ **(Proved)**

(b) If one root of the equation $x^2 + px + q = 0$ is the square of the other, prove that $p^3 - q(3p - 1) + q^2 = 0$

**Solution:** Let $\alpha$, $\alpha^2$ be the roots of the equation $x^2 + px + q = 0$

$\therefore \alpha+\alpha^2 = -p$ -------- (1), and $\alpha\cdot\alpha^2 = q$ or, $\alpha^3 = q$ -------- (2)

Cubing both sides of (1) we get,

$$\left(\alpha+\alpha^2\right)^3 = (-p)^3 \text{ or, } \alpha^3+\left(\alpha^2\right)^3+3\alpha\cdot\alpha^2\left(\alpha+\alpha^2\right) = -p^3 \text{ or, } \alpha^3+\left(\alpha^3\right)^2+3\alpha^3\left(\alpha+\alpha^2\right) = -p^3$$

or, $q + q^2 + 3q(-p) = -p^3$ or, $p^3 - 3pq + q + q^2 = 0$ or, $p^3 - q(3p - 1) + q^2 = 0$ **(Proved)**

**3.** (a) Find the scalar area of the triangle the position vectors of whose vertices are $i + j + k$, $i + 2j + 3k$, $2i + 3j + k$.

**Solution :** With respect to O as origin let the position vectors of the vertices A, B, C of triangle ABC are respectively $i + j + k$, $i + 2j + 3k$, $2i + 3j + k$.

Then $\overrightarrow{OA} = i + j + k$, $\overrightarrow{OB} = i + 2j + 3k$, $\overrightarrow{OC} = 2i + 3j + k$

Therefore, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (i + 2j + 3k) - (i + j + k) = j + 2k$

Therefore, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (2i + 3j + k) - (i + j + k) = i + 2j$

Therefore, the vector area of the triangle ABC

$$\frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{AB}\right) = \frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \frac{1}{2}[i(4-0)-j(2-0)+k(1-0)] = \frac{1}{2}(4i-2j+k)$$

Required area $= \frac{1}{2}|4i-2j+k| = \frac{1}{2}\sqrt{4^2+(-2)^2+1^2} = \frac{1}{2}\sqrt{21}$ sq. unit. **(Ans)**

(b) Split $\frac{s^2+2}{(s+1)(s^2+1)}$ into partial fraction.

**Solution :** Given, $\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{(s+1)^2}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}$ (say)

or, $A(s^2 + 1) + B(s + 1)(s^2 + 1) + (Cs + D)(s + 1)^2 = s$ ......................... (1)

Putting $s = -1$ we get, $2A = -1$ or, $A = -\frac{1}{2}$ ......................... (2)

Equating the coefficient of $s^3$ from both sides we get, $B + C = 0$ ......................... (3)

Equating constant terms from both sides we get, $A + B + D = 0$ or, $B + D = -A$ or, $B + D = \frac{1}{2}$ ................. (4)

Equating the coefficient of $s^2$ from both sides we get, $A + B + 2C + D = 0$ or, $(B + D) + A + 2C = 0$

or, $\frac{1}{2} - \frac{1}{2} + 2C = 0$ or, $C = 0$ [from (2) and (4)]

From (3) we get, $B + 0 = 0$ or, $B = 0$ and from (4), we get, $D = \frac{1}{2}$

Therefore, $\frac{s}{(s+1)^2(s^2+1)} = -\frac{1}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$ **(Ans)**

**4.** (a) If $x = \log_a bc, y = \log_b ca$, $z = \log_c ab$ show that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$

**Solution :** Given, $x = \log_a(bc)$, or, $x + 1 = \log_a(bc) + \log_a a = \log_a(abc)$

$\therefore \frac{1}{x+1} = \log_{abc} a$ .

Similarly, $\frac{1}{y+1} = \log_{abc} b$, $\frac{1}{z+1} = \log_{abc} c$

$\therefore \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} abc = 1$

$\therefore \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$ **(Proved)**

(b) If the 3rd and the 4th terms in the expansion of $\left(2x + \frac{1}{8}\right)^{10}$ are equal, find x.

**Solution :** Given, $\left(2x + \frac{1}{8}\right)^{10}$

Third term, $t_3 = {}^{10}C_2(2x)^{10-2}\left(\frac{1}{8}\right)^2$ and 4th term, $t_4 = {}^{10}C_3(2x)^{10-3}\left(\frac{1}{8}\right)^3$

By the problem, ${}^{10}C_2(2x)^{10-2}\left(\frac{1}{8}\right)^2 = {}^{10}C_3(2x)^{10-3}\left(\frac{1}{8}\right)^3$

or, $\frac{10!}{2!\,8!}\cdot(2x)^8 = \frac{10!}{3!\,7!}\cdot(2x)^7\cdot\frac{1}{8}$ or, $\frac{1}{8}\cdot 2x = \frac{1}{3}\cdot\frac{1}{8}$ or, $x = \frac{1}{6}$ .**(Ans)**

**5.** (a) If **α**, **β** and **γ** are three unit vectors satisfying the condition $\alpha + \beta + \gamma = 0$ show that $\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha = -\frac{3}{2}$

**Solution :** By the problem, $|\alpha| = 1$, $|\beta| = 1$, $|\gamma| = 1$.

Now $\alpha + \beta + \gamma = 0$ gives $|\alpha + \beta + \gamma|^2 = 0$. or $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + 2(\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha) = 0$

or $1 + 1 + 1 + 2(\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha) = 0$

Therefore, $\alpha\cdot\beta + \beta\cdot\gamma + \gamma\cdot\alpha = -\frac{3}{2}$ **(Proved)**

(b) If $\bar{a} = 3i + j + 2k$, $\vec{b} = 2i - 2j + 4k$; find the unit vector perpendicular to the plane containing $\bar{a}$ and $\vec{b}$

**Solution :** Vector perpendicular to both the vectors $\bar{a} = 3i + j + 2k$ and $\vec{b} = 2i - 2j + 4k$ is

$\bar{a}\times\vec{b} = (3i + j + 2k)\times(2i - 2j + 4k)$

$$= \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = (4+4)i-(12-4)j+(-6-2)k = 8i-8j-8k$$

Therefore the required unit vector

$$\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \pm\frac{8(i-j-k)}{|8i-8j-8k|} = \pm\frac{8(i-j-k)}{8\sqrt{1^2+(-1)^2+(-1)^2}} = \pm\frac{1}{\sqrt{3}}(i-j-k) \textbf{ (Ans)}$$

**Group – B**

**6.** (a) A right prism stands on a base which is a regular hexagon of side 15 cm. If the area of its lateral surface be 5400 sq. cm, find its volume. Find also the area of its whole surface.

**Solution:** Area of the base of the prism $= \frac{na^2}{4}.\cot\frac{\pi}{n}$, $n = 6$, $a = 15$ here.

$$= \frac{6\times 15^2}{4}.\cot\frac{\pi}{6} \text{ sq.cm.} = \frac{3\times 225}{2}\times\sqrt{3} \text{ sq.cm.}$$

Perimeter of the base = 6 × 15 cm.

Let h be the height of the prism. $\therefore$ $6\times 15\times h = 5400 \Rightarrow h = 60$ cm.

$\therefore$ height of the prism = 60 cm.

Volume of the prism = Area of the base × height $= \frac{3\times 225}{2}\times\sqrt{3}\times 60$ cu.cm. $= 20250\sqrt{3}$ cu.cm. **(Proved)**

and area of the whole surface $= 4\times\frac{\sqrt{3}\times 12^2}{4} = 144\sqrt{3}$ sq. cm. **(Ans)**

(b) If $\tan\alpha : \tan\beta = 1 : 4$, then prove that $\tan(\beta-\alpha) = \frac{3\sin 2\alpha}{5-3\cos 2\alpha}$

**Solution :** Given, $\tan\alpha : \tan\beta = 1 : 4$ or, $\frac{\tan\alpha}{\tan\beta} = \frac{1}{4}$ or, $\tan\beta = 4\tan\alpha$

$$\text{L. H. S} = \tan(\beta-\alpha) = \frac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha} = \frac{4\tan\alpha-\tan\alpha}{1+4\tan^2\alpha} = \frac{3\tan\alpha}{1+4\tan^2\alpha} = \frac{\frac{3\sin\alpha}{\cos\alpha}}{1+\frac{4\sin^2\alpha}{\cos^2\alpha}} = \frac{3\sin\alpha\cos\alpha}{\cos^2\alpha+4\sin^2\alpha}$$

$$= \frac{3(2\sin\alpha\cos\alpha)}{2\cos^2\alpha+4(2\sin^2\alpha)} = \frac{3\sin 2\alpha}{1+\cos 2\alpha+4(1-\cos 2\alpha)} = \frac{3\sin 2\alpha}{1+\cos 2\alpha+4-4\cos 2\alpha} = \frac{3\sin 2\alpha}{5-3\cos 2\alpha} \textbf{ (Ans)}$$

**7.** (a) A circle touches both the axes of co-ordinates and both the co-ordinates of its centre are positive. If the circle also touches the straight line $5x - 12y + 10 = 0$, then find its equation.

**Solution :** By the problem, let centre of the circle is (a, a) and radius = a

Now the perpendicular distance of the straight line $5x - 12y + 10 = 0$ from (a, a)

$$= \frac{5a-12a+10}{\sqrt{5^2+(-12)^2}} = \frac{10-7a}{\sqrt{25+144}} = \frac{10-7a}{13} = a, \text{ the radius of the circle}$$

or, $13a = 10 - 7a$ or, $20a = 10$ or, $a = \frac{1}{2}$.

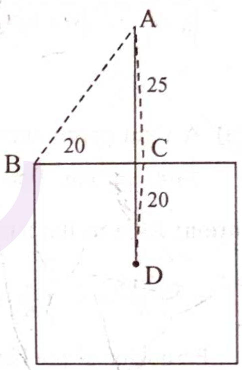
Therefore, the required equation of the circle is $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

or, $x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{4}$ or, $x^2 + y^2 - x - y + \frac{1}{4} = 0$ or, $4x^2 + 4y^2 - 4x - 4y + 1 = 0$ **(Ans)**

(b) The base of a right pyramid is a square of side 40 cm and its slant height is 25 cm. Find its height. If the volume of a cube is equal to the volume of a pyramid, find the length of the side of the cube.

**Solution :** Area of the square base of the pyramid = $(40)^2$ sq. cm. = 1600 sq. cm.

Slant height (CA) = 25 cm.




$\therefore$ height of the pyramid (AD) $= \sqrt{25^2 - 20^2} = \sqrt{625-400} = \sqrt{225} = 15$ cm.

$\therefore$ volume of the pyramid $= \frac{1}{3} \times$ area of the base $\times$ height

$= \frac{1}{3} \times 1600 \times 15$ cu.cm. = 8000 cu.cm.

Let a be the side of the cube. $\therefore a^3 = 8000 = (20)^3 \therefore a = 20$

Therefore, length of the side of the cube = 20 cm. **(Ans)**

**8.** (a) If $\cos^{-1}x + \cos^{-1}y = \theta$, prove that $x^2 - 2xy\cos\theta + y^2 = \sin^2\theta$. 5

**Solution :** Given, $\cos^{-1}x + \cos^{-1}y = \theta$

or, $\cos^{-1}\left\{xy - \sqrt{(1-x^2)(1-y^2)}\right\} = \theta$ or, $xy - \sqrt{(1-x^2)(1-y^2)} = \cos\theta$

or, $xy - \cos\theta = \sqrt{(1-x^2)(1-y^2)}$ or, $(xy - \cos\theta)^2 = (1 - x^2)(1 - y^2)$

or, $x^2y^2 - 2xy\cos\theta + \cos^2\theta = 1 - x^2 - y^2 + x^2y^2$ or, $x^2 - 2xy\cos\theta + y^2 = \sin^2\theta$ **(Proved)**

(b) Find the foot of the perpendicular from the point (–2, 6) on the straight line $2x + 3y = 1$.

**Solution :** Equation of any straight line perpendicular to the straight line

$2x + 3y = 1$ is $3x - 2y = k$ ................ (1)

If it passes through the point (–2, 6) then, $-6 - 12 = k$ or, $k = -18$

From (1) we get, $3x - 2y = -18$ or, $3x - 2y + 18 = 0$

Now solving, $2x + 3y - 1 = 0$ and $3x - 2y + 18 = 0$ we get,

$\frac{x}{54-2} = \frac{y}{-3-36} = \frac{1}{-4-9}$ or, $\frac{x}{52} = \frac{y}{-39} = \frac{1}{-13}$

$\therefore x = -\frac{52}{13} = -4,\ y = \frac{-39}{-13} = 3$

Therefore, (–4, 3) is the required foot of the perpendicular. **(Ans)**

## Group – C

**9.** (a) Find : (i) $\underset{x\to 0}{\text{Lt}} \frac{\sqrt{1+x}-1}{\log_e(1+x)}$ (ii) $\underset{x\to 0}{\text{Lt}} \frac{e^{2x}-1}{\sin 3x}$

**Solution : (i)** $\underset{x\to 0}{\text{Lt}} \frac{\sqrt{1+x}-1}{\log_e(1+x)} = \underset{x\to 0}{\text{Lt}} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)\log_e(1+x)} = \underset{x\to 0}{\text{Lt}} \frac{1+x-1}{(\sqrt{1+x}+1)\log_e(1+x)} = \underset{x\to 0}{\text{Lt}} \frac{x}{(\sqrt{1+x}+1)\log_e(1+x)}$

$= \underset{x\to 0}{\text{Lt}} \frac{1}{\sqrt{1+x}+1} \times \frac{1}{\underset{x\to 0}{\text{Lt}} \frac{1}{x}\log_e(1+x)} = \frac{1}{1+1} \times \frac{1}{1} = \frac{1}{2}$ **(Ans)**

**Solution : (ii)** $\underset{x\to 0}{\text{Lt}}\dfrac{e^{2x}-1}{\sin 3x}$

$$= \underset{2x\to 0}{\text{Lt}}\frac{e^{2x}-1}{2x}\times\frac{1}{\underset{3x\to 0}{\text{Lt}}\frac{\sin 3x}{3x}}\times\frac{2}{3}\left[\because x\to 0 \therefore 2x\to 0, 3x\to 0\right] = 1\times\frac{1}{1}\times\frac{2}{3}=\frac{2}{3}\ \textbf{(Ans)}$$

(b) If $x^m y^n = (x + y)^{m+n}$, prove that $\dfrac{dy}{dx}=\dfrac{y}{x}$

**Solution :** $x^m y^n = (x + y)^{m+n}$

$\log(x^m y^n) = \log(x + y)^{m+n}$ [taking logarithm of both sides]

or, $m\log x + n\log y = (m + n)\log(x + y)$

Differentiating both sides with respect to x we get,

$$m\frac{d}{dx}(\log x)+n\frac{d}{dx}(\log y)=(m+n)\frac{d}{dx}\{\log(x+y)\}\ \text{or,}\ m\times\frac{1}{x}+n\times\frac{1}{y}\frac{dy}{dx}=(m+n)\times\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\text{or,}\ \frac{m}{x}+\frac{n}{y}\times\frac{dy}{dx}=\frac{m+n}{x+y}+\frac{m+n}{x+y}\times\frac{dy}{dx}\ \text{ or,}\ \left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx}=\frac{m+n}{x+y}\to\frac{m}{x}$$

$$\text{or,}\ \frac{nx+ny-my-ny}{y(x+y)}\times\frac{dy}{dx}=\frac{mx+nx-mx-my}{x(x+y)}$$

$$\text{or,}\ \frac{nx-my}{y(x+y)}\times\frac{dy}{dx}=\frac{nx-my}{x(x+y)}\ \text{ or,}\ \frac{1}{y}\times\frac{dy}{dx}=\frac{1}{x}\ \text{ or,}\ \frac{dy}{dx}=\frac{y}{x}\ \textbf{(Proved)}$$

**10.** (a) If $y = \sin(m\sin^{-1}x)$, then prove that $(1 - x^2)y_2 - xy_1 + m^2y = 0$

**Solution :** $y = \sin(m\sin^{-1}x)$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx}=\cos\left(m\sin^{-1}x\right)\times\frac{d}{dx}\left(m\sin^{-1}x\right)=\cos\left(m\sin^{-1}x\right)\times\frac{m}{\sqrt{1-x^2}}$$

$$\text{or,}\ \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=m^2\cos^2\left(m\sin^{-1}x\right)=m^2\left\{1-\sin^2\left(m\sin^{-1}x\right)\right\}=m^2-m^2y^2$$

Differentiating again both sides with respect to x we get,

$$\left(1-x^2\right)\times 2\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+(-2x\times\left(\frac{dy}{dx}\right)^2=0-2m^2y\frac{dy}{dx}$$

$$\text{or,}\ \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-m^2y\ \text{ or,}\ (1-x^2)y_2-xy_1+m^2y=0\ \textbf{(Proved)}$$

(b) Show that the maximum value of the function $x+\dfrac{1}{x}$ is less than its minimum value. 5

**Solution :** Let $f(x)=x+\dfrac{1}{x}$ $\quad\therefore f'(x)=1-\dfrac{1}{x^2},\ f''(x)=\dfrac{2}{x^3}$

$f'(x) = 0$ gives, $1-\dfrac{1}{x^2}=0$ or, $x^2 = 1 \Rightarrow x = \pm 1$

At $x = 1$, $f''(x) = 2 > 0$; At $x = -1$, $f''(x) = -2 < 0$

$\therefore$ at $x = 1$, $f(x)$ is minimum and at $x = -1$, $f(x)$ if maximum.

The maximum value $= -1+\frac{1}{-1}=-2$ and the minimum value $= 1+\frac{1}{1}=2$.

Clearly, maximum value < minimum value. Hence, the result.

**11.** (a) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$

**Solution :** Given, $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

Differentiating both sides with respect to $\theta$ we get,

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta+\sin\theta) = a(1+\cos\theta) \text{ and } \frac{dy}{d\theta} = a\frac{d}{d\theta}(1-\cos\theta) = a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) = \sec^2\frac{\theta}{2}\cdot\frac{1}{2}\frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2}\frac{1}{a(1+\cos\theta)} = \frac{1}{2a}\sec^2\frac{\theta}{2}\frac{1}{2\cos^2\frac{\theta}{2}} = \frac{1}{4a}\sec^4\frac{\theta}{2} \quad \textbf{(Ans)}$$

(b) If the area of a circle changes uniformly with respect to time, show that the rate of change of its circumference varies inversely as its radius.

**Solution :** Let r be the radius of the circle.

Then by the problem, $\frac{d}{dt}(\pi r^2) = k$, constant

or, $2\pi r\frac{dr}{dt} = k$ or, $\frac{dr}{dt} = \frac{k}{2\pi r}$ ..................... (1)

Now circumference of the circle, $C = 2\pi r$

$$\therefore \frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi\left(\frac{k}{2\pi r}\right) = \frac{k}{r}$$

$\therefore \frac{dC}{dt} \propto \frac{1}{r}$, which shows that the rate of change of its circumference varies inversely as its radius. **(Proved)**

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# DECEMBER 2015

# MATHEMATICS

Time Allowed : 3 Hour. Full Marks : 70

Answer to Question No.1 compulsory and to be answered first. This answer is to be made in separate loose script(s) provided for the purpose. Maximum time allowed is 45 minutes, after which the loose answer scripts will be collected and fresh answer scripts for answering the remaining part of the question No.1, a student will get the remaining script earlier.

1. Answer any twenty questions : 20×1

i) Find the value of x if log (x+2) + log (x-3) = log2

**Solution :** Given, $\log(x+2)+\log(x-3)=\log 2$ or, $\log\{(x+2)(x+3)\}=\log 2$ or, $(x+2)(x+3)=2$

or, $x^2+5x+6-2=0$ or, $x^2+5x+4=0$ or, $(x+4)(x+1)=0$ which gives, $x=-4,-1$

But $x=-4$ does not satisfy the given equation.

Therefore $x=-1$ **(Ans)**

ii) Find the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{10}$ is – (a) 251, (b) 252 (c) 250 (d) 253

**Solution :** Let (r + 1)th term is independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{10}$

Here, (r + 1)th term, $t_{r+1} = {}^{10}C_r\cdot(x)^{10-r}\cdot\left(\frac{1}{x}\right)^r = {}^{10}C_r\cdot x^{10-2r}$

$\therefore\ 10-2r=0 \Rightarrow r=5$

$\therefore$ (r + 1)th term = (5 + 1)th = 6th term is independent of x.

The term is = ${}^{10}C_5=\frac{10!}{5!.5!}=\frac{10.9.8.7.6.5!}{5.4.3.2.5!}=252$

**Ans.** (b)

iii) $\underset{x\to 0}{\text{Lt}}\ \frac{e^{ax}-1}{\log(1+ax)}$ is – (a) 2, (b) 1, (c) $\frac{1}{2}$ (d) 0.

**Solution :** $\underset{x\to 0}{\text{Lt}}\ \frac{e^{ax}-1}{\log(1+ax)} = \underset{x\to 0}{\text{Lt}}\left[\frac{e^{ax}-1}{x}\times\frac{x}{\log(1+ax)}\right]$ $[\because x\neq 0]$

$= \underset{ax\to 0}{\text{Lt}}\ \frac{e^{ax}-1}{ax}\times a\times \underset{x\to 0}{\text{Lt}}\ \frac{x}{\log(1+ax)}$ $[\because x\to 0, \therefore ax\to 0]$

$= 1\times a\times\frac{1}{\underset{ax\to 0}{\text{Lt}}\ \frac{1}{ax}\log(1+ax)}\times\frac{1}{a} = a\times\frac{1}{1}\times\frac{1}{a}=1$

**Ans.** (b)

iv) If $\cos 2\theta = -\frac{1}{2}$ then $\cos\theta$ is – (a) 2, (b) 1, (c) $\frac{1}{2}$ (d) 0

**Solution :** $\cos 2\theta = -\frac{1}{2} = \cos 120^0 \quad \therefore 2\theta = 120^0$ or, $\theta = 60^0 \quad \therefore \cos\theta = \cos 60^0 = \frac{1}{2}$ .

**Ans.** (c)

(v) The rinimum value of $16\cos^2\theta + 9\sec^2\theta$ is – (a) 20 (b) 22 (c) 24 (d) none

**Solution :** Let, $A = 16\cos^2\theta + 9\sec^2\theta = (4\cos\theta - 3\sec\theta)^2 + 2 \cdot 4\cos\theta \cdot 3\sec\theta = (4\cos\theta - 3\sec\theta)^2 + 24$

$\because (4\cos\theta - 3\sec\theta)^2$ is a perfect square, its minimum value is 0.

Hence minimum value of the given expression is 24.

**Ans.** (c)

(vi) If x and y are two distinct real positive numbers, then the relation $\sec\theta = \frac{2xy}{x^2+y^2}$ betrue ?

**Solution :** Given, $\sec\theta = \frac{2xy}{x^2+y^2}$ or, $\cos\theta = \frac{x^2+y^2}{2xy}$

Now, we know, $-1 \le \cos\theta \le 1$ or, $-1 \le \frac{x^2+y^2}{2xy} \le 1$ or, $-2xy \le x^2+y^2 \le 2xy$

Now, $x^2+y^2 \le 2xy$ gives, $x^2+y^2-2xy \le 0$ or, $(x-y)^2 \le 0$

which is not true, since x, y are two unequal positive real numbers.

Therefore, the given statement is not true.**(Ans)**

(vii) The vectors $2i + 7j$; $2i - 5j + 6k$ and $-i + 2j - k$ are coplanar or not?

**Solution :** Let $\vec{a} = 2i + 7j$, $\vec{b} = 2i - 5j + 6k$, $\vec{c} = i - 2j - k$

Here, $\vec{b}\times\vec{c} = \begin{vmatrix} i & j & k \\ 2 & -5 & 6 \\ -1 & 2 & -1 \end{vmatrix} = i(5-12) - j(-2+6) + k(4-5) = -7i - 4j - k$

$\therefore \quad \vec{a}\bullet(\vec{b}\times\vec{c}) = (2i+7j)\bullet(-7i-4j-k) = -14-28 = -42 \ne 0$

Hence given vecters are not coplaner. **(Ans)**

(viii) If $v = \frac{A}{r} + B$ then $\frac{d^2v}{dr^2} + \frac{2dv}{rdr} = 0$ is - (a) True, (b) False

**Solution :** $v = \frac{A}{r} + B$ then $\frac{dv}{dr} = -\frac{A}{r^2}$ and $\frac{d^2v}{dr^2} = \frac{2A}{r^3}$

$$\therefore \frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = \frac{2A}{r^3} + \frac{2}{r}\left(-\frac{A}{r^2}\right) = \frac{2A}{r^3} - \frac{2A}{r^3} = 0$$

**Ans.** (a)

(ix) If $\frac{2+i}{2-3i} = A + iB$ (A, B are real, $i = \sqrt{-1}$), find the value of $A^2 + B^2$.

**Solution :** Given, $\frac{2+i}{2-3j} = A + iB$ or, $|A+iB| = \frac{|2+i|}{|2-3i|}$ or, $\sqrt{A^2+B^2} = \frac{\sqrt{2^2+1^2}}{\sqrt{2^2+(-3)^2}}$

or, $A^2+B^2 = \frac{5}{13}$ **(Ans).**

(x) The square root of (3+4i) is - (a) $\sqrt{3}+i$ (b) $2-i$ (c) $2+i$ (d) none of these.

**Solution :** Given, $3 + 4i = 4 + 4i - 1 = 2^2 + 2.2.i + i^2 = (2 + i)^2$

$\therefore$ a square root is $= 2 + i$

**Ans.** (c)

(xi) If $\alpha$ and $\beta$ are the roots of $x^2 - px + q = 0$ then the value of $\alpha^2 + \beta^2 + \alpha\beta$ is -

(a) $q^2 - p$, (b) $q - p^2$ (c) $p^2 - q$ (d) $p - q^2$

**Solution :** Since $\alpha$, $\beta$ are the roots of $x^2 - px + q = 0$ $\therefore$ $\alpha + \beta = p$, $\alpha\beta = q$

Now, $\alpha^2 + \beta^2 + \alpha\beta = (\alpha+\beta)^2 - 2\alpha\beta + \alpha\beta = (\alpha+\beta)^2 - \alpha\beta = p^2 - q$

**Ans.** (c)

(xii) Find the fraction of $\frac{1}{(x-4)(x-3)}$

**Solution :** Let $\frac{1}{(x-4)(x-3)} = \frac{A}{x-4} + \frac{B}{x-3}$ ........................(1)

or, $A(x-3) + B(x-4) = 1$

Putting $x = 3$ we get, $B(3-4) = 1$ or, $B = -1$ and putting $x = 4$ we get, $A(4-3) = 1$ or, $A = 1$

From (i) we get, $\frac{1}{(x-4)(x-3)} = \frac{1}{x-4} - \frac{1}{x-3}$ **(Ans)**

(xiii) The value of $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ}$ is - (a) 1, (b) 2, (c) 3, (d) 4.

**Solution :** $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = \frac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$

$= \frac{4(\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ)}{2\sin 10^\circ - \cos 10^\circ} = \frac{4\sin 20^\circ}{\sin 20^\circ} = 4$

**Ans.** (d)

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(xiv) For what value of p the straight lines $5x+y-4=0$, $px+2y+8=0$ and $2x-y-1=0$ are concurrent.

**Solution :** Given, $5x+y-4=0$ ....... (1), $px+2y+8=0$ ..... (2) $2x-y-1=0$ .... (3)

Adding (1) and (3) we get, $7x-5=0$ or, $x=\frac{5}{7}$

From (3), $y=2x-1$ or, $y=2\times\frac{5}{7}-1=\frac{3}{7}$

Purring $x=\frac{5}{7}$, $y=\frac{3}{7}$ in (2) we get, $\frac{5}{7}p+2\times\frac{3}{7}+8=0$ or, $5p+6+56=0$

or, $5p=-62$ $\therefore$ $p=-\frac{62}{5}$ **[Ans]**

(xv) If $y=\cot^{-1}x$ find $\frac{d^2y}{dx^2}$

**Solution :** Given, $y=\cot^{-1}x$ $\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\cot^{-1}x\right)$ $=\frac{1}{1+x^2}$

$$\therefore \frac{d^2y}{dx^2}=+\frac{1}{\left(1+x^2\right)^2}\times\frac{d}{dx}\left(1+x^2\right)=\frac{2x}{\left(1+x^2\right)^2} \text{ [Ans]}$$

(xvi) The perpendicular distance between the two parallel lines $4x+3y+12=0$ and $4x+3y=8$ is -

(a) 2 units (b) 4 units (c) 6 units (d) 8 units

**Solution :** Given parallel lines are $4x+3y+12=0$, $4x+3y=8$

$$\therefore \text{ required distance } = \frac{8}{\sqrt{4^2+3^2}}-\frac{-12}{\sqrt{4^2+3^2}}=\frac{8}{5}+\frac{12}{5}=\frac{20}{5}=4 \text{ units.}$$

**Ans.** (b)

(xvii) Volume of a right pyramid of height 9 cm stands on a square base of side 8cm its -

(a) 126cc, (b) 280 cc (c) 192 cc (d) None

**Solution :** Height of the pyramid (h) = 9 cm.

Area of the square base of side (a) = 8 cm. is ($a^2$) = $8^2$ sq.cm. = 64 sq.cm.

Volume of the pyramid = $\frac{1}{3}\times a^2\times h=\frac{1}{3}\times 64\times 9$ = 192 sq.cm.

**Ans.** (c)

(xviii) Eliminate the constants A and B, if $y=Ae^{-mx}+Be^{mx}$

**Solution :** Given, $y=Ae^{-mx}+Be^{mx}$

Differentiate both sides with respect to x we get, $\frac{dy}{dx}=-Ame^{-mx}+Bme^{mx}$

Again differentiate both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = Am^2e^{-mx} + Bm^2e^{mx} = m^2\left(Ae^{-mx} + Be^{mx}\right) = m^2y$$

$$\therefore \frac{d^2y}{dx^2} - m^2y = 0$$ **[Ans]**

(xix) If $f(x) = \frac{ax-b}{bx-a}$, find the value of $f(x) \times f\left(\frac{1}{x}\right)$

**Solution :** Given, $f(x) = \frac{ax-b}{bx-a}$ Replacing x by $\frac{1}{x}$ we get,

$$f\left(\frac{1}{x}\right) = \frac{\frac{a}{x} - b}{\frac{b}{x} - a} = \frac{a - bx}{b - ax} = \frac{bx - a}{ax - b} \qquad \therefore f(x) \times f\left(\frac{1}{x}\right) = \frac{ax-b}{bx-a} \times \frac{bx-a}{ax-b} = 1$$

(xx) Find the angles between the vectors $2i + 4j - 7k$ and $3i + 2j + 2k$

**Solution :** Given vectors are $2i + 4j - 7k$ and $3i + 2j + 2k$

Let $\theta$ be the required angle

$$\therefore \cos\theta = \frac{(2i+4j-7k) \bullet (3i+2j+2k)}{|2i+4j-7k||3i+2j+2k|} \text{ or, } \cos\theta = \frac{6+8-14}{\sqrt{2^2+4^2+(-7)^2}\sqrt{3^2+2^2+2^2}} \text{ or, } \cos\theta = 0$$

Therefore, $\theta = 90^0$

Hence the reqired angle is $90^0$ **[Ans]**

(xxi) Examine the following function as even or odd :

$$f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$$

**Solution :** $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

$$\therefore f(-x) = \sqrt{1-x+(-x)^2} - \sqrt{1+x+(-x)^2} = \sqrt{1-x+x^2} - \sqrt{1+x+x^2}$$

$$= -\left(\sqrt{1+x+x^2} - \sqrt{1-x+x^2}\right) = -f(x)$$

Hence given function is an odd function. **(Ans)**

(xxii) If $\Phi$ (x) = log sinx and $\Psi$ (x) = log cosx, then $e^{2\Phi(x)} + e^{2\Psi(x)}$ is - (a) 0, (b) 2 (c) 1, (d) none

**Solution :** Given, $\phi(x) = \log\sin x$ and $\psi(x) = \log\cos x$

Therefore, $e^{2\phi(x)} + e^{2\psi(x)}$

$= e^{2\log\sin x} + e^{2\log\cos x} = e^{\log\sin^2 x} + e^{\log\cos^2 x} = \sin^2 x + \cos^2 x = 1$

**Ans. (c)**

(xxiii) Mod of $\frac{(8\mathrm{l}+1)}{(1+8\mathrm{l})}$ is – (a) 1, (b) $\sqrt{2}$ (c) $\pm 1$ (d) none

**Solution :** Mod of $\frac{8i+1}{1+8i} = \left|\frac{8i+1}{1+8i}\right| = \frac{|8i+1|}{|1+8i|} = \frac{\sqrt{8^2+1}}{\sqrt{1+8^2}} = \frac{\sqrt{65}}{\sqrt{65}} = 1$

**Ans.** (a)

(xxiv) $\underset{x\to 0}{\mathrm{Lt}} \frac{\sin 3x}{\sin 2x}$ - (a) 3/2 (b) 2/3 (c) 1 (d) none

**Solution :** $\underset{x\to 0}{\mathrm{Lt}} \frac{\sin 3x}{\sin 2x} = \underset{x\to 0}{\mathrm{Lt}} \frac{\frac{\sin 3x}{3x}\times 3}{\frac{\sin 2x}{2x}\times 2} = \frac{3}{2} \frac{\underset{3x\to 0}{\mathrm{Lt}} \frac{\sin 3x}{3x}}{\underset{2x\to 0}{\mathrm{Lt}} \frac{\sin 2x}{2x}} = \frac{3}{2}\times\frac{1}{1} = \frac{3}{2}$

**Ans.** (a)

(xxv) Volume of a right regular hexagonal prism of side 2 cm and height $2\sqrt{3}$ cm is -

(a) $12\sqrt{3}$ cc (b) 24cc (c) 36cc (d) none

**Solution :** Here n = 6, a = 2, h = $2\sqrt{3}$

Area of the base of the prism = $\frac{na^2}{4}\cot\frac{\pi}{n} = \frac{6\times 4}{4}\cot\frac{\pi}{6} = 6\sqrt{3}$ sq. cm.

Therefore, volume = $6\sqrt{3}\times h = 6\sqrt{3}\times 2\sqrt{3} = 36$ cu. cm.

**Ans.** (c)

(xxvi) sin 105° + cos105° is - (a) 1/2 (b) –1/2, (c) $1/1\sqrt{2}$ (d) none

**Solution :** Given expression :

$\sin 15^0 \sin 75^0 = \frac{1}{2}(2\sin 75^0 \sin 15^0) = \frac{1}{2}[\cos(75^0 - 15^0) - \cos(75^0 + 15^0)]$

$= \frac{1}{2}\left(\cos 60^0 - \cos 90^0\right) = \frac{1}{2}\times\left(\frac{1}{2} - 0\right) = \frac{1}{4}$

**Ans.** (d)

(xxvii) If vectors a = 2i – 3j + k and b = 4i + mj + 2k are collinear then 'm' is - (a) 0 (b) 3 (c) –3 (d) none

**Solution :** Given a = 2i – 3j + k , b= 4i + mj + 2k are collinear.

Therefore, a = xb, x is a scalar. or, (2j – 3j + k) = x (4i + mj + 2k)

$\therefore$ 4x = 2, mx = –3, 2x = 1 or, $x = \frac{1}{2}$

From mx = –3 we get, $m\times\frac{1}{2} = -3$ or, m = –6

**Ans.** (d)

(xxviii) The area of triangle formed by the axes of co-ordinates and the straight line 2x + 3y = 6 is – (a) 6 (b) 36 (c) 3 (d) none

**Solution :** Given equation of straight line is 2x + 3y = 6 or, $\frac{x}{3}+\frac{y}{2}=1$

Required area of $\Delta$OAB = $\frac{1}{2}\times OB\times OA = \frac{1}{2}\times 2\times 3 = 3$ sq. unit.

**Ans.** (c)

(xxix) The distance between the lines $3x + 4y = 9$ and $6x + 8y = 15$ is - (a) 3/2 (b) 3/10 (c) 6 (d) none

**Solution :** Distance between the lines $3x + 4y = 9$ and $6x + 8y = 15$ is

$$\left|\frac{9}{\sqrt{3^2+4^2}} - \frac{15}{\sqrt{6^2+8^2}}\right| = \left|\frac{9}{5} - \frac{15}{10}\right| = \frac{18-15}{10} = \frac{3}{10} \text{ unit.}$$

**Ans.** (b)

(xxx) Angle between straight lines $x\sqrt{3} - y = 2$ and $y\sqrt{3} + x = 1$ is – (a) 0 (b) $\frac{\pi}{2}$ (c) $\pi$ (d) none

**Solution :** Gradient of $x\sqrt{3} - y = 2$ is $(m_1) = \sqrt{3}$ and that of $x + y\sqrt{3} = 1$ is $(m_2) = -\frac{1}{\sqrt{3}}$

Here $m_1 m_2 = \sqrt{3} \times \left(-\frac{1}{\sqrt{3}}\right) = -1$

**Ans.** (b)

2. (a) An empty vessel in the shape of right regular hexagonal prism each side 24cm is filled with 100 liter of water. What will be the leyel of the water in the vessel?

**Solution :** Given each side of the regular hexagonal prism = 24 cm.

Area of the base of the prism = $\frac{na^2}{4}\cot\frac{\pi}{n}$, n = 6 here. $= \frac{6\times 24^2}{4}\cot\frac{\pi}{6} = \frac{6\times 24^2}{4}\times\sqrt{3} = 864\sqrt{3}$ sq. cm.

Let h be the height of the level of the water in the vessel.

Then by the problem, $864\sqrt{3}$ h c.c = 100 liters = 100000 c.c

or, $h = \frac{100000}{864\sqrt{3}} = \frac{100000\sqrt{3}}{2592} = 38.58\times 1.73 = 66.74$ cm. **(Ans)**

(b) If $A+B+C = \pi$ and $\cos A = \cos B \cos C$ show that ten $A = \tan B + \tan C$ $\cot C = \frac{1}{2}$.

**Solution :** Given, $A + B + C = \pi \therefore B + C = \pi - A$

or, $\cos(B + C) = \cos(\pi - A)$ or, $\cos B \cos C - \sin B \sin C = -\cos A$

or, $\cos B \cos C = \sin B \sin C - \cos B \cos C$ $[\because \cos A = \cos B \cos C]$

or, $2\cos B \cos C = \sin B \sin C$

or, $\frac{\cos B \cos C}{\sin B \sin C} = \frac{1}{2}$ or, $\cot B . \cot C = \frac{1}{2}$ **[(ii) Proved]**

Again, $\tan A = \frac{\sin A}{\cos A} = \frac{\sin\{\pi - (B + C)\}}{\cos A}$ $[\because A + B + C = \pi \therefore A = \pi - (B + C)]$

$= \frac{\sin(B + C)}{\cos A} = \frac{\sin B \cos C + \cos B \sin C}{\cos B \cos C} = \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C}$

$\therefore \tan A = \tan B + \tan C$ **(Proved)**

3. (a) I f x+iy = $\sqrt{\frac{a+ib}{c+id}}$, then show that $(x^2+y^2)=\frac{a^2+b^2}{c^2+d^2}$.

**Solution :** Given, $x+iy=\sqrt{\frac{a+ib}{c+id}}$

or, $(x+iy)^2=\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}$ or, $x^2-y^2+2ixy=\frac{ac+ibc-iad-i^2bd}{c^2-i^2d^2}$

or, $(x^2-y^2)+2ixy=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$, $[i^2=-1]$ or, $(x^2-y^2)+2ixy=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}$

$\therefore x^2-y^2=\frac{ac+bd}{c^2+d^2}$, $2xy=\frac{bc-ad}{c^2+d^2}$ [Equating real and imaginary parts from both sides]

$$\therefore (x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{bc-ad}{c^2+d^2}\right)^2$$

$$=\frac{(ac+bd)^2+(bc-ad)^2}{(c^2+d^2)^2}=\frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2}=\frac{c^2(a^2+b^2)+d^2(a^2+b^2)}{(c^2+d^2)^2}=\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}$$

$\therefore (x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$ **(Proved)**

**(b)** Find the co-efficient if x in the expansion : $(1-2x^3+3x^5)(1+\frac{1}{x})^{10}$ **5+5**

**Solution :** The given expression, $(1-2x^3+3x^5)(1+\frac{1}{x})^{10}=(1-2x^3+3x^5)$

$$\times\left(1+{}^{10}C_1\frac{1}{x}+{}^{10}C_2\frac{1}{x^2}+{}^{10}C_3\frac{1}{x^3}+{}^{10}C_4\frac{1}{x^4}+{}^{10}C_5\frac{1}{x^5}+{}^{10}C_6\frac{1}{x^6}+......\right)$$

Now, in the multiplication $-2x^3\times{}^{10}C_2x^2=-2\times{}^{10}C_2\,x$, $3x^5\times{}^{10}C_4x^4=3\times{}^{10}C_4\,x$

and there is no other term containing x in the product.

$\therefore$ the required coefficient of x

$$=-2\times{}^{10}C_2+3\times{}^{10}C_4=-2\times\frac{10!}{2!\,8!}+3\times\frac{10!}{4!\,6!}==-2\times\frac{10\times9}{2}+3\times\frac{10\times9\times8\times7}{4\times3\times2\times1}=-90+630=540$$ **(Ans)**

4. (a) Find the partial fractions $\frac{4x+3}{(x^2+1)(x+1)}$

**Solution :** Let $\frac{A}{x+1}+\frac{Bx+C}{x^2+1}=\frac{4x+3}{(x+1)(x^2+4)}$ or, $A(x^2+1)+(Bx+C)(x+1)=4x+3$

Putting x = –1 we get, $A(1+1)=-4+3\Rightarrow A=-\frac{1}{2}$

and A + B = 0, or, $B=\frac{1}{2}$ [equating cocfficient of $x^2$]

and A – C = 3, or, C = 3 – A $\Rightarrow C=3-\frac{1}{2}=\frac{5}{2}$ [equating constant terms]

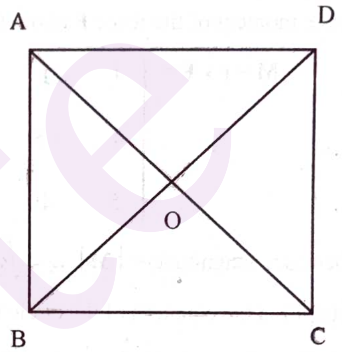
Therefore, $\frac{x}{(x+1)(x^2+1)}=\frac{1}{2}\cdot\frac{1}{x+1}+\frac{1}{2}\cdot\frac{x+5}{x^2+1}$ **(Ans)**

**(b)** Using vector method show that the quadrilateral whose diagonals bisect each other is a parallelogram.

**Solution :**

Let ABCD be a quadrilateral. Diagonals AC and BD meet at O. With respect to O as origin let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are the position vectors of the vertices A, B, C and D.

$\therefore \overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}, \overrightarrow{OD} = \vec{d}$




By the problen, $OB = OD \therefore \vec{b} = -\vec{d}$ ........ (1)

and $OA = OC \therefore \vec{a} = -\vec{c}$ ......... (2)

Now, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{c} - \vec{b} = -\vec{a} + \vec{d}$ [from (1) and (2)]

$= \overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{AD}$

$\therefore \overrightarrow{AD} = \overrightarrow{BC}$ which gives AD and BC are parallel.

or, $|\overrightarrow{AD}| = |\overrightarrow{BC}|$ or, $AD = BC$ which gives AD and BC are equal.

i. e., AD and BC are parallel and cqual.

Hence ABCD is a parallelogram. **[Propved]**

**5. (a)** Show that the vectors 2i – j + k, i – 3j – 5k, 3i – 4j – 4k form the sides of a right angled triangle.

**Solution :** Let, $\vec{a} = 2i - j + k$, $\vec{b} = i - 3j - 5k$, $\vec{c} = 3i - 4j - 4k$

$\therefore |\vec{a}| = |2i - j + k| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

$|\vec{b}| = |i - 3j - 5k|$ or, $|\vec{b}| = \sqrt{1^2 + (-3^2) + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

$|\vec{c}| = |3i - 4j - 4k| = \sqrt{3^2 + (-4)^2 + (-4)^2} = \sqrt{9+16+16} = \sqrt{41}$

**Since,** $|\vec{a}|^2 + |\vec{b}|^2 = (\sqrt{6})^2 + (\sqrt{35})^2 = 6 + 35 = 41 = (\sqrt{41})^2 = |\vec{c}|^2$

**i.e.,** $|\vec{a}|^2 + |\vec{b}|^2 = |\vec{c}|^2$

**Hence the given three vectors are the sides of a right angled triangle. [Proved]**

(b) A force of 15 units acts in the directions of the vector i – 2j + 2k and passes through a points 2i –2j + 2k. Find the moment of the force about the point i + j + k.

**Solution :** Let F be the force of magnitude 15 units and acting in the direction i – 2j + 2k.

**Unit vector in the direction of i-2j+2k is**

$$= \frac{i-2j+2k}{\sqrt{(1)^2 + (-2)^2 + 2^2}} = \frac{i-2j+2k}{\sqrt{9}} = \frac{i-2j+2k}{3}$$

Then $\mathbf{F} = 15.\dfrac{i-2j+2k}{3} = 5(i-2j+2k)$

Let $\overrightarrow{OP} = 2i - 2j + 2k$ and $\overrightarrow{OA} = i + j + k$

Therefore, $\mathbf{r} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OP} = (2i - 2j + 2k) - (i + j + k) = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Hence moment of the force **F** about the given point is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 5 & -10 & -10 \end{vmatrix} = 40i + 15j + 5k$$

Therefore, magnitude $= |\mathbf{M}| = 5\sqrt{8^2 + 3^2 + 1^2} = 5\sqrt{74}$ units. **(Ans)**

**6. (a)** Find the equation of the circle which passed through (5, 3), (6, –4) and radius is 5.

**Solution :** Let (a, b) be the centre of the circle. Given, radius of the circle is 5.

Therefore, $(a-5)^2 + (b-3)^2 = (a-6)^2 + (b+4)^2 = 25$ ..... (1)

Now, $(a-5)^2 + (b-3)^2 = (a-6)^2 + (b+4)^2$ or, $a^2 - 10a + 25 + b^2 - 6b + 9 = a^2 - 12a + 36 + b^2 + 8b + 16$

or, $-10a - 6b + 34 = -12a + 8b + 52$ or, $2a = 14b + 18$ or, $a = 7b + 9$ ..... (2)

From (1) we get,

$(a-6)^2 + (b+4)^2 = 25$ or, $(7b + 9 - 6)^2 + (b+4)^2 = 25$

or, $(7b+3)^2 + (b+4)^2 = 25$ or, $49b^2 + 42b + 9 + b^2 + 8b + 16 = 25$

or, $50b^2 + 50b = 0$ or, $b^2 + b = 0$ or, $b(b+1) = 0$ or, $b = 0, -1$

From (2) we get, For $b = 0$, $a = 9$

For $b = -1$, $a = -7 + 9 = 2$

Therefore, centre of the circle : (9, 0) or (2, –1)

Hence, equation of the circle is

$(x-9)^2 + (y-0)^2 = 25$ or, $x^2 + y^2 - 18x + 81 = 25$ or, $x^2 + y^2 - 18x + 56 = 0$ **(Ans)**

and $(x-2)^2 + (y+1)^2 = 25$ or, $x^2 + y^2 - 4x + 2y + 5 = 25$ or, $x^2 + y^2 - 4x + 2y - 20 = 0$ **(Ans)**

**(b)** Find the equation of the straight line passed through (–3, 4) and making equal angles with the axes.

**Solution :** Since the straight line makes equal angle with the axes, intercepts on the axes by the straight line will be equal and let it be a.

Then the equation of the straight line be $x + y = a$ ..................... (1)

If it passes through the point (–3, 4) then $-3 + 4 = a$ or, $a = 1$

Hence the required equation of the straight line is $x + y = 1$ **(Ans)**

7. (a) The base of a right pyramid is a rectangular hexagon of side a. Prove that the volume of the pyramid is $a^2$

$a^2 \dfrac{(12l^2 - 9a^2)^{\frac{1}{2}}}{4}$ where l is the slant height of the pyramid.

**Solution :** Area of the hexagonal base

$= \frac{1}{4} na^2 \cot\frac{\pi}{n}, n = 6$ here. $= \frac{1}{4}.6a^2 \cot\frac{180^\circ}{6} = \frac{3}{2}a^2 \cot 30^\circ = \frac{3\sqrt{3}}{2}a^2$

Let r be the radius of the inscribed circle of regular hexagon.

Then area of the base, $nr^2 \tan\frac{180^\circ}{n} = \frac{3\sqrt{3}}{2}a^2$ or, $6r^2 \tan 30^\circ = \frac{3\sqrt{3}}{2}a^2$

or, $6r^2.\frac{1}{\sqrt{3}} = \frac{3\sqrt{3}}{2}a^2$ or, $r^2 = \frac{3}{4}a^2$

$\because$ $l$ is the slant height, $\therefore$ height of the pyramid, $h = \sqrt{l^2 - r^2} = \sqrt{l^2 - \frac{3}{4}a^2} = \sqrt{\frac{4l^2 - 3a^2}{4}}$

$\therefore$ volume of the pyramid $= \frac{1}{3} \times$ area of the base $\times$ height.

$= \frac{1}{3} \times \frac{3\sqrt{3}}{2}a^2 \times \frac{\sqrt{4l^2 - 3a^2}}{2} = a^2 \dfrac{(12l^2 - 9a^2)^{\frac{1}{2}}}{4}$ **(Proved)**

(b) Find the equation of the straight line which passes through the center of the circle $x^2 + y^2 + 2x + 2y - 23 = 0$ and is perpendicular to the straight line $x - y + 8 = 0$ 5+5

**Solution :** Centre of the given circle $x^2 + y^2 + 2x + 2y - 23 = 0$ is $(-1, -1)$

Now, equation of any straight line perpendicular to the line $x - y + 8 = 0$ is $x + y = k$.

If it passes through $(-1,-1)$ then $-1 - 1 = k$ or, $k = -2$

Hence the required equation of the straight line is $x + y = -2$ or, $x + y + 2 = 0$ **(Ans)**

8. (a) Discuss the continuity of the following function at $x = 2$.

$$f(x) = x^2 + 2, \quad 0 \le x \le 2$$
$$= x + 4, \quad 2 \le x \le 3$$
$$= 6, \quad x = 2$$

Here,

$\underset{x \to 2-}{\text{Lt}} f(x) = \underset{x \to 2-}{\text{Lt}} (x^2 + 2) = 2^2 + 2 = 6$ $[\because f(x) = x^2 + 2$, when $x < 2]$

$\underset{x \to 2+}{\text{Lt}} f(x) = \underset{x \to 2+}{\text{Lt}} (x + 4) = 2 + 4 = 6$ $[\because f(x) = x + 4$, when $x > 2]$

and $f(2) = 2 + 4 = 6$

$\because \underset{x \to 2-}{\text{Lt}} f(x) = \underset{x \to 2+}{\text{Lt}} f(x) = f(2$ i.e., $\underset{x \to 2}{\text{Lt}} f(x) = f(2)$

Hence, from definition of continuity $f(x)$ is continuous at $x = 2$.

(b) Evaluate $\underset{x \to a}{\text{Lt}} \dfrac{x\sin a - a\sin x}{x - a}$

**Solution :** Given, $\underset{x \to a}{\text{Lt}} \dfrac{x\sin a - a\sin x}{x - a}$

$$= \underset{x \to a}{\text{Lt}} \frac{(x-a)\sin a - a\sin x + a\sin a}{x-a} = \underset{x \to a}{\text{Lt}} \frac{(x-a)\sin a - a(\sin x - \sin a)}{x-a}$$

$$= \sin a - a \underset{x \to a}{\text{Lt}} \frac{\sin x - \sin a}{x-a} = \sin a - a \underset{x \to a}{\text{Lt}} \frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a}$$

$$= \sin a - a \underset{\frac{x-a}{2} \to 0}{\text{Lt}} \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} \underset{x \to a}{\text{Lt}} \cos\frac{x+a}{2} \left[\because x \to \therefore \frac{x-a}{2} \to 0\right]$$

$$= \sin a - a\cos\frac{a+a}{2} = \sin a - a\cos a \quad \textbf{(Ans)}$$

5+5

9. (a) If $y = \left(x+\sqrt{1+x^2}\right)^m$, then prove that, $(1+x^2)\, y_2 + xy_1 = m^2 y$

**Solution :** $y = \left(x+\sqrt{1+x^2}\right)^m$

$\log y = m\log\left(x+\sqrt{1+x^2}\right)$ [taking logarithm of both sides]

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y) = m\frac{d}{dx}\left\{\log\left(x+\sqrt{1+x^2}\right)\right\} \text{ or, } \frac{1}{y}\frac{dy}{dx} = m \times \frac{1}{x+\sqrt{x^2+1}} \times \left(1 + \frac{1}{2\sqrt{x^2+1}} \times 2x\right)$$

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \frac{m}{x+\sqrt{x^2+1}} \times \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} = \frac{m}{\sqrt{x^2+1}}$$

$$\text{or, } \sqrt{x^2+1}\,\frac{dy}{dx} = my \qquad \text{or, } \left(x^2+1\right)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

$$\text{or, } \left(x^2+1\right) \times 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 2m^2y\frac{dy}{dx}$$

$$\text{or, } \left(x^2+1\right)\cdot\frac{d^2y}{dx^2} + x\frac{dy}{dx} = m^2y \quad \text{or, } (1+x^2)\, y_2 + xy_1 - m^2y = 0 \ \textbf{(Proved)}$$

(b) Find $\frac{dy}{dx}$; if $y = \cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ 5+5

**Solution :** Given, $y = \cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ or, $y = \cot^{-1}\left(\frac{\frac{\cos x - \sin x}{\sin x}}{\frac{\cos x + \sin x}{\sin x}}\right)$ or, $y = \cot^{-1}\left(\frac{\cot x - 1}{\cot x + 1}\right)$

$$\text{or, } y = \cot^{-1}\left(\frac{\cot\frac{\pi}{4}\cot x - 1}{\cot\frac{\pi}{4} + \cot x}\right) \text{ or, } y = \cot^{-1}\cot\left(\frac{\pi}{4} + x\right)$$

$$\text{or, } y = \frac{\pi}{4} + x \quad \text{or, } \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + x\right) = 1 \ \textbf{(Ans)}$$

10. (a) If $ax^2 + 2hxy + by^2 = 1$, Prove that $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$

**Solution :** $ax^2 + 2hxy + by^2 = 1$ ..... (1)

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(ax^2) + 2h\frac{d}{dx}(xy) + \frac{d}{dx}(by^2) = \frac{d}{dx}(1)$$

or, $2ax + 2h\left(y.1 + x\frac{dy}{dx}\right) + 2by\frac{dy}{dx} = 0$

or, $ax + hy + hx\frac{dy}{dx} + by\frac{dy}{dx} = 0$ or, $(hx + by)\frac{dy}{dx} = -(ax + hy)$

or, $\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$ ..... (1)

Again, differentiating both sides with respect to x we get,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{ax+hy}{hx+by}\right)$$

$$= -\frac{(hx+by)\frac{d}{dx}(ax+hy) - (ax+hy)\frac{d}{dx}(hx+by)}{(hx+by)^2}$$

$$= -\frac{(hx+by)\left(a+h\frac{dy}{dx}\right) - (ax+hy)\left(h+b\frac{dy}{dx}\right)}{(hx+by)^2}$$

$$= -\frac{\frac{dy}{dx}(h^2x+bhy-abx-bhy) + (ahx+aby-ahx-h^2y)}{(hx+by)^2}$$

$$= -\frac{(h^2-ab)x\frac{dy}{dx} - (h^2-ab)y}{(hx+by)^2} = \frac{(h^2-ab)\left(y-x\frac{dy}{dx}\right)}{(hx+by)^2} = \frac{(h^2-ab)\left(y+x\cdot\frac{ax+hy}{hx+by}\right)}{(hx+by)^2}$$

$$= \frac{(h^2-ab)(hxy+by^2+ax^2+hxy)}{(hx+by)^3} = \frac{(h^2-ab)(ax^2+2hxy+by^2)}{(hx+by)^3} = \frac{(h^2-ab)\cdot 1}{(hx+by)^3}$$

$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$ **(Proved)** [$\because ax^2 + 2hxy + by^2 = 1$

10. (a) Find the maximum and minimum value of $x^3 + \frac{1}{x^3}$

**Solution :** Let, $u = x^3 + \frac{1}{x^3}$ -------------(1)

Differentiating both sides with respect to x we get,

$$\frac{du}{dx} = 3x^2 - \frac{3}{x^4}$$

Again differentiating both sides with respect to x we get,

$$\frac{d^2u}{dx^2} = 6x + \frac{12}{x^5}$$

Now $\frac{du}{dx} = 0$ gives,

$3x^2 - \frac{3}{x^4} = 0$ or, $x^6 - 1 = 0$ or, $(x^2 - 1)(x^4 + x^2 + 1) = 0$

Therefore, $x^2 - 1 = 0$ or, $x = \pm 1$

Again, $x^4 + x^2 + 1 = 0$

or, $x^2 = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$ which are not real.

Therefore, $x = 1$ and $x = -1$

At $x = 1$, $\frac{d^2u}{dx^2} = 6 + 12 = 18 > 0$

At $x = -1$, $\frac{d^2u}{dx^2} = -6 - 12 = -18 < 0$

Therefore at $x = 1$ the given function is minimum and the minimum value is $1 + \frac{1}{1} = 1 + 1 = 2$ **(Ans)**

and at $x = -1$ the given function is maximum and the maximum value is $-1 - \frac{1}{1} = -1 - 1 = -2$ **(Ans)**

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N. Tute

December, 2016

**MATHEMATICS**

Time Allowed: 3 Hours  Full Marks : 70

Answer to Question No.1 is compulsory and is to be answered first.

This answer is to be made in separate loose script(s) provided for the purpose.

Maximum time allowed is 45 minutes, after which the loose answer scripts will be collected and fresh scripts for answering the remaining part of the question will be provided.

On early submission of answer scripts of Question No.1, a student will get the remaining script earlier.

Answer any <u>five</u> Questions taking <u>at least one</u> from Group – A, B & C.

**1.** Answer **<u>any twenty</u>** questions from the following: $20 \times 1 = 20$

(i) When $\log_2 3 = x$ the value of $\log_8 27 = ?$ (a) x (b) 2x (c) $x^2$ (d) None of these.

**Solution :** Given $\log_2 3 = x$

Now, $\log_8 27 = \dfrac{\log_e 27}{\log_e 8} = \dfrac{\log_e 3^3}{\log_e 2^3} = \dfrac{3\cdot\log_e 3}{3\cdot\log_e 2} = \dfrac{\log_e 3}{\log_e 2} = \log_2 3 = x$

**Ans.** (a)

(ii) Amplitude of the complex number $\frac{1}{1-i}$ is – (a) $\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{4}$

**Solution :** Let $z = \dfrac{1}{1-i} = \dfrac{1+i}{(1+i)(1-i)} = \dfrac{1+i}{1-i^2} = \dfrac{1+i}{1+1} = \dfrac{1}{2} + i\cdot\dfrac{1}{2}$

Let amp(z) = $\theta$, then $\tan\theta = \dfrac{\frac{1}{2}}{\frac{1}{2}} = 1$ where $0 < \theta \le \frac{\pi}{2}$ or, $\tan\theta = \tan\frac{\pi}{4}$ or, $\theta = \frac{\pi}{4}$

**Ans.** (c)

(iii) If $z = 2 + i\sqrt{3}$ find $z\bar{z}$

**Solution :** Given $z = 2 + i\sqrt{3}$ $\therefore \bar{z} = 2 - i\sqrt{3}$

$\therefore z\bar{z} = \left(2 + i\sqrt{3}\right)\left(2 - i\sqrt{3}\right) = 4 - \left(i\sqrt{3}\right)^2 = 4 + 3 = 7$ **(Ans)**

(iv) If one root of $5x^2 + 13x + k = 0$ is reciprocals of the other, then k is – (a) 0, (b) 6, (c) 5, (d) none of these.

**Solution :** Let $\alpha, \frac{1}{\alpha}$ be the roots of $5x^2 + 13x + k = 0$ Then $\alpha \times \dfrac{1}{\alpha} = \dfrac{k}{5}$ or, k = 5

**Ans.** (c)

(v) The coefficient of $6^{th}$ term in the expansion of $\left(a - \frac{1}{2a}\right)^{10}$ is – (a) $-\frac{63}{8}$, (b) -63, (c) $\frac{245}{8}$, (d) none of these.

**Solution :** The coefficient of $6^{th}$ term in the expansion of $\left(a - \frac{1}{2a}\right)^{10}$ is

$${}^{10}C_5\, a^{10-5}\left(-\frac{1}{2a}\right)^5 = -\frac{10!}{5!\,5!}\cdot\frac{1}{2^5} = -\frac{10\times9\times8\times7\times6\times5!}{5\times4\times3\times2\times1\times5!}\cdot\frac{1}{2^5} = -9\times7\cdot\frac{1}{2^3} = -\frac{63}{8}$$

**Ans.** (a)

(vi) The term independent of x in the expansion $\left(x-\frac{1}{x}\right)^{10}$ is - (a) $4^{th}$ (b) $5^{th}$ (c) $6^{th}$ (d) none of these

**Solution :** Let (r + 1) th term is independent of x.

Now (r + 1) th term in the given expansion is $t_{r+1} = {}^{10}C_r \cdot x^r\left(-\frac{1}{x}\right)^{10-r}$

$$=(-)^{10-r}.{}^{10}C_r \cdot x^{r-10+r} = (-)^{10-r}.{}^{10}C_r \cdot x^{2r-10}$$

By the problem,, 2r – 10 = 0 or, r = 5

Therefore, $6^{th}$ term is independent of x.

**Ans.** (c)

(vii) Split into partial fraction : $\frac{s}{(s-3)(s-1)}$

**Solution :** Let, $\frac{A}{s-3}+\frac{B}{s-1}=\frac{s}{(s-3)(s-1)}$ or, A(s – 1) + B(s – 3) = s

Putting s = 1 we get, –2B = 1 or, $B=-\frac{1}{2}$

Putting s = 3 we get, 2A = 3 or, $A=\frac{3}{2}$

Therefore, $\frac{s}{(s-3)(s-1)}=\frac{3}{2}\cdot\frac{1}{s-3}-\frac{1}{2}\cdot\frac{1}{s-1}$ **(Ans)**

(viii) If the vectors 2i – 3j + k and 4i + mj + 2k are collinear, then m = ? (a) 3 (b) 0 (c) –6 (d) none of these.

**Solution :** By the problem, 4i + mj + 2k = x(2i – 3j + k)

Equating the coefficients of i, j, k from both sides, we get, 2x = 4, –3x = m, x = 2

Therefore, m = -6

**Ans:** (c)

(ix) The vectors 2i – 5j + 6k and –i + 2j + 2k are perpendicular to each other – (a) true (b) false

**Solution :** Given vectos are 2i – 5j + 6k and –i + 2j + 2k

Since, (2i – 5j + 6k) . (–i + 2j + 2k) = –2 – 10 12 = 0 therefore, the given two vectors are perpendicular to each other.

**Ans.** (a)

(x) If roots of the equation $x^2 - px + q = 0$ are equal then (a) p = q (b) $p^2 = 4q$ (c) $q^2 = 4p$ (d) pq = 1

**Solution :** Let α, α be the roots of the equation $x^2 - px + q = 0$

Therefore, α + α = p or, 2α = p ⇒ $\alpha=\frac{p}{2}$

and α .α = q or, $\alpha^2 = q$ or, $\left(\frac{p}{2}\right)^2 = q$ or, $p^2 = 4q$

**Ans.** (b)

(xi) Value of $\sin(-1755^0)$ is – (a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) none of these

**Solution :** $\sin(-1755^0) = -\sin(90^0 \times 19 + 45^0) = +\cos45^0 = \dfrac{1}{\sqrt{2}}$

**Ans.** (b)

(xii) The value of $\log\tan1^0 + \log\tan2^0 + \log\tan3^0 + ...... + \log\tan88^0 + \log\tan89^0$ is:

(a) 1 (b) 0 (c) $\tan1^0$ (d) none of these

**Solution :** Given, $\log\tan1^0 + \log\tan2^0 + \log\tan3^0 + ...... + \log\tan88^0 + \log\tan89^0$

$= \log(\tan1^0 \tan2^0 \tan3^0 ........\tan88^0 \tan89^0)$

$= \log(\tan1^0 \tan89^0 \tan2^0 \tan88^0 ............ \tan44^0 \tan46^0 \tan45^0)$

$= \log(\tan1^0 \cot1^0 \tan2^0 \cot2^0 ............ \tan44^0 \cot44^0 \tan45^0)$

$= \log(1 . 1 . 1 .......... ^{1 \cdot 1}) = \log 1 = 0$

**Ans.** (b)

(xiii) The value of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega)$, $\omega$ being the cube root of unity,is (a) 4 (b) -4 (c) 2 (d) none of these

**Solution :** Given, $(1 - \omega + \omega^2)(1 - \omega^2 + \omega)$

$= (1 + \omega + \omega^2 - 2\omega)(1 + \omega + \omega^2 - 2\omega^2) = (-2\omega)(-2\omega^2) = 4\omega^3 = 4$

**Ans.** (a)

(xiv) If $10\alpha = \frac{\pi}{2}$, then $\tan3\alpha \tan5\alpha \tan7\alpha$ is (a) 1 (b) 0 (c) 2 (d) none of these

**Solution :** Given, $10\alpha = \frac{\pi}{2}$

Therefore, $\tan3\alpha \tan5\alpha \tan7\alpha$

$= \tan\frac{3\pi}{20} \cdot \tan\frac{5\pi}{20} \cdot \tan\frac{7\pi}{20} = \tan\frac{3\pi}{20} \cdot \tan\frac{\pi}{4} \cdot \tan\left(\frac{\pi}{2} - \frac{3\pi}{20}\right) = \tan\frac{3\pi}{20} \cdot 1 \cdot \cot\frac{3\pi}{20} = \tan\frac{3\pi}{20} \cdot \cot\frac{3\pi}{20} = 1$

**Ans.** (a)

(xv) Find the value of $\sin\left\{\cos^{-1}\left(-\frac{1}{2}\right)\right\}$

**Solution :** Given expression : $\sin\left\{\cos^{-1}\left(-\frac{1}{2}\right)\right\}$ [Let $\cos^{-1}\left(-\frac{1}{2}\right) = \theta$, $\therefore \cos\theta = -\frac{1}{2} = \cos120^0$ $\therefore \theta = 120^0$]

$= \sin 120^0 = \frac{\sqrt{3}}{2}$ **(Ans)**

(xvi) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$ = (a) $\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) none of these

**Solution :** We have, $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2} = \tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

**Ans.** (c)

(xvii) The polar coordinates of the point whose cartesian co-ordinates are (–1, 1) are:

(a) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$, (b) $\left(\sqrt{2}, -\frac{3\pi}{4}\right)$, (c) $\left(\sqrt{2}, \frac{\pi}{4}\right)$, (d) none of these.

**Solution :** Let $(r, \theta)$ be the polar co-ordinates of the point whose cartesian co-ordinates are (–1, 1).

Then $r\cos\theta = -1$, and $r\sin\theta = 1$

$\therefore\ r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + (1)^2$ or, $r^2 = 1 + 1 = 2\ \therefore\ r = \sqrt{2}$

and $\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{-1}{1} = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right)$ [$\because$ the point (–1, 1) lies in the 2nd quadrant.]

or, $\theta = \frac{3\pi}{4}$

Therefore, the polar co-ordinates of the point (–1, 1) are $\left(\sqrt{2}, \frac{3\pi}{4}\right)$.

**Ans.** (a)

(xviii) The area of a square inscribed in a circle of diameter $2\sqrt{2}$ cm is :

(a) 4 sq. unit (b) 8 sq. unit (c) $4\sqrt{2}$ sq. unit (d) none of these.

**Solution :** Given, diameter of the circle = $2\sqrt{2}$ = diagonal of the inscribed square.

Therefore, length of the side of the square = 2 cm

Hence required area = $2^2$ sq. unit = 4 sq. unit.

**Ans.** (a)

(xix) The area of the lateral surface of a riht prism with regular octagonal base of side 2 cm and height 6 cm is : (a) $12\sqrt{3}$ sq cm (b) $36\sqrt{3}$ sq. cm (c) 96 sq. cm (d) 72 sq. cm

**Solution :** Lateral surface of right regular octagonal prism of side 2 cm and height 6 cm is = $2\times6\times8$ cm$^2$ = 96 cm$^2$.

**Ans.** (c)

(xx) The latus rectum of the parabola $y^2 = -48x$ is – (a) 12, (b) –12, (c) 48, (d) –48

**Solution :** The latus rectum of the parabola $y^2 = -48x$ is 48

**Ans.** (c)

(xxi) The ratio of the volumes of a cylinder and a cone on same base and of same height is –

(a) 1 : 41, (b) 1 : 2, (c) 3 : 1, (d) 4 : 1

**Solution :** Let height = h and area of the base = A in both the cases.

Then, volume of the cylinder : volume of the cone = Ah : $\frac{1}{3}$Ah = 3 : 1

**Ans.** (c)

(xxii) $\underset{x\to0}{\text{Lt}} \frac{\tan 3x}{\tan x}$ is – (a) 0 (b) 3 (c) 1 (d) $\frac{1}{3}$

**Solution :** $\underset{x\to0}{\text{Lt}} \frac{\tan 3x}{\tan x} = \underset{x\to0}{\text{Lt}} \frac{\frac{\tan 3x}{x}}{\frac{\tan x}{x}}$ $[\because x \neq 0 \text{ as } x \to 0]$

$$= \frac{\underset{3x\to0}{\text{Lt}} \frac{\tan 3x}{3x}\times 3}{\underset{x\to0}{\text{Lt}} \frac{\tan x}{x}} [\because x\to0 \therefore 3x\to0] = \frac{1\times3}{1} = 3$$

**Ans.** (b)

(xxiii) $\frac{d}{dx}\left(\tan^{-1}x\right)$ is (a) $\frac{1}{x}$ (b) $\frac{1}{1+x^2}$ (c) $\frac{1}{1-x^2}$ (d) $\sqrt{\frac{1}{1+x^2}}$

**Solution :** $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$

**Ans.** (b)

(xxiv) If $y = \log x^2$, find $\frac{d^2y}{dx^2}$

**Solution :** Given, $y = \log x^2$

or, $\frac{dy}{dx}=\frac{d}{dx}\left(\log x^2\right)=\frac{1}{x^2}\times 2x=\frac{2}{x}$ [Differentiating both sides w.r.t. x]

or, $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{2}{x}\right)=2\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{2}{x^2}$ [Again differentiating both sides w.r.t. x] **(Ans)**

(xxv) A ball travels s cm in t sec such that $s = 8t + 15t^2$, the velocity of the ball when t = 2 sec is (a) 60 cm/sec (b) 16 cm/sec (c) 68 cm/sec (d) None of these.

**Solution :** Given, $s = 8t + 15t^2$,

Therefore, velocity at t, $\frac{ds}{dt}=\frac{d}{dt}\left(8t+15t^2\right)=8+30t$

At t = 2, $\frac{ds}{dt}=8+30\times 2=68$

**Ans.** (c)

(xxvi) If $f(x) = \log(\sin 3x)$ find $f'(x)$

**Solution :** Given, $f(x) = \log(\sin 3x)$

Therefore, $f'(x)=\frac{d}{dx}\{\log(\sin 3x)\}=\frac{1}{\sin 3x}\frac{d}{dx}(\sin 3x) = \frac{\cos 3x}{\sin 3x}\frac{d}{dx}(3x)=3\cot 3x$ **(Ans)**

(xxvii) If $v=\frac{A}{r}+B$ then $\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}$ is, – (a) 1 (b) $r^2$ (c) 0 (d) $\frac{1}{r^2}$

**Solution :** $v=\frac{A}{r}+B$ then $\frac{dv}{dr}=-\frac{A}{r^2}$ and $\frac{d^2v}{dr^2}=\frac{2A}{r^3}$

$\therefore \frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=\frac{2A}{r^3}+\frac{2}{r}\left(-\frac{A}{r^2}\right)=\frac{2A}{r^3}-\frac{2A}{r^3}=0$

**Ans. :** (c)

## Group – A

**2.** (a) If $\frac{\log x}{y+z} = \frac{\log y}{z+x} = \frac{\log z}{x+y}$ prove that, $\left(\frac{x}{y}\right)^z \times \left(\frac{y}{z}\right)^x \times \left(\frac{z}{x}\right)^y = 1$

**Solution :** Given, $\frac{\log x}{y+z} = \frac{\log y}{z+x} = \frac{\log z}{x+y} = k$ (say)

$\therefore$ $\log x = k(y + z)$ ------------(1) $\log y = k(z + x)$ ------------(2) $\log z = k(x + y)$ ------------(3)

$\therefore$ $(y - z).\log x = k(y^2 - z^2)$, $(z - x)\log y = k(z^2 - x^2)$ and $(x - y)\log z = k(x^2 - y^2)$

$\therefore$ $(y - z)\log x + (z - x)\log y + (x - y)\log z = k(y^2 - z^2 + z^2 - x^2 + x^2 - y^2) = 0$

or, $\log x^{y-z} + \log y^{z-x} + \log z^{x-y} = 0$ or, $\log(x^{y-z}.y^{z-x}.z^{x-y}) = 0$ or, $x^{y-z}.y^{z-x}.z^{x-y} = e^0 = 1$

or, $\frac{x^y}{x^z}.\frac{y^z}{y^x}.\frac{z^x}{z^y} = 1$ or, $\frac{x^z}{y^z}.\frac{y^x}{z^x}.\frac{z^y}{x^y} = 1$ or, $\left(\frac{x}{y}\right)^z \times \left(\frac{y}{z}\right)^x \times \left(\frac{z}{x}\right)^y = 1$ **(Proved)**

(b) If $\alpha$ and $\beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$. find the quadratic equation whose roots are $\frac{1}{\alpha+1}$ and $\frac{1}{\beta+1}$

**Solution:** $\because$ $\alpha, \beta$ are the roots of $ax^2 + bx + c = 0$

$\therefore$ $\alpha+\beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

$\therefore \frac{1}{\alpha+1} + \frac{1}{\beta+1} = \frac{\beta+1+\alpha+1}{(\alpha+1)(\beta+1)} = \frac{\alpha+\beta+2}{\alpha\beta+\alpha+\beta+1} = \frac{-\frac{b}{a}+2}{\frac{c}{a}-\frac{b}{a}+1} = \frac{2a-b}{c-b+a}$ and

$\left(\frac{1}{\alpha+1}\right)\left(\frac{1}{\beta+1}\right) = \frac{1}{(\alpha+1)(\beta+1)} = \frac{1}{\alpha\beta+\alpha+\beta+1} = \frac{1}{\frac{c}{a}-\frac{b}{a}+1} = \frac{a}{c-b+a}$

$\therefore$ the required equation is, $x^2$ – (sum of the roots)x + product of the roots = 0

or, $x^2 - \frac{2a-b}{c-b+a}x + \frac{a}{c-b+a} = 0$ or, $(c - b + a)x^2 - (2a - b)x + a = 0$ **(Ans)**

**3.** (a) If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\phi$, using De Moivre's theorem, show that $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

**Solution :** Given $x + \frac{1}{x} = 2\cos\theta$, or, $2x\cos\theta = x^2 + 1$

or, $x^2 - 2x\cos\theta + 1 = 0$ $\therefore$ $x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\times1\times1}}{2\times1}$ or, $x = \frac{2\cos\theta \pm 2\sqrt{-(1-\cos^2\theta)}}{2} = \cos\theta \pm \sin\theta$

Taking +ve sign only in place of $\pm$, we get $x = \cos\theta + i\sin\theta$,

Similarly, $y = \cos\phi + i\sin\phi$

Now $x^m \cdot y^n = (\cos\theta + i\sin\theta)^m(\cos\phi + i\sin\phi)^n = (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$[By De Moivre's Theorem]

$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$

$\therefore$ $x^{-m} \cdot y^{-n} = [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1} = \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$[By De Moivre's Theorem]

$\therefore$ $x^m y^n + x^{-m}y^{-n} = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$

or, $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$ **(Proved)**

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(b) If the third and fourth terms in the expansion of $\left(2x+\frac{1}{8}\right)^{10}$ are equal find the value of x.

**Solution :** Third term and fourth term in the expansion of $\left(2x+\frac{1}{8}\right)^{10}$ are respectively ${}^{10}C_2(2x)^{10-2}\left(\frac{1}{8}\right)^2$ and ${}^{10}C_3(2x)^{10-3}\left(\frac{1}{8}\right)^3$

By the problem, ${}^{10}C_2(2x)^{10-2}\left(\frac{1}{8}\right)^2 = {}^{10}C_3(2x)^{10-3}\left(\frac{1}{8}\right)^3$ or, ${}^{10}C_2(2x)^8 = {}^{10}C_3(2x)^7\left(\frac{1}{8}\right)$

or, $\frac{10!}{2!\ 8!}\cdot 2x = \frac{10!}{3!\ 7!}\cdot\frac{1}{8}$ or, $\frac{1}{8}\cdot 2x = \frac{1}{3}\cdot\frac{1}{8}$ or, $2x = \frac{1}{3}$ or, $x = \frac{1}{6}$ **(Ans)**

**4.** (a) Find the **unit vector** perpendicular to both the vectors (3i + j + 2k) and (2i – 2j + 4k) and find the angle between them.

**Solution :** Let $\bar{a}$ = 3i + j + 2k) and $\bar{b}$ = 2i – 2j + 4k

Then, $\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = (4+4)i-(12-4)j+(-6-2)k = 8i-8j-8k$

Now $|\bar{a}\times\bar{b}| = |8i-8j-8k| = \sqrt{8^2+8^2+8^2} = 8\sqrt{3}$

Then the unit vector perpendicular to both the given vectors is $\pm\frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|} = \pm\frac{8i-8j-8k}{8\sqrt{3}} = \pm\frac{\sqrt{3}}{3}(i-j-k)$

Now, $|\bar{a}| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$ and $|\bar{b}| = \sqrt{2^2+(-2)^2+4^2} = \sqrt{24}$

Let θ be the angle between the given vectors.

Then $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \frac{6-2+8}{\sqrt{14}\times\sqrt{24}} = \frac{12}{\sqrt{14}\times\sqrt{24}} = \frac{\sqrt{84}}{14}$

or, $\theta = \cos^{-1}\left(\frac{\sqrt{84}}{14}\right)$ = $\cos^{-1}(0.65465367) = 49.1^0$ (approximately).

(b) If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then show that $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$

**Solution :** The roots of the quadratic equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ will be equal, if

Discriminant $(B^2 - 4AC) = 0$

or, $\{b(c-a)\}^2 - 4a(b-c).\ c(a-b) = 0$ or, $b^2(c-a)^2 - 4ac(b-c)(a-b) = 0$

or, $b^2\{(c+a)^2 - 4ac\} - 4ac(b-c)(a-b) = 0$ or, $b^2(c+a)^2 - 4acb^2 - 4ac(ab - ac - b^2 + bc) = 0$

or, $b^2(c+a)^2 - 4ac(b^2 + ab - ac - b^2 + bc) = 0$ or, $\{b(c+a)\}^2 - 4ac(ab + bc) + 4a^2c^2 = 0$

or, $\{b(a+c)\}^2 - 2b(a+c).2ac + (2ac)^2 = 0$ or, $\{b(a+c) - 2ac\}^2 = 0$ or, $b(a+c) - 2ac = 0$

or, $b(a+c) = 2ac$ or, $\frac{a+c}{ac} = \frac{2}{b}$ or, $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$ **(Proved)**

**5.** (a) Find the work done by the forces $2i + 2j + k$ and $3i + 4j - 2k$ acting on a particle displaced from the point $(4, -3, -2)$ to $(6, 1, -3)$.

**Solution :** Given forces are $(2i + 2j + k)$ and $(3i + 4j - 2k)$.

Therefore, the resultant force $= (2i + 2j + k) + (3i + 4j - 2k) = 5i + 6j - k$.

Since, the particle is displaced from the point $(4, -3, -2)$ to $(6, 1, -3)$, the displacement vector

$= (6 - 4)i + (1 + 3)j + (-3 + 2)k = 2i + 4j - k$

Hence the required work done

$= (5i + 6j - k) . (2i + 4j - k) = 10 + 24 + 1 = 35$ unit of work **(Proved)**

(b) Find the partial fraction of $\dfrac{3s}{s^2 + 2s - 8}$ 5

**Solution :** Given, $\dfrac{3s}{s^2 + 2s - 8} = \dfrac{3s}{(s+4)(s-2)} = \dfrac{A}{s+4} + \dfrac{B}{s-2}$ (say)

or, $A(s-2) + B(s+4) = 3s$ ................................ (1)

Putting $s = 2$ we get, $6B = 6$ or, $B = 1$

Putting $s = -4$ we get, $-6A = -12$ or, $A = 2$

Therefore, $\dfrac{3s}{s^2 + 2s - 8} = 2 \cdot \dfrac{1}{s+4} + \dfrac{1}{s-2}$ **(Ans)**

## Group – B

**6.** (a) If $\sin^4 x + \sin^2 x = 1$ show that $\cot^4 x + \cot^2 x = 1$

**Solution:** Given $\sin^4 x + \sin^2 x = 1$ or $\sin^4 x = 1 - \sin^2 x = \cos^2 x$

$\therefore \cot^4 x + \cot^2 x = \dfrac{\cos^4 x}{\sin^4 x} + \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{\cos^4 x}{\cos^2 x} + \dfrac{\sin^4 x}{\sin^2 x} = \cos^2 x + \sin^2 x = 1$ **(Proved)**

(b) If $\tan\frac{\theta}{2} = \tan^3\frac{\phi}{2}$ and $\tan\phi = 2\tan\alpha$, then prove that $\theta + \phi = 2\alpha$

**Solution :** We know, $\tan\left(\frac{\theta}{2} + \frac{\varphi}{2}\right) = \dfrac{\tan\frac{\theta}{2} + \tan\frac{\varphi}{2}}{1 - \tan\frac{\theta}{2} . \tan\frac{\varphi}{2}} = \dfrac{\tan^3\frac{\varphi}{2} + \tan\frac{\varphi}{2}}{1 - \tan^3\frac{\varphi}{2} . \tan\frac{\varphi}{2}}$ $\left[\because \tan\frac{\theta}{2} = \tan^3\frac{\varphi}{2}\right]$

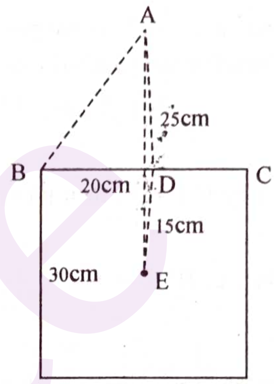
$$= \frac{\tan\frac{\varphi}{2}\left(\tan^2\frac{\varphi}{2} + 1\right)}{1 - \tan^4\frac{\varphi}{2}} = \frac{\tan\frac{\varphi}{2}\left(1 + \tan^2\frac{\varphi}{2}\right)}{\left(1 + \tan^2\frac{\varphi}{2}\right)\left(1 - \tan^2\frac{\varphi}{2}\right)} = \frac{\tan\frac{\varphi}{2}}{1 - \tan^2\frac{\varphi}{2}}$$

$$= \frac{\dfrac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}}{1 - \dfrac{\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}} = \frac{\dfrac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}}{\dfrac{\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}} = \frac{2\sin\frac{\varphi}{2} . \cos\frac{\varphi}{2}}{2\cos\varphi} = \frac{\sin\varphi}{2\cos\varphi} = \frac{\tan\varphi}{2} = \frac{2\tan\alpha}{2} \quad [\because \tan\varphi = 2\tan\alpha]$$

$\therefore \tan\left(\frac{\theta}{2} + \frac{\varphi}{2}\right) = \tan\alpha \quad \therefore \frac{\theta}{2} + \frac{\varphi}{2} = \alpha \Rightarrow \theta + \varphi = 2\alpha$ **(Proved)**

**7.** (a) The height of a right pyramid is 20 cm and its base is a square of side 30 cm. Find the volume and area of the lateral surface.

**Solution :** Height of the pyramid (AE) = 20 cm and DE = 15 cm.




Therefore,

$$AD = \sqrt{20^2 + 15^2} = \sqrt{400 + 225} = \sqrt{625} = 25 \text{ cm.}$$

Area of the lateral surface

$$= 4 \times \frac{1}{2} \times 30 \times 25 = 2 \times 750 = 1500 \text{ sq. cm.}$$

Volume of the pyramid

$$= \frac{1}{3} \times \text{area of the base} \times \text{height}$$

$$= \frac{1}{3} \times 30^2 \times 20 \text{ cu. cm} = 6000 \text{ cc.}$$ **(Ans)**

(b) A circle passes through the points (–3, 4) and (1, 0) and its centre lies on the x-axis. Find the equation of the circle.

**Solution :** Let $x^2 + y^2 + 2gx + 2fy + c = 0$ ............ (1)

be the equation of the circle.

Equation (1) passes through the points (–3, 4) and (1, 0).

$\therefore\ 9 + 16 - 6g + 8f + c = 0$ or, $-6g + 8f + c = -25$ ............ (2)

and, $1 + 0 + 2g + c = 0$ or, $2g + c = -1$ ............ (3)

Again centre of the circle (1) $(-g, -f)$ lies on x-axis i.e., on $y = 0$

$\therefore\ -f = 0$ or, $f = 0$ ............ (4)

From (2) we get, $-6g + c = -25$ ............ (5)

Subtracting (5) from (3) we get, $2g + c + 6g - c = -1 + 25$

or, $8g = 24$ or, $g = 3$

From (5) we get, $c = 6g - 25 = 18 - 25 = -7$

$\therefore$ from (1) we get

$x^2 + y^2 + 6x - 7 = 0$ which is the required equation of the circle. **(Ans)**

**8.** (a) A right prism of height 12 cm. stands on a base which is regular hexagon. If the area of the whole surface of the prism be $1152\sqrt{3}$ sq. cm., find the volume of the prism..

**Solution :** Let a be the length of a side of the regular hexagon.

Then area of the side-faces of the prism = $6ah = 6a \times 12 = 72a$ sq. cm.

Area of the base of the prism = $\frac{na^2}{4}\cot\frac{\pi}{n}$, n = 6 here.

$$= \frac{6a^2}{4}\cot\frac{\pi}{6} = \frac{3a^2}{2} \times \sqrt{3} = \frac{3\sqrt{3}}{2}a^2 \text{ sq. cm.}$$

Area of the whole surface of the prism = $72a + 2 \times \frac{3\sqrt{3}}{2}a^2$

Therefore, $72a + 2 \times \frac{3\sqrt{3}}{2}a^2 = 1152\sqrt{3}$

or, $a^2 + 8\sqrt{3}a - 384 = 0$ [Dividing both sides by $3\sqrt{3}$.]

or, $a^2 + 16\sqrt{3}a - 8\sqrt{3}a - 384 = 0$ or, $\left(a + 16\sqrt{3}\right)\left(a - 8\sqrt{3}\right) = 0$

$\therefore$ either, $a = -16\sqrt{3}$ or, $a = 8\sqrt{3}$

But $a = -16\sqrt{3}$ is not possible. Therefore, $a = 8\sqrt{3}$

Therefore the required volume of the prism = area of its base $\times$ its height

$$= \frac{3\sqrt{3}}{2}a^2 \times 12 = \frac{3\sqrt{3}}{2}\left(8\sqrt{3}\right)^2 \times 12 = 3456\sqrt{3} \text{ c.c. } \textbf{(Ans)}$$

(b) If xy = 1 + $a^2$, then prove that $\tan^{-1}\left(\frac{1}{a+x}\right) + \tan^{-1}\left(\frac{1}{a+y}\right) = \tan^{-1}\left(\frac{1}{a}\right), x + y + 2a \neq 0$

**Solution :** L. H. S. = $\tan^{-1}\left(\frac{1}{a+x}\right) + \tan^{-1}\left(\frac{1}{a+y}\right) = \tan^{-1}\left[\frac{\frac{1}{a+x} + \frac{1}{a+y}}{1 - \left(\frac{1}{a+x}\right)\left(\frac{1}{a+y}\right)}\right]$

$$= \tan^{-1}\left[\frac{\frac{a+y+a+x}{(a+x)(a+y)}}{\frac{(a+x)(a+y)-1}{(a+x)(a+y)}}\right] = \tan^{-1}\left[\frac{x+y+2a}{a^2 + ay + ax + xy - 1}\right]$$

$$= \tan^{-1}\left[\frac{x+y+2a}{a^2 + ay + ax + a^2}\right] \quad [\text{ since, } xy = 1 + a^2 \text{ }]$$

$$= \tan^{-1}\left[\frac{x+y+2a}{a(x+y+2a)}\right] = \tan^{-1}\left(\frac{1}{a}\right) \quad [\because x + y + 2a \neq 0] \textbf{ (Proved).}$$

## Group – C

**9.** (a) Find : (i) A function is defined by

$$F(x) = |x|, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Test the continuity of the function at x = 0.

(ii) $\underset{x \to 0}{\text{Lt}} \frac{e^{3x} - 1}{\sin 5x}$

**Solution : (i)** Given function is

$$f(x) = |x|, \quad x \neq 0$$
$$= 0, \quad x = 0$$

i. e., $f(x) = x, \quad x > 0$
$= -x, \quad x < 0$
$= 0, \quad x = 0$

Test of continuity at x = 0

$$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} (-x) = 0 \qquad [\because f(x) = -x \text{ for } x < 0]$$

$$\underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} x = 0 \qquad [\because f(x) = x, \text{ for } x > 0]$$

and f(0) = 0

$$\therefore \underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} f(x) = f(0) \qquad \text{i.e., } \underset{x \to 0}{\text{Lt}} f(x) = f(0)$$

Hence, from definition of continuity, $f(x)$ is continuous at x = 0. **(Ans)**

**Solution : (ii)** $\underset{x\to 0}{\text{Lt}} \frac{e^{3x}-1}{\sin 5x}$

$= \underset{3x\to 0}{\text{Lt}} \frac{e^{3x}-1}{3x} \times \frac{1}{\underset{5x\to 0}{\text{Lt}} \frac{\sin 5x}{5x}} \times \frac{3}{5}$ $[\because x\to 0 \therefore 3x\to 0, 5x\to 0]$

$= 1\times\frac{1}{1}\times\frac{3}{5} = \frac{3}{5}$ **(Ans)**

(b) Find $\frac{dy}{dx}$ if $y = (\tan x)^{\sin x}$

**Solution :** Given, $y = (\tan x)^{\sin x}$

or, logy = sinx log(tanx) [taking logarithm of both sides]

or, $\frac{d}{dx}(\log y) = \frac{d}{dx}\{\sin x \log(\tan x)\}$ [Differentiating both sides with respect to x,]

or, $\frac{1}{y}\frac{dy}{dx} = \sin x \frac{1}{\tan x}\frac{d}{dx}(\tan x) + \log(\tan x)\frac{d}{dx}(\sin x)$

or, $\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{\tan x}\cdot \sec^2 x + \cos x \log(\tan x) = \frac{\sin x \cdot \cos x}{\sin x \cdot \cos^2 x} + \cos x \log(\tan x) = \sec x + \cos x \log(\tan x)$

or, $\frac{dy}{dx} = y\{\sec x + \cos x \log(\tan x)\}$ or, $\frac{dy}{dx} = (\tan x)^{\sin x}\{\sec x + \cos x \log(\tan x)\}$ **(Ans)**

10. (a) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, show that : $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

**Solution :** Given, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

or, $\sqrt{1-\cos^2\theta} + \sqrt{1-\cos^2\varphi} = a(\cos\theta - \cos\varphi)$ [ Let $x = \cos\theta$, $y = \cos\varphi$ ]

or, $\sin\theta + \sin\varphi = a(\cos\theta - \cos\varphi)$

or, $2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} = 2a\sin\frac{\theta+\varphi}{2}\sin\frac{\varphi-\theta}{2}$ or, $\cos\frac{\theta-\varphi}{2} = -a\sin\frac{\theta-\varphi}{2}$

or, $\frac{\cos\frac{\theta-\varphi}{2}}{\sin\frac{\theta-\varphi}{2}} = -a$ or, $\cot\frac{\theta-\varphi}{2} = -a$ or, $\frac{\theta-\varphi}{2} = \cot^{-1}(-a)$

or, $\theta - \varphi = 2\cot^{-1}(-a)$ or, $\cos^{-1}x - \cos^{-1}y = 2\cot^{-1}(-a)$

Differentiating both sides with respect to x we get,

$-\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0$ or, $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **(Proved)**

(b) Differentiate $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$.

**Solution :** Let $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$, let $x = \tan\theta$

$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$

$\therefore u = \frac{1}{2}\tan^{-1}x = \frac{1}{2}v$ [let $v = \tan^{-1}x$]

$\therefore \frac{du}{dv} = \frac{1}{2}$ **(Ans)** [differentiating with respect to v]

**11.** (a) If $y = (\sin^{-1}x)^2$, prove that $(1 - x^2)y_2 - xy_1 = 2$ where $\frac{dy}{dx} = y_1$, $\frac{d^2y}{dx^2} = y_2$ 5

**Solution :** Given, $y = (\sin^{-1}x)^2$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = 2(\sin^{-1}x) \times \frac{d}{dx}(\sin^{-1}x) = 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$$

or, $\sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1}x$ or, $(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1}x)^2$ [squaring both sides]

or, $(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y$

Again, differentiating both sides with respect to x we get,

$\frac{d}{dx}\left\{(1-x^2)\left(\frac{dy}{dx}\right)^2\right\} = 4\frac{dy}{dx}$ or, $(1-x^2) \times 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \times (-2x) = 4\frac{dy}{dx}$

or, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$ or, $(1 - x^2)y_2 - xy_1 = 2$ **(Proved)**

(b) Show that sinx + cosx will be maximum at $x = \frac{\pi}{4}$ 5

**Solution :** Let f(x) = sinx + cosx

$\therefore$ $f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$

Now at $x = \frac{\pi}{4}$, $f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

and $f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

Hence, at $x = \frac{\pi}{4}$ the given function sinx + cosx ic maximum **(Proved)**

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December, 2017

**MATHEMATICS**

Time Allowed: 3 Hours | Full Marks : 70

Answer to Question No.1 is compulsory and is to be answered first.

This answer is to be made in separate loose script(s) provided for the purpose.

Maximum time allowed is 45 minutes, after which the loose answer scripts will be collected and fresh scripts for answering the remaining part of the question will be provided.

On early submission of answer scripts of Question No.1, a student will get the remaining script earlier.

Answer any <u>five</u> Questions from the rest..

**1.** Answer **<u>any twenty</u>** questions with minimum justification: $20 \times 1 = 20$

(i) The quadratic equation whose sum of the roots is '$-$ p' and product of the roots is 'q', is –

(a) $x^2 - px + q = 0$ (b) $x^2 + px + q = 0$ (c) $x^2 + px - q = 0$ (d) $x^2 - px - q = 0$

**Solution :** The quadratic eqation whose sum of the roots is $-$ p and product of the roots is q is

$x^2$ – (Sum of the roots) x + Product of the roots = 0

or, $x^2 - (-p)x + q = 0$ or, $x^2 + px + q = 0$

**Ans.** (b)

(ii) Conjugate of a complex number $3 - 2i$ is – (a) $-3 - 2i$ (b) $-3 + 2i$ (c) $3 - 2i$ (d) $3 + 2i$

**Solution :** Conjugate of a complex number $3 - 2i$ is $3 + 2i$

**Ans.** (d)

(iii) If $p\cos\theta - q\sin\theta + r = 0$, then the value of $p\sin\theta + q\cos\theta$ in terms of p, q and r is –

(a) $\pm\sqrt{p^2+q^2+r^2}$ (b) $\pm\sqrt{p^2+q^2-r^2}$ (c) $\pm\sqrt{p^2-q^2+r^2}$ (d) none.

**Solution :** Given $p\cos\theta - q\sin\theta + r = 0$

or, $p\cos\theta - q\sin\theta = -r$ or, $(p\cos\theta - q\sin\theta)^2 = (-r)^2$ [ sqauring both sides ]

or, $p^2\cos^2\theta + q^2\sin^2\theta - 2pq\sin\theta\cos\theta = r^2$ or, $p^2(1-\sin^2\theta) + q^2(1-\cos^2\theta) - 2pq\sin\theta\cos\theta = r^2$

or, $p^2\sin^2\theta + q^2\cos^2\theta + 2pq\sin\theta\cos\theta = p^2 + q^2 - r^2$

or, $(p\sin\theta + q\cos\theta)^2 = p^2 + q^2 - r^2$

or, $p\sin\theta + q\cos\theta = \pm\sqrt{p^2+q^2-r^2}$

**Ans :** (b)

(iv) If $\sec\theta + \cos\theta = \sqrt{3}$, then the value of $\sec^3\theta + \cos^3\theta$ is – (a) 0 (b) 2 (c) [illegible] (d) 3

**Solution :** $\sec\theta + \cos\theta = \sqrt{3}$ or, $(\sec\theta + \cos\theta)^3 = (\sqrt{3})^3$ [ cubing both sides ]

or, $\sec^3\theta + \cos^3\theta + 3\sec\theta\cos\theta(\sec\theta + \cos\theta) = 3\sqrt{3}$

or, $\sec^3\theta + \cos^3\theta + 3\sqrt{3} = 3\sqrt{3}$ or, $\sec^3\theta + \cos^3\theta = 0$

**Ans.** (a)

(v) The area of the triangle formed by the co-ordinate axes and the straight line $2x + 3y = 6$ is –

(a) 1 (b) 9 (c) 6 (d) 3

**Solution :** Given, the straight line is $2x + 3y = 6$ or, $\frac{x}{3}+\frac{y}{2}=1$

The straight line intersects the co-ordinate axes at A(3, 0) and B(0, 2).

$\therefore$ OA = 3, OB = 2

$\therefore$ the area of the triangle OAB is $\frac{1}{2}\times OA\times OB=\frac{1}{2}\times 3\times 2=3$ square unit.

**Ans.** (d)

(vi) If $\cos\theta + \sin\theta = 2$, then $\sin 2\theta =$ (a) 1 (b) 0 (c) 3 (d) 2

**Solution :** Given, $\cos\theta + \sin\theta = 2$ or, $(\cos\theta + \sin\theta)^2 = 4$ [ sqauring both sides ]

or, $\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 4$

or, $1 + 2\sin\theta\cos\theta = 4$ or, $\sin 2\theta = 4 - 1$ or, $\sin 2\theta = 3$

**Ans.** (c)

(vii) The value of $\cos 1^0\cos 2^0\cos 3^0\cos 4^0 ...\cos 89^0 ...\cos 95^0\cos 96^0\cos 97^0$ = (a) 0 (b) 1 (c) – 1 (d) none of these

**Solution :** Given, $\cos 1^0\cos 2^0\cos 3^0\cos 4^0 ...\cos 89^0 ...\cos 95^0\cos 96^0\cos 97^0$

$= \cos 1^0\cos 2^0\cos 3^0\cos 4^0 ...\cos 89^0\cos 90^0\cos 91^0 ...\cos 95^0\cos 96^0\cos 97^0 = 0$ [Since, $\cos 90^0 = 0$]

**Ans.** (a)

(viii) If $\log_2 x^2 = 2$, then the value of x is – (a) 2 (b) – 2 (c) $\pm 2$ (d) 4

**Solution :** Given, $\log_2 x^2 = 2$

or, $\frac{\log x^2}{\log 2}=2$ or, $\log x^2 = 2\log 2 = \log 4$ or, $x^2 = 4$ or, $x = \pm 2$

**Ans.** (c)

(ix) If $\frac{x}{(x-2)(x-1)}=\frac{p}{x-2}+\frac{q}{x-1}$, then the value of (p, q) is – (a) (1, 1) (b) (2, 1) (c) (2, –1) (d) (–2, –1)

**Solution :** Given, $\frac{x}{(x-2)(x-1)}=\frac{p}{x-2}+\frac{q}{x-1}$ or, $x = p(x - 1) + q(x - 2)$

Putting $x = 1$ we get, $1 = q(1 - 2)$ or, $q = -1$

Putting $x = 2$ we get, $2 = p(2 - 1)$ or, $p = 2$

Therefore, $p = 2$ and $q = -1$

**Ans.** (c)

(x) If $y = ae^{mx} + be^{-mx}$, then which relation is correct – (a) $\frac{d^2y}{dx^2}=m^2y$ (b) $\frac{d^2y}{dx^2}=-m^2y$ (c) $\frac{d^2y}{dx^2}=my$ (d) none

**Solution :** Given, $y = ae^{mx} + be^{-mx}$

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = ame^{mx} - bme^{-mx} \quad \text{or,} \quad \frac{dy}{dx} = m\left(ae^{mx} - be^{-mx}\right)$$

Again differentiating both sides with respect to x, we get,

$$\frac{d^2y}{dx^2} = m\left(ame^{mx} + bme^{-mx}\right) = m^2\left(ae^{mx} + be^{-mx}\right) = m^2y$$

**Ans.** (a)

(xi) The modulus of $\frac{4-3i}{3+4i}$ is – (a) 5 (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) 1

**Solution :** Modulus of $\frac{4-3i}{3+4i} = \left|\frac{4-3i}{3+4i}\right| = \frac{|4-3i|}{|3+4i|} = \frac{\sqrt{4^2+(-3)^2}}{\sqrt{3^2+4^2}} = \frac{\sqrt{25}}{\sqrt{25}} = 1$

**Ans.** (d)

(xii) The dot product of $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + \vec{j} + 2\vec{k}$ is – (a) 3 (b) 0 (c) 5 (d) 2.

**Solution :** The dot product of $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + \vec{j} + 2\vec{k}$ is

$$\vec{a} \cdot \vec{b} = \left(2\vec{i} - \vec{j} + \vec{k}\right) \cdot \left(\vec{i} + \vec{j} + 2\vec{k}\right) = 2 - 1 + 2 = 3$$

**Ans.** (a)

(xiii) The point (–2, 5) lies in – (a) 1st quadrant (b) 2nd quadrant (c) 3rd quadrant (d) 4th quadrant.

**Solution :** The point (–2, 5) lies in 2nd quadrant

**Ans.** (b)

(xiv) If the distance from the point (2, 3) to the point on the x-axis is 3, then the point on x-axis is –

(a) 3 (b) 2 (c) 1 (d) none of these

**Solution :** Let, any point on the x-axis is (a, 0).

Therefore, by the problem, $\sqrt{(2-a)^2 + (3-0)^2} = 3$

or, $(2 - a)^2 + 9 = 3^2 = 9$ or, $(2 - a)^2 = 0$ or, a = 2

**Ans.** (b)

(xv) The co-ordinates of the centroid of a triangle having vertices (–1, 2), (–1, –1) and (2, –1) are –

(a) (0, 0) (b) (1, 1) (c) (2, –1) (d) (3, 3)

**Solution :** Co-ordinates of the centroid of the triangle whose vertices are (–1, 2), (–1, –1) and (2, –1) are

$$\left(\frac{-1-1+2}{3}, \frac{2-1-1}{3}\right) = (0, 0)$$

**Ans :** (a)

(xvi) If $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ then the value of $\omega^{2015} + \omega^{2016} + \omega^{2017}$ is – (a) $\omega^2$ (b) $1+\omega$ (c) 1 (d) 0

**Solution:** Given, $\omega^{2015} + \omega^{2016} + \omega^{2017} = \omega^{2015}(1 + \omega + \omega^2) = 0 \quad [\because 1 + \omega + \omega^2 = 0]$

**Ans.** (d)

(xvii) If $f(x) = |x-5|$ then the value of f(3) is – (a) – 8 (b) – 2 (c) 2 (d) – 5

**Solution :** Given, $f(x) = |x-5|$

Therefore, $f(3) = |3-5| = |-2| = 2$

**Ans.** (c)

(xviii) If $\underset{x\to 0}{\text{Lt}} \frac{\log_e(1+mx)}{3x} = 1$ then the value of m is – (a) 1 (b) $\frac{1}{3}$ (c) 3 (d) none of these

**Solution :** Given, $\underset{x\to 0}{\text{Lt}} \frac{\log_e(1+mx)}{3x} = 1$ or, $\underset{mx\to 0}{\text{Lt}} \frac{\log_e(1+mx)}{mx} \times \frac{m}{3} = 1$ $[\because x \to 0 \therefore mx \to 0]$

or, $1 \times \frac{m}{3} = 1$ or, $\frac{m}{3} = 1$ or, m = 3

**Ans.** (c)

(xix) If $y = e^{\log_e x}$ then $\frac{dy}{dx}$ = (a) 1 (b) 0 (c) $\frac{1}{x}$ (d) none of these

**Solution :** Given, $y = e^{\log_e x} = x$

Differentiating both sides with respect to x, we get, $\frac{dy}{dx} = 1$

**Ans.** (a)

(xx) The number of terms in the expansion of $(5x + 11y)^7$ is – (a) 6 (b) 7 (c) 8 (d) none of these.

**Solution :** The number of terms in the expansion of $(5x + 11y)^7$ is (7 + 1) = 8

**Ans.** (c)

(xxi) A ball travels s ft. in t sec. where $s = 8t + 10t^2$. The velocity of the ball when t = 2 sec. – (a) 8 ft./sec. (b) 18 ft./sec.(c) 28 ft./sec.(d) 48 ft./sec.

**Solution :** Given, $s = 8t + 10t^2$

Differentiating both sides with respect to t, we get, $\frac{ds}{dt} = 8 + 20t$

Therefore, at t =2 sec., $\frac{ds}{dt} = 8 + 20 \times 2 = 48$

**Ans.** (d)

(xxii) If x is real then the least (minimum) value of $f(x) = 4x^2 - 4x + 1$ is – (a) 0 (b) – 1(c) 1 (d) $\frac{1}{2}$

**Solution :** Given, $f(x) = 4x^2 - 4x + 1 = (2x - 1)^2$

Since, $(2x - 1)^2$ is perfect square, therefore the minimum value of f(x) is 0.

**Ans.** (a)

(xxiii) The ratio of volumes of a cylinder to a cone on the same base and same height is –

(a) 1 : 3 (b) 3 : 1 (c) 2 : 3 (d) 2 : 1

**Solution :** The volume of a cylinder and a cone are $\pi r^2 h$ and $\frac{1}{3}\pi r^2 h$ respectively where r is the radius of the base and h is the height of both of them..

Therefore, the ratio of volumes of a cylinder to a cone on the same base and same height is

$\pi r^2 h : \frac{1}{3}\pi r^2 h = 1 : \frac{1}{3} = 3 : 1$

**Ans.** (b)

(xxiv) The area of the triangle having vertices (0, 0), (a, 0) and (0, b) is – (a) ab (b) $\frac{1}{4}$ab (c) $\frac{1}{8}$ab (d) $\frac{1}{2}$ab

**Solution :** The area of the triangle having vertices (0, 0), (a, 0) and (0, b) is

$\frac{1}{2}[0(0-b) + a(b-0) + 0(0-0)] = \frac{1}{2}ab$

**Ans.** (d)

(xxv) If the vectors $2\vec{i} - \vec{j} + m\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$ are perpendicular to each other, then the value of m is –

(a) 0 (b) – 3 (c) 3 (d) none of these

**Solution :** Given, the vectors $2\vec{i} - \vec{j} + m\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$ are perpendicular to each other.

Therefore, $(2\vec{i} - \vec{j} + m\vec{k}) \bullet (\vec{i} - \vec{j} - \vec{k}) = 0$ or, $2 + 1 - m = 0$ or, $m = 3$

**Ans.** (c)

(xxvi) If $\log_x \log_2 \log_3 81 = 1$, then x = (a) 2 (b) 3 (c) 1 (d) none of these

**Solution :** Given, $\log_x \log_2 \log_3 81 = 1$ or, $\log_x \log_2 \log_3 3^4 = 1$ or, $\log_x \log_2 4 = 1$ [since, $\log_3 3 = 1$]

or, $\log_x \log_2 2^2 = 1$ or, $\log_x 2 = 1$ [since, $\log_2 2 = 1$] or, x = 2

**Ans.** (a)

(xxvii) The eqation $x = a\cos\theta$, $y = b\sin\theta$ represents the parametric eqation of $\theta$ –

(a) ellipse (b) hyperbola (c) circle (d) none of these

**Solution :** Given, $x = a\cos\theta$, $y = b\sin\theta$

or, $\frac{x}{a} = \cos\theta$, $\frac{y}{b} = \sin\theta$ or, $\frac{x^2}{a^2} = \cos^2\theta$, $\frac{y^2}{b^2} = \sin^2\theta$ $\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$

or, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which represents the equation of an ellipse.

**Ans.** (a)

(xxviii) The value of $\sin(\sin^{-1}x + \sec^{-1}\frac{1}{x})$ is – (a) 0 (b) 1 (c) 2 (d) none of these

**Solution :** Given, $\sin(\sin^{-1}x + \sec^{-1}\frac{1}{x}) = \sin(\sin^{-1}x + \cos^{-1}x) = \sin\frac{\pi}{2} = 1$

**Ans.** (b)

**2.**(a) If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, then show that $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$

**Solution :** Given, $x = \log_a(bc)$ or, $x + 1 = \log_a(bc) + \log_a a = \log_a(abc)$

$$\therefore \frac{1}{x+1} = \log_{abc} a$$

Similarly, $\frac{1}{y+1} = \log_{abc} b$, $\frac{1}{z+1} = \log_{abc} c$

$$\therefore \frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1} = \log_{abc}a + \log_{abc}b + \log_{abc}c = \log_{abc}abc = 1$$

$\therefore \frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$ **(Proved)**

(b) If $\alpha$ and $\beta$ be the roots of the eqation $3x^2 - 6x + 3 = 0$, then find the value of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+2\alpha\beta+3(\frac{1}{\alpha}+\frac{1}{\beta})+1$

**Solution:** Given, $\alpha$ and $\beta$ be the roots of the eqation $3x^2 - 6x + 3 = 0$

$\therefore \alpha+\beta = 2$ -------- (1), and $\alpha\beta = 1$ -------- (2)

Now, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+2\alpha\beta+3(\frac{1}{\alpha}+\frac{1}{\beta})+1 = \frac{\alpha^2+\beta^2}{\alpha\beta}+2\alpha\beta+3\left(\frac{\alpha+\beta}{\alpha\beta}\right)+1$

$$= \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}+2\alpha\beta+3\left(\frac{\alpha+\beta}{\alpha\beta}\right)+1 = \frac{2^2-2\times 1}{1}+2\times 1+3\times\frac{2}{1}+1 = 4-2+2+6+1 = 11$$ **(Ans)**

**3.** (a) Find the middle term in the expansion of $\left(3x-\frac{1}{2x}\right)^8$

**Solution :** The number of terms in the expansion of $\left(3x-\frac{1}{2x}\right)^8$ is $(8 + 1) = 9$, which is an odd number.

Therefore, the middle term is $\left(\frac{8}{2}+1\right)$ th = 5th term.

$$\therefore t_5 = t_{4+1} = {}^8C_4 \cdot (3x)^{8-4} \cdot \left(-\frac{1}{2x}\right)^4 = {}^8C_4 \cdot (3x)^4 \cdot \left(\frac{1}{2x}\right)^4 = {}^8C_4 \cdot 3^4 \cdot \frac{1}{2^4}$$

$$= \frac{8!}{4!\,4!}\times\frac{81}{16} = \frac{8\times7\times6\times5\times4!}{4\times3\times2\times1\times4!}\times\frac{81}{16} = \frac{7\times5\times81}{8} = \frac{2835}{8}$$ **(Ans)**

(b) (i) Find the value of $i^4 + i^5 + i^6 + i^7$, where $i = \sqrt{-1}$

**Solution :** Given, $i^4 + i^5 + i^6 + i^7 = (i^2)^2 + i\,(i^2)^2 + (i^2)^3 + i\,(i^2)^3$

$= (-1)^2 + i(-1)^2 + (-1)^3 + i(-1)^3 = 1 + i - 1 - i = 0$ **(Ans)**

(ii) If $\sqrt[3]{x+iy} = a + ib$, then show that $\frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$

**Solution :** Given, $\sqrt[3]{x+iy} = a + ib$

or, $x + iy = (a + ib)^3 = a^3 + 3a^2ib + 3ai^2b^2 + i^3b^3 = a^3 + 3ia^2b - 3ab^2 - ib^3 = (a^3 - 3ab^2) + i(3a^2b - b^3)$

Equating the real and imaginary parts from both sides, we get,

$$x = a^3 - 3ab^2 \text{ and } y = 3a^2b - b^3$$

Now, $\frac{x}{a} + \frac{y}{b} = \frac{a^3 - 3ab^2}{a} + \frac{3a^2b - b^3}{b} = a^2 - 3b^2 + 3a^2 - b^2 = 4(a^2 - b^2)$ **(Proved)**

**4.** (a) If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are three vectors such that $\vec{\alpha} + \vec{\beta} + \vec{\gamma} = 0$ and $|\vec{\alpha}| = 2$, $|\vec{\beta}| = 4$, $|\vec{\gamma}| = 6$ then show that $\vec{\alpha}.\vec{\beta} + \vec{\beta}.\vec{\gamma} + \vec{\gamma}.\vec{\alpha} = -28$

**Solution:** By the problem, $|\vec{\alpha}| = 2$, $|\vec{\beta}| = 4$, $|\vec{\gamma}| = 6$.

Now $\vec{\alpha} + \vec{\beta} + \vec{\gamma} = 0$ gives $|\vec{\alpha} + \vec{\beta} + \vec{\gamma}|^2 = 0$.

or $|\vec{\alpha}|^2 + |\vec{\beta}|^2 + |\vec{\gamma}|^2 + 2(\vec{\alpha}\cdot\vec{\beta} + \vec{\beta}\cdot\vec{\gamma} + \vec{\gamma}\cdot\vec{\alpha}) = 0$ or $4 + 16 + 36 + 2(\vec{\alpha}\cdot\vec{\beta} + \vec{\beta}\cdot\vec{\gamma} + \vec{\gamma}\cdot\vec{\alpha}) = 0$

or $2(\vec{\alpha}\cdot\vec{\beta} + \vec{\beta}\cdot\vec{\gamma} + \vec{\gamma}\cdot\vec{\alpha}) = -56$

Therefore, $\vec{\alpha}\cdot\vec{\beta} + \vec{\beta}\cdot\vec{\gamma} + \vec{\gamma}\cdot\vec{\alpha} = -28$ **(Proved)**

(b) (i) The two forces $3\vec{i} + 2\vec{j} + 5\vec{k}$ and $2\vec{i} + \vec{j} - 3\vec{k}$ act on a particle and displaced from $2\vec{i} - \vec{j} - 3\vec{k}$ to the point $4\vec{i} - 3\vec{j} + 7\vec{k}$. Calculate the total work done.

**Solution:** Let $\vec{F}_1 \cdot \vec{F}_2$ be the given two forces.

Then, $\vec{F}_1 = 3\vec{i} + 2\vec{j} + 5\vec{k}$, $\vec{F}_2 = 2\vec{i} + \vec{j} - 3\vec{k}$

The resultant force, $\vec{F} = \vec{F}_1 + \vec{F}_2 = (3\vec{i} + 2\vec{j} + 5\vec{k}) + (2\vec{i} + \vec{j} - 3\vec{k}) = 5\vec{i} + 3\vec{j} + 2\vec{k}$

Given two points are $2\vec{i} - \vec{j} - 3\vec{k}$ and $4\vec{i} - 3\vec{j} + 7\vec{k}$

Therefore displacement $\vec{d} = (4\vec{i} - 3\vec{j} + 7\vec{k}) - (2\vec{i} - \vec{j} - 3\vec{k}) = 2\vec{i} - 2\vec{j} + 10\vec{k}$

Hence work done by the forces

$= \vec{F}\cdot\vec{d} = (5\vec{i} + 3\vec{j} + 2\vec{k})\cdot(2\vec{i} - 2\vec{j} + 10\vec{k}) = 10 - 6 + 20 = 24$ unit. **(Ans)**

(ii) Find $\vec{\alpha} \times \vec{\beta}$, where $\vec{\alpha} = 3\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{\beta} = \vec{j} - 3\vec{k}$

**Solution :** Given, $\vec{\alpha} = 3\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{\beta} = \vec{j} - 3\vec{k}$

Therefore, $\vec{\alpha} \times \vec{\beta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 5 \\ 0 & 1 & -3 \end{vmatrix} = \vec{i}(-6-5) + \vec{j}(0+9) + \vec{k}(3-0) = -11\vec{i} + 9\vec{j} + 3\vec{k}$ **(Ans)**

**5.**(a) Find the eqation of the straight line which passes through the point of intersection of the straight lines $x + y + 9 = 0$ and $3x - 2y + 2 = 0$ and is perpendicular to the straight line $4x + 5y + 1 = 0$.

**Solution :** Given eqation of the straight lines are

$x + y + 9 = 0$ ................ (1) and $3x - 2y + 2 = 0$ ...........(2)

Using rule of cross-multiplication we get,

$\frac{x}{2+18}=\frac{y}{27-2}=\frac{1}{-2-3}$ or, $\frac{x}{20}=\frac{y}{25}=\frac{1}{-5} \Rightarrow x=-4,\ y=-5$

$\therefore$ point of intersection of (1) and (2) is $(-4, -5)$.

Now equation of any straight line perpendicular to $4x + 5y + 1 = 0$ is $5x - 4y = k$

If it passes through the point $(-4, -5)$ then $5 \times (-4) - 4 \times (-5) = k$ or, $k = 0$

$\therefore$ equation of the required straight line is $5x - 4y = 0$ **(Ans).**

(b) (i) Find the equation of the circle passing through (4, 5) having the centre at (2, 2).

**Solution :** Given centre of the circle is (2, 2) and it passes through (4, 5).

$\therefore$ radius of the circle $= \sqrt{(2-4)^2+(2-5)^2} = \sqrt{4+9} = \sqrt{13}$ unit.

Hence the required equation of the circle is

$(x-2)^2 + (y-2)^2 = \left(\sqrt{13}\right)^2$ or, $x^2 + y^2 - 4x - 4y + 4 + 4 = 13$

or, $x^2 + y^2 - 4x - 4y = 5$ **(Ans)**

(ii) Find the equation of the circle which touches both the axes and passes through the point (6, 3).

**Solution :** Let (a, a) be the centre of the circle and radius is a. [since the circle touches both the axes.]

Then the equation of the circle is $(x-a)^2 + (y-a)^2 = a^2$ ............... (1)

If it passes through the point (6, 3) we get,

$(6-a)^2 + (3-a)^2 = a^2$ or, $36 - 12a + a^2 + 9 - 6a + a^2 = a^2$

or, $a^2 - 18a + 45 = 0$ or, $(a-15)(a-3) = 0$ or, $a = 3, 15$

Therefore, the required equation of the circle is

$(x-3)^2 + (y-3)^2 = 9$ or, $(x-15)^2 + (y-15)^2 = 225$ **(Ans)**

6. (a) The height of a right pyramid is 20 cm and its base is a square of a side 30 cm. Find the volume and area of the lateral surface.

**Solution:** Area of the base of the pyramid $= (30)^2 = 900\ cm^2$

Height of the pyramid = 20 cm.

Therefore, volume of the pyramid $= \frac{1}{3} \times 900 \times 20 = 6000\ cm^3$ **(Ans)**

Let, height of the lateral surface = h cm

Therefore, $h^2 = 20^2 + 15^2 = 400 + 225 = 625$ or, $h = 25$ cm.

Therefore, area of the lateral surface $= 4 \times \frac{1}{2} \times 30 \times 25 = 1500\ cm^2$ **(Ans)**

(b) If $\sin^4\theta + \sin^2\theta = 1$, then show that $\cot^4\theta + \cot^2\theta = 1$

**Solution :** Given, $\sin^4\theta + \sin^2\theta = 1$

or, $\sin^4\theta = 1 - \sin^2\theta = \cos^2\theta$

or, $\frac{\cos^2\theta}{\sin^4\theta}=1$ or, $\cot^2\theta\operatorname{cosec}^2\theta=1$ or, $\cot^2\theta\left(\cot^2\theta+1\right)=1$

or, $\cot^4\theta+\cot^2\theta=1$ **(Proved)**

7. (a) If $2\tan\alpha=3\tan\beta$, show that $\tan(\alpha-\beta)=\frac{\sin 2\beta}{5-\cos 2\beta}$

**Solution :** Given, $2\tan\alpha=3\tan\beta$ or, $\tan\alpha=\frac{3}{2}\tan\beta$

$$\therefore\ \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}=\frac{\frac{3}{2}\tan\beta-\tan\beta}{1+\frac{3}{2}\tan\beta\cdot\tan\beta}=\frac{3\tan\beta-2\tan\beta}{2+3\tan^2\beta}$$

$$=\frac{\tan\beta}{2+3\tan^2\beta}=\frac{\frac{\sin\beta}{\cos\beta}}{2+\frac{3\sin^2\beta}{\cos^2\beta}}=\frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta}=\frac{2\sin\beta\cos\beta}{2\left(2\cos^2\beta\right)+3\left(2\sin^2\beta\right)}$$

$$=\frac{\sin 2\beta}{2(1+\cos 2\beta)+3(1-\cos 2\beta)}=\frac{\sin 2\beta}{2+2\cos 2\beta+3-3\cos 2\beta}=\frac{\sin 2\beta}{5-\cos 2\beta}$$ **(Proved)**

(b) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, show that $xy+yz+zx=1$

**Solution :** Given, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$

$$\text{or, }\tan^{-1}\left(\frac{x+y+z-xyz}{1-yz-zx-xy}\right)=\frac{\pi}{2}\quad\text{or, }\frac{x+y+z-xyz}{1-yz-zx-xy}=\tan\frac{\pi}{2}$$

or, $1-yz-zx-xy=0$ or, $xy+yz+zx=1$ **(Proved)**

8. (a) **Evaluate:** (i) $\underset{x\to 0}{\mathrm{Lt}}\frac{e^{2x}-e^{3x}}{\log_e(1+x)}$

**Solution :** Given, $\underset{x\to 0}{\mathrm{Lt}}\frac{e^{2x}-e^{3x}}{\log_e(1+x)}=\underset{x\to 0}{\mathrm{Lt}}\frac{\frac{e^{2x}-e^{3x}}{x}}{\frac{\log_e(1+x)}{x}}$ $[\because\ x\neq 0]$

$$=\frac{\underset{x\to 0}{\mathrm{Lt}}\frac{e^{2x}-e^{3x}}{x}}{\underset{x\to 0}{\mathrm{Lt}}\frac{\log_e(1+x)}{x}}=\frac{\underset{x\to 0}{\mathrm{Lt}}\frac{\left(e^{2x}-1\right)-\left(e^{3x}-1\right)}{x}}{1}=\underset{x\to 0}{\mathrm{Lt}}\frac{e^{2x}-1}{x}-\underset{x\to 0}{\mathrm{Lt}}\frac{e^{3x}-1}{x}$$

$$=2\times\underset{2x\to 0}{\mathrm{Lt}}\frac{e^{2x}-1}{2x}-3\times\underset{3x\to 0}{\mathrm{Lt}}\frac{e^{3x}-1}{3x}\quad[\because x\to 0\ \therefore 2x\to 0,3x\to 0]$$

$=2\times 1-3\times 1=2-3=-1$ **(Ans)**

(ii) **Evaluate :** $\underset{x\to 0}{\text{Lt}}\dfrac{2^x-1}{\sqrt{1+x}-1}$

**Solution :** Given, $\underset{x\to 0}{\text{Lt}}\dfrac{2^x-1}{\sqrt{1+x}-1}=\underset{x\to 0}{\text{Lt}}\left[\dfrac{\frac{2^x-1}{x}}{\frac{\sqrt{1+x}-1}{x}}\right]$ $[\because x\neq 0]$

$$=\frac{\underset{x\to 0}{\text{Lt}}\frac{2^x-1}{x}}{\underset{x\to 0}{\text{Lt}}\frac{\sqrt{1+x}-1}{x}}=\frac{\log_e 2}{\underset{x\to 0}{\text{Lt}}\frac{(\sqrt{1+x}+1)(\sqrt{1+x}-1)}{x(\sqrt{1+x}+1)}}=\frac{\log_e 2}{\underset{x\to 0}{\text{Lt}}\frac{1+x-1}{x(\sqrt{1+x}+1)}}$$

$$=\frac{\log_e 2}{\underset{x\to 0}{\text{Lt}}\frac{x}{x(\sqrt{1+x}+1)}}=\frac{\log_e 2}{\underset{x\to 0}{\text{Lt}}\left(\frac{1}{\sqrt{1+x}+1}\right)}\ [\because x\neq 0]=\frac{\log_e 2}{\frac{1}{1+1}}=2\log_e 2 \ \textbf{(Ans)}$$

(b) (i) If $\alpha(x)=\log_e\sin x$ and $\beta(x)=\log_e\cos x$, then find the value of $e^{2\alpha(x)}+e^{2\beta(x)}$

**Solution:** Given, $\alpha(x)=\log_e\sin x$ and $\beta(x)=\log_e\cos x$

Therefore, $e^{2\alpha(x)}+e^{2\beta(x)}=e^{2\log_e\sin x}+e^{2\log_e\cos x}$

$=e^{\log_e\sin^2 x}+e^{\log_e\cos^2 x}=\sin^2 x+\cos^2 x=1$ **(Ans)**

(ii) Discuss the continuity of the following function at x = 2, $f(x)=\begin{cases}x^2+2, & 0\le x\le 2\\ x+4, & 2\le x\le 3\\ 2, & x=2\end{cases}$

**Solution:** Here,

$\underset{x\to 2-}{\text{Lt}}f(x)=\underset{x\to 2-}{\text{Lt}}(x^2+2)=2^2+2=6$ $[\because f(x)=x^2+2, \text{ when } x<2]$

$\underset{x\to 2+}{\text{Lt}}f(x)=\underset{x\to 2+}{\text{Lt}}(x+4)=2+4=6$ $[\because f(x)=x+4, \text{ when } x>2]$

and $f(2)=2+4=6$

$\because \underset{x\to 2-}{\text{Lt}}f(x)=\underset{x\to 2+}{\text{Lt}}f(x)=f(2)$ i.e., $\underset{x\to 2}{\text{Lt}}f(x)=f(2)$

Hence, from definition of continuity f(x) is continuous at x = 2.

**9.** (a) If $f(x)=\left(\frac{a+x}{1+x}\right)^x$, find the value of $f'(0)$

**Solution :** Given, $f(x)=\left(\frac{a+x}{1+x}\right)^x$ or, $\log f(x)=x\log\left(\frac{a+x}{1+x}\right)$ [taking log of both sides]

or, $\log f(x)=x\{\log(a+x)-\log(1+x)\}$

Differentiating both sides with respect to x we get,

$$\frac{1}{f(x)}\cdot f'(x) = x\left(\frac{1}{a+x} - \frac{1}{1+x}\right) + \log\left(\frac{a+x}{1+x}\right)$$

Putting x = 0, we get, $\frac{1}{f(0)}\cdot f'(0) = 0\times\left(\frac{1}{a} - 1\right) + \log a = \log a$

or, $f'(0) = f(0)\log a = \log a \quad \left[\because f(x) = \left(\frac{a+x}{1+x}\right)^x \therefore f(0) = a^0 = 1\right]$ **(Ans)**

(b) (i) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

**Solution:** Given, $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}} = \tan^{-1}\sqrt{\tan^2\frac{x}{2}} = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$

Now, differentiating both sides with respect to x, we get,

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2}$ **(Ans)**

(ii) Find the derivative of sinx with respect to $\log_e x$.

**Solution:** Let, U = sinx and V = $\log_e x$ or, x = $e^V$

Therefore, U = $\sin e^V$ [Since, x = $e^V$]

Now, differentiating both sides with respect to V, we get,

$\frac{dU}{dV} = \frac{d}{dV}\left(\sin e^V\right) = e^V \cos e^V = x \cos x$

Therefore, $\frac{dU}{dV} = x \cos x$ **(Ans)**

**10.** (a) If $y = e^{a\sin^{-1}x}$, then prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

**Solution :** Given, $y = e^{a\sin^{-1}x}$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = e^{a\sin^{-1}x} \times \frac{d}{dx}\left(a\sin^{-1}x\right) = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

$\left(1-x^2\right)\times 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = 2a^2y\frac{dy}{dx}$ [Again, differentiating both sides with respect to x ]

or, $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$ or, $\left(1-x^2\right)y_2 - xy_1 - a^2y = 0$ **(Proved)**

b) A particle is moving along a straight line, on which O is a point. After time t, the distance of the particle from O is given by acosnt + bsinnt (a, b, n are constants). Prove that the acceleration of the particle is proportional to its distance from O.

**Solution :** Let, $s = a\cos nt + b\sin nt$

Differentiating both sides with respect to t, we get,

$$\frac{ds}{dt} = an(-\sin nt) + bn\cos nt = n\,(b\cos nt - a\sin nt)$$

Again, differentiating both sides with respect to t, we get,

$$\frac{d^2s}{dt^2} = n\,(-bn\sin nt - an\cos nt) = -n^2(a\sin nt + b\cos nt) = -n^2 s$$

or, $\frac{d^2s}{dt^2} \propto s$ which shows that, the acceleration of the particle is proportional to its distance from O. **(Proved)**

**11.** (a) If $A + B = 45^0$, prove that $(1 + \tan A)(1 + \tan B) = 2$. Hence show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$

**Solution :** Given, $A + B = 45°$ or, $\tan(A + B) = \tan 45°$ or, $\frac{\tan A + \tan B}{1 - \tan A.\tan B} = 1$

or, $\tan A + \tan B = 1 - \tan A.\tan B$ or, $\tan A + \tan B + \tan A\tan B = 1$

or, $(1 + \tan A) + \tan B(1 + \tan A) = 2$ or, $(1 + \tan A)(1 + \tan B) = 2$——(1) **(Proved)**

Now, putting B = A in A + B = 45° we get,

$A + A = 45°$ or, $2A = 45° \Rightarrow A \quad \therefore\ A = \frac{45^0}{2} = 22\frac{1}{2}^0$

And from (1) we get, $(1 + \tan A)(1 + \tan A) = 2$ or, $(1 + \tan A)^2 = 2$

or, $1 + \tan A = \sqrt{2}$ or, $\tan 22\frac{1}{2}^{\circ} = \sqrt{2} - 1$ or, $\tan\frac{\pi}{8} = \sqrt{2} - 1$ **(Proved)**

(b) If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then find the value of $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

**Solution:** Given, $x = a\cos^3\theta$ .......(1), $y = a\sin^3\theta$ .........(2)

Differentiating both sides of (1) and (2) with respect to $\theta$ we get,

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\sin\theta\cos^2\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Therefore, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{3a\sin\theta\cos^2\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

Now, $$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (-\tan\theta)^2} = \sqrt{1 + \tan^2\theta} = \sec\theta \quad \textbf{(Ans)}$$

December, 2018

**MATHEMATICS**

Time Allowed: 3 Hours Full Marks : 70

Answer to Question No.1 is compulsory and is to be answered first.

This answer is to be made in separate loose script(s) provided for the purpose.

Maximum time allowed is 45 minutes, after which the loose answer scripts will be collected and fresh scripts for answering the remaining part of the question will be provided.

On early submission of answer scripts of Question No.1, a student will get the remaining script earlier.

Answer any five questions from Group - A, B and C, taking at least one from each group.

**1.** Answer questions with minimum justification (any twenty):

$20 \times 1 = 20$

(i) The logarithm of 729 to the base 3 is – (a) 6 (b) 5 (c) 7 (d) none of these

**Solution :** The logarithm of 729 to the base 3 is $= \log_3 729 = \log_3 3^6 = 6\log_3 3 = 6 \times 1 = 6$

**Ans.** (a)

(ii) If one root of the equation $x^2 - 6x + m = 0$ be double of the other, then m is – (a) 4 (b) 6 (c) 8 (d) none

**Solution :** Let a, 2a be the roots of $x^2 - 6x + m = 0$

Therefore, $a + 2a = 6$ or, $3a = 6$ or, $a = 2$ So two roots are 2, 4

Therefore, product of the roots, $2 \times 4 = m$ or, $m = 8$

**Ans.** (c)

(iii) The coefficient of $x^{-3}$ in the expansion of $\left(1-\frac{1}{x}\right)^{10}$ is – (a) 120 (b) 45 (c) 60 (d) none of these.

**Solution :** Let (r + 1)th term contains $x^{-3}$ in the expansion of $\left(1-\frac{1}{x}\right)^{10}$

Here, (r + 1)th term, $t_{r+1} = {}^{10}C_r \cdot (1)^{10-r} \cdot \left(-\frac{1}{x}\right)^r = (-1)^r . {}^{10}C_r \cdot x^{-r} \quad \therefore -r = -3 \Rightarrow r = 3$

$\therefore$ (r + 1)th term = (3 + 1)th = 4th term contains $x^{-3}$.

The coefficient of $x^{-3} = (-1)^3 \, {}^{10}C_3 = -\frac{10!}{3!.7!} = -\frac{10.9.8.7!}{3.2.1.7!} = -120$

**Ans :** (d)

(iv) The square root of i is – (a) $\pm\frac{1+i}{\sqrt{2}}$ (b) $\pm\frac{1-i}{\sqrt{2}}$ (c) 1 (d) none of these

**Solution :** Since, $i = \frac{1}{2}(2i) = \frac{1}{2}(1+2i-1) = \frac{1}{2}(1^2+2i+i^2) = \frac{1}{2}(1+i)^2$

$\therefore$ the required square roots are $= i = \frac{1}{2}(2i) = \frac{1}{2}(1+2i-1) = \frac{1}{2}(1^2+2i+i^2) = \frac{1}{2}(1+i)^2$

**Ans.** (a)

(v) The amp. of $\sqrt{3}-i$ is – (a) $60^0$ (b) $-30^0$ (c) $30^0$ (d) none of these

**Solution :** In the z- plane the point $z=\sqrt{3}-i$ lies in the 4th quadrant.

Let amp(z) = θ, $\therefore\ \tan\theta=\frac{-1}{\sqrt{3}}$ where $-\frac{\pi}{2}<\theta<0$

or, $\tan\theta=\tan\left(-\frac{\pi}{6}\right)$ or, $\theta=-\frac{\pi}{6}=-30^0$ $\therefore$ the required amplitude is $-30^0$

**Ans.** (b)

(vi) The angle between the two vectors i – 3j – 5k and 2i – j + k is – (a) $120^0$ (b) $45^0$ (c) $90^0$ (d) none of these

**Solution :** Scalar product of the given vectors i – 3j – 5k and 2i – i + k = (i – 3j – 5k)·(2i – i + k)

$$= 1\times2 + (-3)\times(-1) + (-5)\times(1) = 2 + 3 + -5 = 0$$

Therefore, given two vectors are perpendicular to each other.

**Ans.** (c)

(vii) The unit vector along 2i – j – 2k is (a) $\frac{2i-j-2k}{3}$ (b) $\frac{2i-j-2k}{\sqrt{3}}$ (c) $\frac{2i-j-2k}{6}$ (d) none of these

**Solution:** Required unit vector along 2i – j – 2k is

$$=\frac{2i-j-2k}{\sqrt{2^2+(-1)^2+(-2)^2}}=\frac{2i-j-2k}{\sqrt{4+1+4}}=\frac{2i-j-2k}{\sqrt{9}}=\frac{2i-j-2k}{3}$$

**Ans.** (a)

(viii) The value of jx(kxi) is – (a) 1 (b) 0 (c) j (d) none of these

**Solution :** ??

(ix) The value of sec($-945^0$) is – (a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $-\frac{1}{\sqrt{2}}$ (d) none of these

**Solution :** Given, sec($-945^0$) = sec$945^0$ [$\because$ sec($-\theta$) = secθ]

$$= \sec(90^0\times10+45^0) = -\sec45^0 = -\sqrt{2}$$

**Ans.** (a)

(x) If tanxtan3x = 1, then the value of tan2x is – (a) 2 (b) 1 (c) $-\sqrt{3}$ (d) none of these.

**Solution :** Given, tanx tan3x = 1 or, tan3x = cotx or, tan3x = tan($90^0$ – x)

or, 3x = $90^0$ – x or, 4x = $90^0$ or, 2x = $45^0$ Therefore, tan2x = tan$45^0$ = 1

**Ans.** (b)

(xi) The minimum value of sinθ + cosθ is – (a) 0 (b) – 1 (c) $-\sqrt{2}$ (d) $\sqrt{2}$

**Solution :** Given expression, sinθ + cosθ = $\sqrt{(\sin\theta+\cos\theta)^2}$

$=\sqrt{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}=\sqrt{1+\sin2\theta}$, which shows that minimum value of sinθ + cosθ is 0, since minimum value of sin2θ = –1

**Ans.** (a)

(xii) Value of $\sin\left[\tan^{-1}x+\tan^{-1}\frac{1}{x}\right]$ is – (a) 0 (b) 1 (c) – 1 (d) $\sqrt{2}$.

**Solution :** Given, $\sin\left[\tan^{-1}x+\tan^{-1}\frac{1}{x}\right]=\sin\left[\tan^{-1}x+\cot^{-1}x\right]=\sin\frac{\pi}{2}=1$

**Ans.** (b)

(xiii) If a + b : b + c : c + a = 13 : 11 : 12, then CosA = (a) $\frac{1}{6}$ (b) 5 (c) $\frac{1}{7}$ (d) $\frac{1}{5}$.

**Solution :** Given, a + b : b + c : c + a = 13 : 11 : 12

or, $\frac{a+b}{13}=\frac{b+c}{11}=\frac{c+a}{12}==k$ (say)

or, a + b = 13k -- (1), b + c = 11k -- (2), c + a = 12k -- (3)

or, 2(a + b + c) = 13k + 11k + 12k = 36k [adding (1), (2), (3)] or, a + b + c = 18k ------- (4)

or, c = 5k [(4) – (1)], a = 7k [(4) – (2)], b = 6k [(4) – (3)]

Now, $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{36k^2+25k^2-49k^2}{60k^2}=\frac{61-49}{60}=\frac{12}{60}=\frac{1}{5}$

**Ans.** (d)

(xiv) The cartesian co-ordinates of the point (3, 30$^0$) are – (a) $\left(\frac{3}{2},\frac{3}{2}\right)$ (b) $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$ (c) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$ (d) none.

**Solution :** Let, (x, y) are the cartesian co-ordinates.

Therefore, $x = 3\cos 30^0=\frac{3\sqrt{3}}{2}$ and $y = 3\sin 30^0=\frac{3}{2}$.

**Ans.** (b)

(xv) The x and y intercept of the straight line 3x + 4y =12 are – (a) 3, 3 (b) 4, 5 (c) 4, 3 (d) none of these

**Solution :** Given, 3x + 4y =12 or, $\frac{x}{4}+\frac{y}{3}=1$ Therefore, x and y intercept are 4, 3

**Ans :** (c)

(xvi) The distance between the lines 3x + 4y =9 and 6x + 8y = 15 is – (a) $\frac{3}{2}$ (b) $\frac{3}{10}$ (c) 6 (d) none of these

**Solution:** Given, two lines are 3x + 4y =9 and 6x + 8y = 15

i. e., 3x + 4y =9 and 3x + 4y = $\frac{15}{2}$

Therefore, the required distance $=\frac{9-\frac{15}{2}}{\sqrt{3^2\;4^2}}=\frac{\frac{3}{2}}{\sqrt{25}}=\frac{\frac{3}{2}}{5}=\frac{3}{10}$

**Ans.** (b)

(xvii) The angle between the lines x + y =3 and y = x – 3 is – (a) 0$^0$ (b) 45$^0$ (c) 60$^0$ (d) 90$^0$

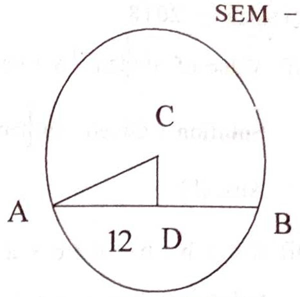
**Solution :** Gradient of the straight line, x + y = 3 is $(m_1)$ = –1 and that of y = x – 3 is $(m_2)$ = 1

Since, $m_1m_2$ = (–1)(1) = – 1, the two straight lines are perpendicular to each other.

**Ans.** (d)

(xviii) A chord of a circle measures 24 cm and its distance from the centre is 4.

Then the radius of the circle is – (a) 16 (b) 12.64 (c) 14 (d) none of these




**Solution :** By the problem,

AB = 24 cm, therefore, AD = 12 cm and CD = 4 cm

Therefore, radius of the circle,

$CA = \sqrt{12^2+4^2} = \sqrt{144+16} = \sqrt{160} = 4\sqrt{10}$ cm

**Ans.** (d)

(xix) A right pyramid of height 15 cm stands on a square base of side 16 cm. Its volume is –

(a) 1260 cc (b) 1280 cc (c)1020 cc (d) none of these

**Solution :** Area of a square of side 16 cm. = $16^2$ = 256 sq. cm.

Given, height of the pyramid = 15 cm.

$\therefore$ Volume $= \frac{1}{3} \times$ area of the base $\times$ height $= \frac{1}{3} \times 256 \times 15$ cc $= 256 \times 5$ cc $= 1280$ cc

**Ans.** (b)

(xx) The volume of a right prism is 36 cft and area of its base is 6 sq. ft. The height of this prism is –

(a) 4 ft (b) 12 ft (c) 6 ft (d) none of these.

**Solution :** Let h be the height of the prism.

Then, Area of the base of the prism $\times h$ = Volume

or, $6 \times h = 36$ or, h = 6

**Ans.** (c)

(xxi) The domain of the function $f(x) = \frac{1}{\sqrt{6-x}}$ is – (a) $[-\infty, 6]$ (b) $(-\infty, 6)$ (c) $(-\infty, 6]$ (d) none of these

**Solution :** Given, $f(x) = \frac{1}{\sqrt{6-x}}$

The domain of the function is given by, 6 – x > 0 or, x < 6 i. e., $-\infty < x < 6$ i. e., $(-\infty, 6)$

**Ans.** (b)

(xxii) The value of $\underset{x \to 0}{\text{Lt}} \frac{\cos x}{\frac{\pi}{2} - x}$ is – (a) 1 (b) 0 (c) 2 (d) –1

**Solution :** Given, $\underset{x \to 0}{\text{Lt}} \frac{\cos x}{\frac{\pi}{2} - x} = \underset{\frac{\pi}{2} - x \to 0}{\text{Lt}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x} = 1$

**Ans.** (a)

(xxiii) If f(x) = $\log e^x + e^{\log x}$ then $f'(x)$ is – (a) 2 (b) 1 (c) 2x (d) x

**Solution :** Given, f(x) = $\log e^x + e^{\log x} = x + x = 2x$ Therefore $\frac{d}{dx}(2x) = 2$

**Ans.** (a)

(xviii)A chord of a circle measures 24 cm and its distance from the centre is 4. Then the radius of the circle is –

(a) 16 (b) 12.64 (c) 14 (d) none of these

**Solution :** By the problem,

AB = 24 cm, therefore, AD = 12 cm and CD = 4 cm

Therefore, Radius of the circle,

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Given, height of the pyramid = 15 cm.

$\therefore$ Volume = $\frac{1}{3}\times$ area of the base $\times$ height = $\frac{1}{3}\times 256\times 15$ cc = $256\times 5$ cc = 1280 cc

**Ans.** (b)

(xx) The volume of a right prism is 36 cft and area of its base is 6 sq. ft. The height of this prism is –

(a) 4 ft (b) 12 ft (c) 6 ft (d) none of these.

**Solution :** Let h be the height of the prism.

Then, Area of the base of the prism $\times h$ = Volume

or, $6\times h = 36$ or, h = 6

**Ans.** (c)

(xxi) The domain of the function $f(x)=\frac{1}{\sqrt{6-x}}$ is – (a) $[-\infty,6]$ (b) $(-\infty,6)$ (c) $(-\infty,6]$ (d) none of these

**Solution :** Given, $f(x)=\frac{1}{\sqrt{6-x}}$

The domain of the function is given by, $6-x>0$ or, $x<6$ i. e., $-\infty<x<6$ i. e., $(-\infty, 6)$

**Ans.** (b)

(xxii) The value of $\underset{x\to 0}{\text{Lt}}\frac{\cos x}{\frac{\pi}{2}-x}$ is – (a) 1 (b) 0 (c) 2 (d) –1

**Solution :** Given, $\underset{x\to 0}{\text{Lt}}\frac{\cos x}{\frac{\pi}{2}-x} = \underset{\frac{\pi}{2}-x\to 0}{\text{Lt}}\frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x} = 1$

**Ans.** (a)

(xxiii)If $f(x) = \log e^x + e^{\log x}$ then $f'(x)$ is – (a) 2 (b) 1 (c) 2x (d) x

**Solution :** Given, $f(x) = \log e^x + e^{\log x} = x + x = 2x$ Therefore $\frac{d}{dx}(2x) = 2$

**Ans.** (a)

(xxiv) If $y = \log_e\left(\tan\frac{x}{2}\right)$ then $\frac{d^2y}{dx^2}$ is – (a) $-\operatorname{cosec}^2 x$ (b) $-\operatorname{cosec}x\cot x$ (c) $\cos^2 x$ (d) none of these

**Solution :** Given, $y = \log_e\left(\tan\frac{x}{2}\right)$

Therefore, $\frac{dy}{dx} = \frac{1}{\tan\frac{x}{2}} \times \sec^2\frac{x}{2} \times \frac{1}{2} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \operatorname{cosec}x$

$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\operatorname{cosec}x)$ or, $\frac{d^2y}{dx^2} = -\operatorname{cosec}x\cot x$

**Ans.** (b)

(xxv) The minimum value of $y = x^2 - x + 2$ is – (a) $\frac{7}{4}$ (b) $\frac{7}{2}$ (c) $\frac{7}{5}$ (d) $\frac{3}{4}$

**Solution :** Given, $y = x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + 2 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4}$,

which is minimum only when $\left(x - \frac{1}{2}\right)^2$ is minimum. Since $\left(x - \frac{1}{2}\right)^2$ is a perfect square its minimum value is 0.

Therefore, the minimum value of the given function is $\frac{7}{4}$.

**Ans.** (a)

(xxvi) A ball travels s cm in t sec. so that $s = 10t + 12t^2$, the velocity of the ball when t = 3 sec. is –

(a) 72 cm/sec (b) 82 cm/sec (c) 92 cm/sec (d) none of these

**Solution :** Given, $s = 10t + 12t^2$ $\therefore \frac{ds}{dt} = 10 + 24t$

$\therefore \left(\frac{ds}{dt}\right)_{t=3} = 10 + 24 \times 3 = 82$ cm/sec

$\therefore$ The required velocity of the ball is 82 cm/sec..

**Ans.** (b).

**2.(a)** If $\frac{\log x}{y - z} = \frac{\log y}{z - x} = \frac{\log z}{x - y}$, then show that $x^{y+z} y^{z+x} z^{x+y} = 1$.

**Solution :** Given, $\frac{\log x}{y - z} = \frac{\log y}{z - x} = \frac{\log z}{x - y} = k(\neq 0)$ (say)

$\therefore \log x = k(y - z), \log y = k(z - x), \log z = k(x - y)$

$\therefore (y + z)\log x + (z + x)\log y + (x + y)\log z = k(y + z)(y - z) + k(z + x)(z - x) + k(x + y)(x - y)$

or, $\log x^{y+z} + \log y^{z+x} + \log z^{x+y} = k(y^2 - z^2 + z^2 - x^2 + x^2 - y^2)$

or, $\log(x^{y+z} z^{x+y} y^{z+x}) = k \times 0$ or, $x^{y+z} z^{x+y} y^{z+x} = e^0 = 1$ **(Proved)**

(b) If $\alpha$ is a root of $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root.

**Solution:** Let $\beta$ be the other root of the equation $4x^2 + 2x - 1 = 0$.

$\therefore \alpha + \beta = -\frac{2}{4} = -\frac{1}{2}$ or, $\beta = -\alpha - \frac{1}{2}$ ............... (1)

Since $\alpha$ be a root of $4x^2 + 2x - 1 = 0$ $\quad \therefore 4\alpha^2 + 2\alpha - 1 = 0$ or, $4\alpha^2 = 1 - 2\alpha$ -------- (2)

Now, $4\alpha^3 - 3\alpha = \alpha(4\alpha^2) - 3\alpha = \alpha(1 - 2\alpha) - 3\alpha = \alpha - 2\alpha^2 - 3\alpha$

$= -2\alpha^2 - 2\alpha = -\frac{1}{2}(4\alpha^2) - 2\alpha = -\frac{1}{2}(1 - 2\alpha) - 2\alpha = -\frac{1}{2} + \alpha - 2\alpha = -\frac{1}{2} - \alpha = \beta$ [from (1)],

which shows that $4\alpha^3 - 3\alpha$ is the other root. **(Proved)**

**3.** (a) Prove that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5........(2n-1)}{n} 2^n x^n$

**Solution :** Number of terms in the expansion of $(1 + x)^{2n}$ is $2n + 1$, which is an odd number.

$\therefore$ only middle term is, $\left(\frac{2n}{2} + 1\right) = (n + 1)$th term.

Here, $t_{n+1} = {}^{2n}C_n \cdot 1^{2n-n} \cdot x^n = \frac{2n!}{n!\,n!} x^n = \frac{2n(2n-1)(2n-2)........6.5.4.3.2.1}{n!\,n!} \cdot x^n$

$= \frac{\{2n(2n-2)........6.4.2\}\{(2n-1)......5.3.1\}}{n!\,n!} \cdot x^n = \frac{2^n.\{n(n-1)......3.2.1\}\{(2n-1)........5.3.1\}}{n!\;n!} \cdot x^n$

$= \frac{n!.\{1.3.5........(2n-1)\}.2^n.x^n}{n!.n!} = \frac{1.3.5.......(2n-1)}{n!}.2^n.x^n$ **(Proved)**

(b) If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\phi$, using De Moivre's theorem, show that $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

**Solution :** Given, $x + \frac{1}{x} = 2\cos\theta$, or, $2x\cos\theta = x^2 + 1$ or, $x^2 - 2x\cos\theta + 1 = 0$

$\therefore x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\times1\times1}}{2\times1}$ or, $x = \frac{2\cos\theta \pm 2\sqrt{-(1-\cos^2\theta)}}{2} = \cos\theta \pm \sin\theta$

Taking +ve sign only in place of $\pm$, we get $x = \cos\theta + i\sin\theta$, Similarly, $y = \cos\phi + i\sin\phi$

Now $x^m \cdot y^n = (\cos\theta + i\sin\theta)^m \cdot (\cos\phi + i\sin\phi)^n$

$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$ [By De Moivre's Theorem] $= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$

$\therefore x^{-m} \cdot y^{-n} = [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1} = \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$ [By De Moivre's Theorem]

$\therefore x^m y^n + x^{-m} y^{-n} = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$

or, $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$ **(Proved)**

**4.** (a) If a and b are two unit vectors and $\theta$ be the angle between them, show that $\sin\frac{\theta}{2} = \frac{1}{2}|a - b|$

**Solution:** By the problem, $|a| = 1$, $|b| = 1$

Now $|a - b|^2 = |a|^2 + |b|^2 - 2\,a \cdot b = 1^2 + 1^2 - 2|a||b|\cos\theta = 2 - 2\cos\theta$ [since, $|a| = 1$, $|b| = 1$]

$= 2(1 - \cos\theta) = 2 \cdot 2\sin^2\frac{\theta}{2}$ Therefore, $4\sin^2\frac{\theta}{2} = |a - b|^2$

or, $2\sin\frac{\theta}{2} = |a - b|$ or, $\sin\frac{\theta}{2} = \frac{1}{2}|a - b|$ **(Proved)**

(b) Find the area of the triangle, the position vestors of whose vertices are i + j + k, i + 2j + 3k, 2i + 3j + k.

**Solution:** With respect to O as origin let the position vectors of the vertices A, B, C of triangle ABC are respectively i + j + k, i + 2j + 3k and 2i + 3j + k.

Then $\overrightarrow{OA} = i + j + k$, $\overrightarrow{OB} = i + 2j + 3k$, $\overrightarrow{OC} = 2i + 3j + k$

Therefore, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (i + 2j + 3k) - (i + j + k) = j + 2k$

Therefore, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (2i + 3j + k) - (i + j + k) = i + 2j$

Therefore, the vector area of the triangle ABC

$$\frac{1}{2}\left(\overrightarrow{AC} \times \overrightarrow{AB}\right) = \frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \frac{1}{2}[i(4 - 0) - j(2 - 0) + k(1 - 0)] = \frac{1}{2}(4i - 2j + k)$$

Required area $= \frac{1}{2}|4i - 2j + k| = \frac{1}{2}\sqrt{4^2 + (-2)^2 + 1^2} = \frac{1}{2}\sqrt{21}$ sq. unit. **(Ans)**

**5.**(a) Find the moment of the force 7i – j + 3k, acting at the point (–1, 1, 2), about the point (2, 1, 4).

**Solution :** Let **F** be the given force acting at the point P(–1, 1, 2), about the point A(2, 1, 4).

Therefore, $\mathbf{F} = 7i - j + 3k$

and $\overrightarrow{OP} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OA} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ where, **i, j, k** are three unit vectors along three rectangular axes.

$\therefore$ $\mathbf{r} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = -3\mathbf{i} - 2\mathbf{k}$

Hence the required moment,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 0 & -2 \\ 7 & -1 & 3 \end{vmatrix}$$

$$= (0 - 2)\mathbf{i} - (-9 + 14)\mathbf{j} + (3 - 0)\mathbf{k} = -2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$$

Therefore, magnitude = $|\mathbf{M}| = \sqrt{(-2)^2+(-5)^2+3^2} = \sqrt{4+25+9} = \sqrt{38}$ units. **(Ans)**

(b) Resolve $\dfrac{1}{(x+1)^2(x^2+1)}$ into partial fractions.

**Solution :** Let, $\dfrac{A}{(x+1)^2}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+1}=\dfrac{1}{(x+1)^2(x^2+1)}$ ------ (1)

or, $A(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x+1)^2=1$

or, $A(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x^2+2x+1)=1$

Putting $x=-1$ we get, $2A=1$ or, $A=\frac{1}{2}$ ---- (2)

Equating the coefficient of $x^3$ from both sides we get, $B+C=0$ ---- (3)

Equating the coefficient of $x^2$ from both sides we get, $A+B+2C+D=0$ ---- (4)

Equating the constant terms from both sides we get,

$A+B+D=1$ or, $B+D=1-A=1-\frac{1}{2}=\frac{1}{2}$ ---- (5)

From (2), (4) and (5) we get,

$\frac{1}{2}+2C+\frac{1}{2}=0$ or, $2C=-1$ or, $C=-\frac{1}{2}$ ---- (6)

From (3) and (6) we get, $B=\frac{1}{2}$ ---- (7)

From (4) and (6) we get, $\frac{1}{2}+D=\frac{1}{2}$ or, $D=0$

Therefore, putting the values of A, B, C and D in (1) we get,

$$\frac{\frac{1}{2}}{(x+1)^2}+\frac{\frac{1}{2}}{x+1}+\frac{-\frac{1}{2}x+0}{x^2+1}=\frac{1}{(x+1)^2(x^2+1)} \quad \text{or,} \quad \frac{1}{(x+1)^2(x^2+1)}=\frac{1}{2(x+1)^2}+\frac{1}{2(x+1)}-\frac{x}{2(x^2+1)} \quad \textbf{(Ans)}$$

**6.** (a) If A and B are positive acute angles and $\cos 2A=\dfrac{3\cos 2B-1}{3-\cos 2B}$, show that $\tan A=\sqrt{2}\tan B$

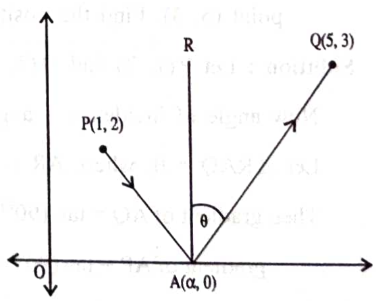
**Solution:** Given, $\cos 2A=\dfrac{3\cos 2B-1}{3-\cos 2B}$ or, $\dfrac{\cos 2A}{1}=\dfrac{3\cos 2B-1}{3-\cos 2B}$

By comp. and div. we get,

$\dfrac{1-\cos 2A}{1+\cos 2A}=\dfrac{3-\cos 2B-3\cos 2B+1}{3-\cos 2B+3\cos 2B-1}=\dfrac{4-4\cos 2B}{2+2\cos 2B}$

or, $\dfrac{1-\cos 2A}{1+\cos 2A}=\dfrac{2(1-\cos 2B)}{1+\cos 2B}$ or, $\dfrac{2\sin^2 A}{2\cos^2 A}=\dfrac{2\cdot 2\sin^2 B}{2\cos^2 B}$

or, $\tan^2 A=2\tan^2 B$ or, $\tan A=\sqrt{2}\tan B$ **(Proved)**




(b) The base of a right pyramid is a regular hexagon. If its height is 10 cm and volume $180\sqrt{3}$ cc, find the length of each side of its base. What is the area of the total slant surface of the pyramid ?

**Solution :** Let, a be the length of each side of regular hexagon.

Then, area of the regular hexagonal base of the pyramid

$= \frac{1}{4}na^2 \cot\frac{\pi}{n} = \frac{1}{4} \times 6a^2 \cot\frac{180^o}{6}$ [$\because$ n = 6] $= \frac{3}{2}a^2 \cot 30^o = \frac{3}{2}a^2 \cdot \sqrt{3} = \frac{3\sqrt{3}}{2}a^2$ sq. cm. ------------- (1)

$\therefore$ volume of the pyramid $= \frac{1}{3} \times$ area of the base $\times$ height $= 180\sqrt{3}$ (given)

or, $\frac{1}{3} \times \frac{3\sqrt{3}}{2}a^2 \times 10 = 180\sqrt{3}$ or, $5a^2 = 180$ or, $a^2 = 36$ or, $a = 6$ cm.

Therefore, the length of each side of its base = 6 cm. **(Ans)**

Again, let r be the radius of the inscribed circle of regular hexagon.
Then area of the base,

$nr^2 \tan\frac{180^o}{n} = \frac{3\sqrt{3}}{2} \times 6^2$ [from (1); a = 6 here] or, $6r^2 \tan\frac{180^o}{6} = \frac{3\sqrt{3}}{2} \times 6^2$

or, $r^2 \tan 30^o = \frac{3\sqrt{3}}{2} \times 6$ or, $r^2 \cdot \frac{1}{\sqrt{3}} = 9\sqrt{3}$ or, $r^2 = 27$

Let p be the slant height of the pyramid. Then, $p^2 = r^2 + h^2 = 27 + 10^2 = 127$

$\therefore$ p = slant height $= \sqrt{127}$ cm.

$\therefore$ Area of the total slant surface of the pyramid

$= \frac{1}{2} \times a \times p \times 6 = \frac{1}{2} \times 6 \times \sqrt{127} \times 6 = 18\sqrt{127}$ sq. cm. **(Ans)**

**7.** (a) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, show that $xy + yz + zx = 1$

**Solution :** Given, Given, $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$

$$\text{or, } \tan^{-1}\left(\frac{x + y + z - xyz}{1 - yz - zx - xy}\right) = \frac{\pi}{2} \text{ or, } \frac{x + y + z - xyz}{1 - yz - zx - xy} = \tan\frac{\pi}{2}$$

or, $1 - yz - zx - xy = 0$ or, $xy + yz + zx = 1$ **(Proved)**

(b) A ray of light passing through the point (1, 2) is reflected at a point A on x-axis and then passes through the point (5, 3). Find the position of A.

**Solution :** Let P(1, 2) and Q(5, 3) be the given points and the co-ordinate of A be (a, 0).

Now angle of incidence = angle of reflection.

Let $\angle RAQ = \theta$, where AR is perpendicular to the x-axis at A.

Then gradient of AQ = $\tan(90^O - \theta) = \cot\theta$ and

gradient of AP = $\tan(90^O + \theta) = -\cot\theta$.

$\therefore$ gradient of AP = – gradient of AQ

or, $\frac{2-0}{1-\alpha} = -\frac{3-0}{5-\alpha}$ or, $\frac{2}{1-\alpha} = -\frac{3}{5-\alpha}$

or, $2a - 10 = 3 - 3a$ or, $5a = 13$ $\therefore a = \frac{13}{5}$

Therefore, the co-ordinates of A are $\left(\frac{13}{5}, 0\right)$ **(Ans)**

**8.** (a) Find the equation of the circle which touches both of the axes and passes through (6, 3).

**Solution :** Let (a, a) be the centre of the circle and radius is a. [since the circle touches both the axes.]

Then the equation of the circle is $(x-a)^2 + (y-a)^2 = a^2$ ................ (1)

If it passes through the point (6, 3) we get, $(6-a)^2 + (3-a)^2 = a^2$

or, $36 - 12a + a^2 + 9 - 6a + a^2 = a^2$ or, $a^2 - 18a + 45 = 0$ or, $(a-15)(a-3) = 0$

or, $a = 3, 15$

Therefore, the required equation of the circle is

$(x-3)^2 + (y-3)^2 = 9$ or $(x-15)^2 + (y-15)^2 = 225$ **(Ans)**

(b) The base of a right prism is a regular pentagon. If its hright be a, volume be b and area of lateral surface be c. Prove that $20ab = c^2\cot 36^0$.

**Solution:** By the problem, a be the height of the prism.

Area of the lateral surface, c = perimeter of the base × height. $= 5x \times a$ [Assuming x be the side of the base.]

or, $x = \frac{c}{5a}$ -------------- (1)

Again, volume of the prism, $b = \frac{nx^2}{4}.\cot\frac{\pi}{n} \times a,\ n = 5$ here.

or, $b = \frac{5x^2}{4}\cdot\cot\frac{180^0}{5} \times a = \frac{5a}{4}\left(\frac{c}{5a}\right)^2 \cdot \cot 36^0$ [from (1)] $= \frac{5a}{4}\cdot\frac{c^2}{25a^2}\cdot\cot 36^0$

or, $b = \frac{1}{4}\cdot\frac{c^2}{5a}\cdot\cot 36^0$ or, $20ab = c^2\cot 36^0$ **(Proved)**

**9.** (a) Evaluate : $\underset{x\to 0}{\mathrm{Lt}}\ \frac{(e^x+1)\log(1+x)}{\sin x}$

**Solution :** Given, $\underset{x\to 0}{\mathrm{Lt}}\ \frac{(e^x+1)\log(1+x)}{\sin x} = \underset{x\to 0}{\mathrm{Lt}}\ \frac{\log(1+x)}{\sin x}\cdot \underset{x\to 0}{\mathrm{Lt}}\ (e^x+1) = \underset{x\to 0}{\mathrm{Lt}}\ \frac{\frac{1}{x}\log(1+x)}{\frac{\sin x}{x}}\cdot(e^0+1)$

$$= \frac{\underset{x\to 0}{\mathrm{Lt}}\ \frac{1}{x}\log(1+x)}{\underset{x\to 0}{\mathrm{Lt}}\ \frac{\sin x}{x}}\cdot(1+1) = \frac{1}{1}\times 2 = 2 \text{ (Ans)}$$

(b) Find wheather the function $f(x) = \frac{|x|}{x}, x \neq 0,$

$= 0,\ x = 0$ is continuous at $x = 0$.

**Solution:** Given, $f(x)=\frac{|x|}{x}, x \neq 0,$

$= 0,\ x = 0$

We know, $|x| = x,\ x > 0$

$= -x,\ x < 0$

Therefore, $f(x) = 1,\ x > 0$

$= -1,\ x < 0$

$= 0,\ x = 0$

Now, $\underset{x\to 0-}{\text{Lt}} f(x) = \underset{x\to 0-}{\text{Lt}} (-1) = -1$ [$\because$ f(x) = –1 for x < 0]

$\underset{x\to 0+}{\text{Lt}} f(x) = \underset{x\to 0+}{\text{Lt}} (1) = 1$ [$\because$ f(x) = 1 for x > 0]

and f(0) = 0

$\therefore \underset{x\to 0-}{\text{Lt}} f(x) \neq \underset{x\to 0+}{\text{Lt}} f(x) \neq f(0)$

Hence, from definition of continuity, f(x) is not continuous at x = 0 **(Ans)**

(c) Find $\frac{dy}{dx}$: $e^{xy} - 4xy = 4$

**Solution :** Given, $e^{xy} - 4xy = 4$

Differentiating both sides with respect to x we get,

$\frac{d}{dx}\left(e^{xy}\right) - 4\frac{d}{dx}(xy) = 0$ or, $e^{xy}\frac{d}{dx}(xy) - 4\frac{d}{dx}(xy) = 0$ or, $\frac{d}{dx}(xy)\left(e^{xy} - 4\right) = 0$ or, $\frac{d}{dx}(xy) = 0$[assuming $e^{xy} \neq 4$]

or, $x\frac{dy}{dx} + y = 0$ or, $\frac{dy}{dx} = -\frac{y}{x}$ **(Ans)**

**10. (a)** Find $\frac{dy}{dx}$: $x^3y^4 = (x+y)^7$

**Solution :** Given, $x^3y^4 = (x + y)^7$ or, $\log(x^3y^4) = \log(x + y)^7$ [taking logarithm of both sides]

or, $3\log x + 4\log y = 7\log(x + y)$

Differentiating both sides with respect to x we get,

$3\frac{d}{dx}(\log x) + 4\frac{d}{dx}(\log y) = 7\frac{d}{dx}\{\log(x+y)\}$ or, $\frac{3}{x} + \frac{4}{y}\frac{dy}{dx} = \frac{7}{x+y}\left(1 + \frac{dy}{dx}\right)$

or, $\frac{3}{x} + \frac{4}{y}\cdot\frac{dy}{dx} = \frac{7}{x+y} + \frac{7}{x+y}\cdot\frac{dy}{dx}$ or, $\left(\frac{4}{y} - \frac{7}{x+y}\right)\frac{dy}{dx} = \frac{7}{x+y} - \frac{3}{x}$

or, $\frac{4x+4y-7y}{y(x+y)}\cdot\frac{dy}{dx} = \frac{7x-3x-3y}{x(x+y)}$ or, $\frac{4x-3y}{y(x+y)}\cdot\frac{dy}{dx} = \frac{4x-3y}{x(x+y)}$ or, $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x}$ or, $\frac{dy}{dx} = \frac{y}{x}$ **(Ans)**

b) If $y = \sin(m\sin^{-1}x)$, prove that $(1 - x^2)y_2 - xy_1 + m^2y = 0$

**Solution :** Given, $y = \sin(m\sin^{-1}x)$ or, $\sin^{-1}y = m\sin^{-1}x$

Differentiating both sides with respect to x we get,

$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}$ or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = m^2\left(1-y^2\right)$

December, 2019

**MATHEMATICS**

Time Allowed: 3 Hours　　　　　　　　　　　　　　　　　　　　Full Marks : 70

Answer to Question No.1 is compulsory and is to be answered first.

This answer is to be made in separate loose script(s) provided for the purpose.

Maximum time allowed is 45 minutes, after which the loose answer scripts will be collected and fresh scripts for answering the remaining part of the question will be provided.

On early submission of answer scripts of Question No.1, a student will get the remaining script earlier.

Answer any five questions from Group - A, B and C, taking at least one from each group.

**1.** Answer questions with minimum justification (any twenty):

$20 \times 1 = 20$

(i) $\log_7 3 \times \log_6 7 \times \log_5 6 \times \log_4 5 \times \log_3 4 =$ (a) 0 (b) 1 (c) 2 (d) None of these

**Solution :** $\log_7 3 \times \log_6 7 \times \log_5 6 \times \log_4 5 \times \log_3 4$

$$= \frac{\log_e 3}{\log_e 7} \times \frac{\log_e 7}{\log_e 6} \times \frac{\log_e 6}{\log_e 5} \times \frac{\log_e 5}{\log_e 4} \times \frac{\log_e 4}{\log_e 3} = \frac{\log_e 3}{\log_e 3} = 1$$

**Ans.** (b)

(ii) If $\log_e 2 \ \log_x 625 = \log_{10} 16 \ \log_2 10$ then x = (a) 4 (b) 5 (c) $\frac{1}{5}$ (d) None of these.

**Solution :** Given, $\log_e 2 . \log_x 625 = \log_{10} 16 . \log_e 10$

or, $\log_e 2 \times \frac{\log_e 5^4}{\log_e x} = \frac{\log_e 2^4}{\log_e 10} \times \log_e 10$ or, $\frac{\log_e 2 \times 4 . \log_e 5}{\log_e x} = 4 . \log_e 2$

or, $\log_e x = \log_e 5 \therefore x = 5$

**Ans.** (b)

(iii) If one root of $3x^2 + px + 3 = 0$ is square of the other, then p =

(i) $\frac{2}{3}$, (b) 3, (c) 1, (d) None of these.

**Solution :** Let a, $a^2$ be the roots of the given equation.

Then, $a + a^2 = -\frac{p}{3}$ -------- (1) and $a . a^2 = 1$ or, $a^3 = 1$ or, $a = 1$

From (1), $1 + 1 = -\frac{p}{3}$ or, $p = -6$

**Ans.** (d)

(iv) If $\alpha$ and $\beta$ be the roots of the equation $x^2 - px + q = 0$ then the value of $\alpha^2 + \beta^2 + \alpha\beta$ is – (a) $q^2 - p$ (b) $q - p^2$ (c) $p^2 - q$ (d) $p - q^2$

**Solution :** Since, $\alpha$ and $\beta$ are the roots of the equation $x^2 - px + q = 0$

Then, $\alpha + \beta = p$ and $\alpha\beta = q$

Now, $\alpha^2 + \beta^2 + \alpha\beta = (\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta = (\alpha + \beta)^2 - \alpha\beta = p^2 - q$

**Ans.** (c)

(v) A right pyramid of height 15 cm stands on a square base of side 16 cm. Its volume is – (a) 1260 cc. (b) 1280 cc. (c) 1020 cc. (d) None of these

**Solution :** Area of a square of side 16 cm. = $16^2$ = 256 sq. cm.

Given, height of the pyramid = 15 cm.

$\therefore$ Volume $= \frac{1}{3} \times$ area of the base $\times$ height $= \frac{1}{3} \times 256 \times 15$ cc $= 256 \times 5$ cc $= 1280$ cc

**Ans.** (b)

(vi) Which of the following functions are proper fractions –

(a) $\frac{x^4 + x^3}{x^2 + 1}$ (b) $\frac{2x^3 + x^2 + 1}{x^3 - 1}$ (c) $\frac{x^2 + 7}{3x^3 + x + x^2}$ (d) None of these

**Solution :** We know, if degree of f(x) < degree of g(x) then $\frac{f(x)}{g(x)}$ is a proper fraction.

Since, degree of $(x^2 + 7)$ < degree of $(3x^3 + x + x^2)$ therefore $\frac{x^2 + 7}{3x^3 + x + x^2}$ is the proper fraction.

**Ans.** (c)

(vii) The value of $\vec{j} \times (\vec{k} \times \vec{i})$ is (a) 1 (b) $\vec{i}$ (c) 0 (d) $\vec{k}$

**Solution:** Given $\vec{j} \times (\vec{k} \times \vec{i}) = \vec{j} \times \vec{j} = 0$

**Ans.** (c)

(viii) The term independent of x in the expansion of $\left(1 - \frac{1}{x}\right)^{10}$ is –

(a) 4th (b) 5th (c) 6th (d) None of these

**Solution :** Let (r + 1)th term is independent of x in the expansion of $\left(1 - \frac{1}{x}\right)^{10}$

Here, (r + 1)th term, $t_{r+1} = {}^{10}C_r \cdot (1)^{10-r} \cdot \left(-\frac{1}{x}\right)^r = (-1)^r \cdot {}^{10}C_r \cdot x^{-r}$ $\therefore -r = 0 \Rightarrow r = 0$

$\therefore$ (r + 1)th term = (0 + 1)th = 1st term is independent of x.

**Ans :** (d)

(ix) The number of terms in the expansion of $\left(x^2-2+\frac{1}{x^2}\right)^6$ is – (a) 6 (b) 7 (c) 5 (d) 13

**Solution :** Given, $\left(x^2-2+\frac{1}{x^2}\right)^6=\left\{\left(x-\frac{1}{x}\right)^2\right\}^6=\left(x-\frac{1}{x}\right)^{12}$

Therefore, the number of terms = 12 + 1 = 13

**Ans.** (d)

(x) $(1-2x)^{-\frac{1}{2}}$ can be expanded in a binomial series if –

(a) $x>\frac{1}{2}$ (b) $-2 < x < 2$ (c) $-\frac{1}{2}<x<\frac{1}{2}$ (d) $-1 < x < 1$

**Solution :** Given, $(1-2x)^{-\frac{1}{2}}$ can be expanded in a binomial series if

$|2x| < 1$ or, $-\frac{1}{2}<x<\frac{1}{2}$

**Ans.** (c)

(xi) If tanx tan3x = 1, then the value of tan2x is – (a) 2 (b) 1 (c) $-\sqrt{3}$ (d) None of these.

**Solution :** Given, tanx tan3x = 1 or, tan3x = cotx or, tan3x = $\tan(90^0 - x)$

or, $3x = 90^0 - x$ or, $4x = 90^0$ or, $2x = 45^0$ Therefore, $\tan 2x = \tan 45^0 = 1$

**Ans.** (b)

(xii) The square root of 3 + 4i is – (a) $\sqrt{3}+i$ (b) 2 – i (c) 2 + i (d) None of these.

**Solution :** Given, $3 + 4i = 4 + 4i - 1 = 2^2 + 2.2.i + i^2 = (2 + i)^2$

Therefore, the square root of 3 + 4i is 2 + i

**Ans.** (c)

(xiii) The angle between the vectors $4\vec{i}+2\vec{j}-7\vec{k}$ and $3\vec{i}+2\vec{j}+2\vec{k}$ is – (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) None of these.

**Solution :** Let θ be the angle between the given vectors $4\vec{i}+2\vec{j}-7\vec{k}$ and $3\vec{i}+2\vec{j}+2\vec{k}$

Then $\cos\theta=\frac{(4\vec{i}+2\vec{j}-7\vec{k})\cdot(3\vec{i}+2\vec{j}+2\vec{k})}{|4\vec{i}+2\vec{j}-7\vec{k}|\,|3\vec{i}+2\vec{j}+2\vec{k}|}=\frac{4\cdot3+2\cdot2-7\cdot2}{\sqrt{4^2+2^2+(-7)^2}\cdot\sqrt{3^2+2^2+2^2}}=\frac{2}{\sqrt{69}\cdot\sqrt{17}}$

$\therefore\ \theta=\cos^{-1}\left(\frac{2}{\sqrt{69}\cdot\sqrt{17}}\right)$

**Ans.** (d)

(xiv) If $\bar{z}$ is conjugate of z then, amp(z) + amp($\bar{z}$) = (a) 0 (b) $\pi$ (c) $2\pi$ (d) None of these.

**Solution :** Let, z = x + iy, then $\bar{z}$ = x – iy.

$$\text{Therefore, } amp(z) + amp(\bar{z}) = \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(-\frac{y}{x}\right) = \tan^{-1}\left(\frac{\frac{y}{x}-\frac{y}{x}}{1+\frac{y^2}{x^2}}\right) = \tan^{-1}0 = 0$$

**Ans.** (a)

(xv) If $10\alpha = \frac{\pi}{2}$ then tan3α tan5α tan7α = (a) 0 (b) 1 (c) 2 (d) None of these

**Solution :** Given, $10\alpha = \frac{\pi}{2}$ $\therefore$ $5\alpha = \frac{\pi}{4}$ and $3\alpha + 7\alpha = \frac{\pi}{2}$ or, $7\alpha = \frac{\pi}{2} - 3\alpha$

$$\text{Therefore, } \tan3\alpha\ \tan5\alpha\ \tan7\alpha = \tan 3\alpha \tan\frac{\pi}{4}\tan\left(\frac{\pi}{2} - 3\alpha\right) = \tan 3\alpha \cdot 1 \cdot \cot 3\alpha = \tan 3\alpha \cot 3\alpha = 1$$

**Ans.** (b)

(xvi) The value of $\sin\left[\tan^{-1}x + \tan^{-1}\frac{1}{x}\right]$ = (a) 0 (b) 1 (c) –1 (d) $\sqrt{2}$

**Solution :** Given, $\sin\left[\tan^{-1}x + \tan^{-1}\frac{1}{x}\right] = \sin\left[\tan^{-1}x + \cot^{-1}x\right] = \sin\frac{\pi}{2} = 1$

**Ans.** (b)

(xvii) The value of $\frac{1}{\sin10^0} - \frac{\sqrt{3}}{\cos10^0}$ is – (a) 1 (b) 2 (c) 3 (d) 4

**Solution :** Given, $\frac{1}{\sin10^0} - \frac{\sqrt{3}}{\cos10^0} = \frac{\cos10^0 - \sqrt{3}\sin10^0}{\sin10^0\cos10^0} = 2\cdot\frac{\frac{1}{2}\cos10^0 - \frac{\sqrt{3}}{2}\sin10^0}{\sin10^0\cos10^0}$

$$= 2\cdot\frac{\sin30^0\cos10^0 - \cos30^0\sin10^0}{\sin10^0\cos10^0} = 4\cdot\frac{\sin\left(30^0 - 10^0\right)}{2\sin10^0\cos10^0} = 4\cdot\frac{\sin20^0}{\sin20^0} = 4$$

**Ans.** (d)

(xviii) The distance between the lines 3x + 4y = 9 and 6x + 8y = 15 is – (a) $\frac{3}{2}$ (b) $\frac{3}{10}$ (c) 6 (d) None of these

**Solution :** Any point on the line 3x + 4y = 9 is (3, 0).

Now, the perpendicular distance from (3, 0) to the line 6x + 8y = 15 is

$$\frac{3\times6+8\times0-15}{\sqrt{6^2+8^2}} = \frac{18-15}{\sqrt{36+64}} = \frac{3}{\sqrt{100}} = \frac{3}{10}$$

**Ans.** (b)

(xix) The centre of the circle $ax^2 + (2a-3)y^2 - 4x - 1 = 0$ is –

(a) $\left(\frac{2}{3}, 0\right)$ (b) $\left(-\frac{2}{3}, 0\right)$ (c) (2, 0) (d) None of these

**Solution :** Given circle is $ax^2 + (2a-3)y^2 - 4x - 1 = 0$

Therefore, $a = 2a - 3$ or, $a = 3$

So the equation of the circle becomes, $3x^2 + (6-3)y^2 - 4x - 1 = 0$

or, $3x^2 + 3y^2 - 4x - 1 = 0$ or, $x^2 + y^2 - \frac{4}{3}x - \frac{1}{3} = 0$

Therefore the centre is $\left(\frac{2}{3}, 0\right)$

**Ans.** (a)

(xx) The vertex of the parabola $y^2 = 8x + 4y + 4$ is –

(a) (–1, –2) (b) (2, 1) (c) (1, –2) (d) (–1, 2)

**Solution :** Given parabola, $y^2 = 8x + 4y + 4$

or, $y^2 - 4y = 8x + 4$ or, $(y-2)^2 = 8(x+1)$

Therefore, vertex is (–1, 2)

**Ans.** (d)

(xxi) The unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ is –

(a) $\vec{a}\times\vec{b}$ (b) $\vec{b}\times\vec{a}$ (c) $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ (d) $\frac{\vec{a}\times\vec{b}}{|\vec{a}\cdot\vec{b}|}$

**Solution :** The unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b}$

**Ans.** (a)

(xxii) The area of a parallelogram, two adjacent sides of which are represented by $\vec{a}$ and $\vec{b}$ is –

(a) $\vec{a}\times\vec{b}$ (b) $\frac{1}{2}\left(\vec{a}\times\vec{b}\right)$ (c) $\vec{a}\cdot\vec{b}$ (d) $\frac{1}{2}\left(\vec{a}\cdot\vec{b}\right)$

**Solution :** The area of a parallelogram, two adjacent sides of which are represented by $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}\left(\vec{a}\times\vec{b}\right)$

**Ans.** (b)

(xxiii) The function $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$ is –

(a) even function (b) odd function (c) periodic function (d) parametric function.

**Solution :** Given function, $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

Therefore, $f(-x) = \sqrt{1+(-x)+(-x)^2} - \sqrt{1-(-x)+(-x)^2}$ [ Replacing x by –x ]

$$= \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -\left(\sqrt{1+x+x^2} - \sqrt{1-x+x^2}\right) = f(x)$$

Therefore, f(–x) = –f(x) and hence f(x) is an odd function.

**Ans.** (b)

(xxiv) If $\phi(x)$ = logsinx and $\psi(x)$ = logcosx then, $e^{2\phi(x)} + e^{2\psi(x)}$ is –

(a) 0 (b) 2 (c) 1 (d) None of these

**Solution :** Given, $\phi(x)$ = logsinx and $\psi(x)$ = logcosx

$\therefore\ e^{2\phi(x)} + e^{2\psi(x)} = e^{2\log\sin x} + e^{2\log\cos x} = e^{\log\sin^2 x} + e^{\log\cos^2 x} = \sin^2 x + \cos^2 x = 1$

**Ans.** (c)

(xxv) The value of $\underset{x \to 0}{\text{Lt}} \dfrac{\cos x}{\frac{\pi}{2} - x}$ is – (a) 1 (b) 0 (c) 2 (d) –1

**Solution :** Given, $\underset{x \to 0}{\text{Lt}} \dfrac{\cos x}{\frac{\pi}{2} - x} = \underset{\frac{\pi}{2} - x \to 0}{\text{Lt}} \dfrac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x} = 1$

**Ans.** (a)

(xxvi) The minimum value of y = $x^2 - x + 2$ is – (a) $\frac{7}{4}$ (b) $\frac{7}{2}$ (c) $\frac{7}{5}$ (d) $\frac{3}{4}$

**Solution :** Given, y = $x^2 - x + 2$ = $\left(x - \frac{1}{2}\right)^2 + 2 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4}$,

which is minimum only when $\left(x - \frac{1}{2}\right)^2$ is minimum. Since $\left(x - \frac{1}{2}\right)^2$ is a perfect square its minimum value is 0. Therefore, the minimum value of the given function is $\frac{7}{4}$.

**Ans.** (a)

(xxvii) The function $f(x) = \sqrt{1-x^2}$ is not defined for –

(a) x = 1 (b) x = 0 (c) |x| > 1 (d) None of these.

**Solution :** Given function $f(x) = \sqrt{1-x^2}$ is not defined for $1 - x^2 < 0$ or, $x^2 > 1$ or, $|x| > 1$

**Ans.** (c)

(xxviii) If $f(x) = \log_e e^x + e^{\log_e x}$ then $f'(x)$ is – (a) 2 (b) $e^x + 1$ (c) $e^x + x$ (d) None of these.

**Solution :** Given, $f(x) = \log_e e^x + e^{\log_e x}$ = x + x = 2x Therefore $f'(x) = \frac{d}{dx}(2x) = 2$

**Ans.** (a)

## Group – A

**2.**(a) If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, then show that $x^{y+z} y^{z+x} z^{x+y} = 1$.

**Solution :** Given, $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k(\neq 0)$ (say)

$\therefore$ $\log x = k(y - z)$, $\log y = k(z - x)$, $\log z = k(x - y)$

$\therefore$ $(y + z) \log x + (z + x) \log y + (x + y) \log z$

$$= k(y + z)(y - z) + k(z + x)(z - x) + k(x + y)(x - y)$$

or, $\log x^{y+z} + \log y^{z+x} + \log z^{x+y} = k(y^2 - z^2 + z^2 - x^2 + x^2 - y^2)$

or, $\log(x^{y+z} z^{x+y} y^{z+x}) = k \times 0$ or, $x^{y+z} z^{x+y} y^{z+x} = e^0 = 1$ **(Proved)**

(b) If the coefficient of $x^7$ in the expansion of $\left(px^2 + \frac{1}{qx}\right)^{11}$ be equal to the coefficient of $x^{-7}$ in the expansion of $\left(px - \frac{1}{qx^2}\right)^{11}$, prove that $pq = 1$.

**Solution :** Let (r + 1)th term contains $x^7$ in the expansion of $\left(px^2 + \frac{1}{qx}\right)^{11}$

Here, (r + 1)th term, $t_{r+1} = {}^{11}C_r \cdot \left(px^2\right)^{11-r} \cdot \left(\frac{1}{qx}\right)^r$

$= {}^{11}C_r \cdot p^{11-r} \cdot \frac{1}{q^r} x^{22-2r-r} = {}^{11}C_r \cdot p^{11-r} \cdot \frac{1}{q^r} x^{22-3r}$

By the problem, $22 - 3r = 7$ or, $3r = 15$ $\therefore$ $r = 5$

$\therefore$ coefficient of $x^7$ in the expansion of $\left(px^2 + \frac{1}{qx}\right)^{11}$ is

${}^{11}C_5 \cdot p^{11-5} \cdot \frac{1}{q^5} = {}^{11}C_5 \cdot p^6 \cdot \frac{1}{q^5}$

Again let (u + 1)th term contains $x^{-7}$ in the expansion of $\left(px - \frac{1}{qx^2}\right)^{11}$

Here, $t_{u+1} = {}^{11}C_u \cdot (px)^{11-u} \cdot \left(-\frac{1}{qx^2}\right)^u$

$= {}^{11}C_u \cdot p^{11-u} \cdot \left(-\frac{1}{q}\right)^u x^{11-u-2r} = {}^{11}C_u \cdot p^{11-u} \cdot \left(-\frac{1}{q}\right)^u x^{11-3u}$

$\therefore\ 11 - 3u = -7$ or, $3u = 18\ \therefore\ u = 6$

$\therefore$ coefficient of $x^{-7}$ in the expansion of $\left(px - \frac{1}{qx^2}\right)^{11}$

$= {}^{11}C_6 \cdot p^{11-6} \cdot \left(-\frac{1}{q}\right)^6 = {}^{11}C_6 \cdot p^5 \cdot \frac{1}{q^6}$

By the problem, ${}^{11}C_5 \cdot \frac{p^6}{q^5} = {}^{11}C_6 \cdot \frac{p^5}{q^6}$ or, $\frac{11!}{5!\cdot 6!} p = \frac{11!}{6!\cdot 5!} \frac{1}{q}$ or, $pq = 1$ **(Proved)**

**3.**(a) If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\phi$, using De Moivre's theorem, show that

$x^3y^4 + \frac{1}{x^3y^4} = 2\cos(3\theta + 4\phi)$

**Solution :** Given, $x + \frac{1}{x} = 2\cos\theta$, or, $2x\cos\theta = x^2 + 1$ or, $x^2 - 2x\cos\theta + 1 = 0$

$\therefore\ x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\times1\times1}}{2\times1}$ or, $x = \frac{2\cos\theta \pm 2\sqrt{-(1-\cos^2\theta)}}{2} = \cos\theta \pm \sin\theta$

Taking +ve sign only in place of ±, we get $x = \cos\theta + i\sin\theta$, Similarly, $y = \cos\phi + i\sin\phi$

Now $x^3 \cdot y^4 = (\cos\theta + i\sin\theta)^3 \cdot (\cos\phi + i\sin\phi)^4$

$= (\cos 3\theta + i\sin 3\theta)(\cos 4\phi + i\sin 4\phi)$ [By De Moivre's Theorem]

$= \cos(3\theta + 4\phi) + i\sin(3\theta + 4\phi)$

$\therefore\ x^{-3} \cdot y^{-4} = [\cos(3\theta + 4\phi) + i\sin(3\theta + 4\phi)]^{-1} = \cos(3\theta + 4\phi) - i\sin(3\theta + 4\phi)$

[By De Moivre's Theorem]

$\therefore\ x^3y^4 + x^{-3}y^{-4} = \cos(3\theta + 4\phi) + i\sin(3\theta + 4\phi) + \cos(3\theta + 4\phi) - i\sin(3\theta + 4\phi)$

or, $x^3y^4 + \frac{1}{x^3y^4} = 2\cos(3\theta + 4\phi)$ **(Proved)**

(b) If $\alpha$ be a root of $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root.

**Solution:** Let $\beta$ be the other root of the equation $4x^2 + 2x - 1 = 0$.

$\therefore \alpha + \beta = -\frac{2}{4} = -\frac{1}{2}$ or, $\beta = -\alpha - \frac{1}{2}$ ---------- (1)

Since $\alpha$ be a root of $4x^2 + 2x - 1 = 0$

$\therefore 4\alpha^2 + 2\alpha - 1 = 0$ or, $4\alpha^2 = 1 - 2\alpha$ -------- (2)

Now, $4\alpha^3 - 3\alpha = \alpha(4\alpha^2) - 3\alpha = \alpha(1 - 2\alpha) - 3\alpha = \alpha - 2\alpha^2 - 3\alpha$

$= -2\alpha^2 - 2\alpha = -\frac{1}{2}(4\alpha^2) - 2\alpha = -\frac{1}{2}(1 - 2\alpha) - 2\alpha = -\frac{1}{2} + \alpha - 2\alpha = -\frac{1}{2} - \alpha = \beta$ [from (1)],

which shows that $4\alpha^3 - 3\alpha$ is the other root. **(Proved)**

**4.**(a) Prove that $(1 + x + x^2 + \ldots\ldots \infty)(1 - x + x^2 - \ldots\ldots \infty) = 1 + x^2 + x^4 + \ldots\ldots \infty$

**Solution :** Given expression :

$(1 + x + x^2 + \ldots\ldots \infty)(1 - x + x^2 - \ldots\ldots \infty)$

$= (1 - x)^{-1}(1 + x)^{-1} = \{(1 - x)(1 + x)\}^{-1} = (1 - x^2)^{-1}$

$= 1 + x^2 + (x^2)^2 + \ldots\ldots \infty = 1 + x^2 + x^4 + \ldots\ldots \infty$ **(Proved)**

(b) If a and b are two unit vectors and $\theta$ be the angle between them, show that $\sin\frac{\theta}{2} = \frac{1}{2}|a - b|$

**Solution:** By the problem, $|a| = 1$, $|b| = 1$

Now $|a - b|^2 = |a|^2 + |b|^2 - 2\,a \cdot b = 1^2 + 1^2 - 2|a||b|\cos\theta = 2 - 2\cos\theta$ [since, $|a| = 1$, $|b| = 1$]

$= 2(1 - \cos\theta) = 2 \cdot 2\sin^2\frac{\theta}{2}$

Therefore, $4\sin^2\frac{\theta}{2} = |a - b|^2$ or, $2\sin\frac{\theta}{2} = |a - b|$ or, $\sin\frac{\theta}{2} = \frac{1}{2}|a - b|$ **(Proved)**

**5.**(a) Is the function $\frac{3x}{x^2 + 2x - 8}$ proper or improper ? Split the fraction into partial fractions.

**Solution :** We know, if degree of f(x) < degree of g(x) then $\frac{f(x)}{g(x)}$ is a proper fraction.

Since, degree of 3x < degree of $(3x^2 + 2x - 8)$ therefore $\frac{3x}{x^2 + 2x - 8}$ is the proper fraction.

Now, $\frac{3x}{x^2 + 2x - 8} = \frac{3x}{x^2 + 4x - 2x - 8} = \frac{3x}{x(x + 4) - 2(x + 4)}$

$= \frac{3x}{(x + 4)(x - 2)} = \frac{A}{x + 4} + \frac{B}{x - 2}$ (say) ---------------- (1)

Therefore, $A(x - 2) + B(x + 4) = 3x$ ---------------- (2)

Putting $x = 2$ and $x = -4$ in (2) successively we get,

$6B = 6$ or, $B = 1$ and $-6A = -12$ or, $A = 2$

From (1) we get, $\dfrac{3x}{x^2+2x-8} = \dfrac{2}{x+4} + \dfrac{1}{x-2}$ **(Ans)**

(b) Find the unit vector perpendicular to both the vectors $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{b} = 7\vec{i} - 5\vec{j} + \vec{k}$

**Solution :** Given, $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{b} = 7\vec{i} - 5\vec{j} + \vec{k}$

$\therefore\ \vec{a}\times\vec{b} = (2\vec{i} - 3\vec{j} + \vec{k})\times(7\vec{i} - 5\vec{j} + \vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 7 & -5 & 1 \end{vmatrix} = (-3+5)\vec{i} - (2-7)\vec{j} + (-10+21)\vec{k} = 2\vec{i} + 5\vec{j} + 11\vec{k}$$

Therefore, the required unit vector

$$= \pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \pm\frac{2\vec{i}+5\vec{j}+11\vec{k}}{|2\vec{i}+5\vec{j}+11\vec{k}|} = \pm\frac{2\vec{i}+5\vec{j}+11\vec{k}}{\sqrt{2^2+5^2+11^2}} = \pm\frac{2\vec{i}+5\vec{j}+11\vec{k}}{\sqrt{150}} = \pm\frac{2\vec{i}+5\vec{j}+11\vec{k}}{5\sqrt{6}} \quad \textbf{(Ans)}$$

## Group – B

**6.**(a) Find the equation of the circle which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5, 5) and whose radius is 5 unit.

**Solution:** Given circle is $x^2 + y^2 - 2x - 4y - 20 = 0$ whose centre is (1, 2) and radius

$= \sqrt{1^2 + 2^2 + 20} = \sqrt{25} = 5$ unit.

Let (a, b) be the centre of the required circle. Then (5, 5) is the middle point of the line segment joining points (1, 2) and (a, b).

Therefore, $\dfrac{1+a}{2} = 5$ and $\dfrac{2+b}{2} = 5$

or, $a + 1 = 10$ and $b + 2 = 10$ or, $a = 9$ and $b = 8$

Therefore, centre of the required circle is (9, 8)

Hence, the equation of the required circle is

$(x-9)^2 + (y-8)^2 = 5^2$ or, $x^2 + y^2 - 18x - 16y + 81 + 64 = 25$

or, $x^2 + y^2 - 18x - 16y + 120 = 0$ **(Ans)**

(b) A right prism of height 12 cm. stands on a base which is regular hexagon. If the area of the whole surface of the prism be $1152\sqrt{3}$ sq. cm., find the volume of the prism.

**Solution :** Let a be the length of a side of the regular hexagon.

Then area of the side-faces of the prism = 6ah = $6a\times 12 = 72a$ sq. cm.

Area of the base of the prism = $\dfrac{na^2}{4}\cot\dfrac{\pi}{n}$, n = 6 here.

$$= \frac{6a^2}{4}\cot\frac{\pi}{6} = \frac{3a^2}{2}\times\sqrt{3} = \frac{3\sqrt{3}}{2}a^2 \text{ sq. cm.}$$

Area of the whole surface of the prism $= 72a + 2\times\frac{3\sqrt{3}}{2}a^2$

Therefore, $72a + 2\times\frac{3\sqrt{3}}{2}a^2 = 1152\sqrt{3}$

or, $a^2 + 8\sqrt{3}a - 384 = 0$ [Dividing both sides by $3\sqrt{3}$ ]

or, $a^2 + 16\sqrt{3}a - 8\sqrt{3}a - 384 = 0$ or, $(a+16\sqrt{3})(a-8\sqrt{3}) = 0$

$\therefore$ either, $a = -16\sqrt{3}$ or, $a = 8\sqrt{3}$

But $a = -16\sqrt{3}$ is not possible. Therefore, $a = 8\sqrt{3}$

Therefore the required volume of the prism = area of its base $\times$ its height

$$= \frac{3\sqrt{3}}{2}a^2\times 12 = \frac{3\sqrt{3}}{2}(8\sqrt{3})^2\times 12 = 3456\sqrt{3} \text{ c.c. } \textbf{(Ans)}$$

**7.**(a) If $\tan\frac{\theta}{2} = \tan^3\frac{\phi}{2}$ and $\tan\phi = 2\tan\alpha$, then prove that $\theta + \phi = 2\alpha$.

Solution : We know, $\tan\left(\frac{\theta}{2}+\frac{\phi}{2}\right) = \frac{\tan\frac{\theta}{2} + \tan\frac{\phi}{2}}{1 - \tan\frac{\theta}{2}.\tan\frac{\phi}{2}} = \frac{\tan^3\frac{\varphi}{2} + \tan\frac{\varphi}{2}}{1 - \tan^3\frac{\varphi}{2}.\tan\frac{\varphi}{2}}$ $\left[\because \tan\frac{\theta}{2} = \tan^3\frac{\varphi}{2}\right]$

$$= \frac{\tan\frac{\varphi}{2}\left(\tan^2\frac{\varphi}{2}+1\right)}{1-\tan^4\frac{\varphi}{2}} = \frac{\tan\frac{\varphi}{2}\left(1+\tan^2\frac{\varphi}{2}\right)}{\left(1+\tan^2\frac{\varphi}{2}\right)\left(1-\tan^2\frac{\varphi}{2}\right)} = \frac{\tan\frac{\varphi}{2}}{1-\tan^2\frac{\varphi}{2}}$$

$$= \frac{\frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}}{1-\frac{\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}} = \frac{\frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}}{\frac{\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}} = \frac{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}{2\cos\phi} = \frac{\sin\varphi}{2\cos\phi} = \frac{\tan\varphi}{2} = \frac{2\tan\alpha}{2} \quad [\because \tan\varphi = 2\tan\alpha]$$

$\therefore \tan\left(\frac{\theta}{2}+\frac{\phi}{2}\right) = \tan\alpha$ $\therefore \frac{\theta}{2}+\frac{\phi}{2} = \alpha \Rightarrow \theta + \varphi = 2\alpha$ **(Proved)**

7.(b) If $2\tan\alpha = 3\tan\beta$, show that $\tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}$

**Solution :** $2\tan\alpha = 3\tan\beta$ or, $\tan\alpha = \frac{3}{2}\tan\beta$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\cdot\tan\beta} = \frac{\frac{3}{2}\tan\beta - \tan\beta}{1+\frac{3}{2}\tan\beta\cdot\tan\beta} = \frac{3\tan\beta - 2\tan\beta}{2+3\tan^2\beta}$$

$$= \frac{\tan\beta}{2+3\tan^2\beta} = \frac{\frac{\sin\beta}{\cos\beta}}{2+\frac{3\sin^2\beta}{\cos^2\beta}} = \frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta} = \frac{2\sin\beta\cos\beta}{2\cdot(2\cos^2\beta)+3\cdot(2\sin^2\beta)}$$

$$= \frac{\sin 2\beta}{2\cdot(1+\cos 2\beta)+3\cdot(1-\cos 2\beta)} = \frac{\sin 2\beta}{2+2\cos 2\beta+3-3\cos 2\beta} = \frac{\sin 2\beta}{5-\cos 2\beta}$$ **(Proved)**

8.(a) Prove that $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right\}=\frac{2b}{a}$

**Solution :** $\text{L.H.S.} = \tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)$

$$= \tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right),\ \theta=\frac{1}{2}\cos^{-1}\frac{a}{b}\ \text{(say)}$$

$$= \frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}.\tan\theta}+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}.\tan\theta} = \frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{(1+\tan\theta)^2+(1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)} = \frac{2\left(1+\tan^2\theta\right)}{1-\tan^2\theta}$$

$$= 2\cdot\frac{1}{\frac{1-\tan^2\theta}{1+\tan^2\theta}} = \frac{2}{\cos 2\theta} = \frac{2b}{a}\quad\left[\because\ \theta=\frac{1}{2}\cos^{-1}\frac{a}{b},\ \therefore\ \cos 2\theta=\frac{a}{b}\right]$$ **(Proved)**

8.(b) Find the equation of the straight line which passes trough the point of intersection of the line $2x + 3y = 5$ and $3x + 5y = 7$ and makes equal intercepts upon co-ordinates axes.

**Solution :** Given lines are, $2x + 3y = 5$ or, $2x + 3y - 5 = 0$................. (1)

and $3x + 5y = 7$ or, $3x + 5y - 7 = 0$ ............................(2)

Using rule of cross-multiplication we get

$$\frac{x}{-21+25} = \frac{y}{-15+14} = \frac{1}{10-9}\quad \text{or}\ \frac{x}{4} = \frac{y}{-1} = \frac{1}{1} \Rightarrow x = 4,\ y = -1$$

Therefore, point of intersection of (1) and (2) is (4, −1).

Let the equation of the required straight line be

$$\frac{x}{\pm k}+\frac{y}{\pm k}=1\ \text{ or }\ x+y=\pm k$$

[since it makes equal positive intercepts upon the co-ordinate axes]

If it pass through the point (4, −1), then $4 - 1 = \pm k$ or $k = \pm 3$.

Therefore, the equation of the required straight line is $x + y = \pm 3$. **(Ans)**

## Group – C

9.(a) A function f(x) is defined by $f(x)=\frac{|x|}{x}, x\neq 0$. Draw a rough sketch of the function. What value may be assigned to f(x) at x = 0, so that f(x) is continuous at x = 0.

**Solution:** Given, $f(x)=\frac{|x|}{x}, x\neq 0$

We know, $|x| = x, x > 0$

$= -x, x < 0$

Therefore, $f(x) = 1, x > 0$

$= -1, x < 0$

Now, $\underset{x\to 0-}{\text{Lt}} f(x) = \underset{x\to 0-}{\text{Lt}} (-1) = -1$ $[\because f(x) = -1 \text{ for } x < 0]$

$\underset{x\to 0+}{\text{Lt}} f(x) = \underset{x\to 0+}{\text{Lt}} (1) = 1$ $[\because f(x) = 1 \text{ for } x > 0]$

$\therefore \underset{x\to 0-}{\text{Lt}} f(x) \neq \underset{x\to 0+}{\text{Lt}} f(x)$

Therefore, $\underset{x\to 0}{\text{Lt}} f(x)$ does not exist and hence f(x) is not continuous at x = 0 **(Ans)**

(b) Evaluate : $\underset{x\to a}{\text{Lt}} \frac{x\sin a - a\sin x}{x-a}$

**Solution :** Given, $\underset{x\to a}{\text{Lt}} \frac{x\sin a - a\sin x}{x-a}$

$$= \underset{x\to a}{\text{Lt}} \frac{(x-a)\sin a - a\sin x + a\sin a}{x-a} = \underset{x\to a}{\text{Lt}} \frac{(x-a)\sin a - a(\sin x - \sin a)}{x-a}$$

$$= \sin a - a \underset{x\to a}{\text{Lt}} \frac{\sin x - \sin a}{x-a} = \sin a - a \underset{x\to a}{\text{Lt}} \frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a}$$

$$= \sin a - a \underset{\frac{x-a}{2}\to 0}{\text{Lt}} \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} \underset{x\to a}{\text{Lt}} \cos\frac{x+a}{2} \quad \left[\because x\to a \therefore \frac{x-a}{2}\to 0\right]$$

$$= \sin a - a\cos\frac{a+a}{2} = \sin a - a\cos a \quad \textbf{(Ans)}$$

**10.** (a) (i) If $y=\tan^{-1}\left[\frac{a\cos x-b\sin x}{a\sin x+b\cos x}\right]$ find $\frac{dy}{dx}$

**Solution :** Given, $y=\tan^{-1}\left[\frac{a\cos x-b\sin x}{a\sin x+b\cos x}\right]$

$$=\tan^{-1}\left(\frac{\frac{a\cos x-b\sin x}{b\cos x}}{\frac{b\cos x+a\sin x}{b\cos x}}\right)=\tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right)=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)=\tan^{-1}\frac{a}{b}-x$$

Differentiating both sides with respect to x we get,

$\frac{dy}{dx}=-1$ **(Ans)** $\left[\frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right)=0\right]$ **(Ans)**

**10.** (a) (ii) If $y = \sin x^0$, find $\frac{dy}{dx}$

**Solution :** Given, $y = \sin x^0$ or, $y=\sin\left(\frac{\pi}{180}x\right)$ [ since, $\pi = 180^0$ ]

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx}=\cos\left(\frac{\pi}{180}x\right)\frac{d}{dx}\left(\frac{\pi}{180}x\right)=\cos x^0\cdot\frac{\pi}{180}=\frac{\pi}{180}\cos x^0 \quad \textbf{(Ans)}$$

b) If $y = \sin(m\sin^{-1}x)$, prove that $(1 - x^2)y_2 - xy_1 + m^2y = 0$

**Solution :** Given, $y = \sin(m\sin^{-1}x)$ or, $\sin^{-1}y = m\sin^{-1}x$

Differentiating both sides with respect to x we get,

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=\frac{m}{\sqrt{1-x^2}} \quad \text{or,} \quad \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=m^2\left(1-y^2\right)$$

Again, differentiating both sides with respect to x we get,

$$\left(1-x^2\right)\times 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+(-2x)\left(\frac{dy}{dx}\right)^2=-2m^2y\frac{dy}{dx} \quad \text{or,} \quad \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-m^2y$$

or, $(1 - x^2)y_2 - xy_1 + m^2y = 0$ **(Proved)**

**11.** (a) An open tank with square base is to be made by metalic sheet of 48 sq. mt. What will be the dimentions of the tank so that it may contain maximum quantity of water ?

**Solution :** Let a be the base and h be the height of the tank.

Therefore, the metalic surface of the tank = $a^2 + 4ah$ [since the upper side of the tank is open.]

By the problem, $a^2 + 4ah = 48$ or, $h=\frac{48-a^2}{4a}$ ............. (1)

Let, v be the volume of the tank, then, $v = a^2h = a^2\left(\frac{48-a^2}{4a}\right)$ or, $v = 12a - \frac{a^3}{4}$

Therefore, $\frac{dv}{da} = 12 - \frac{3}{4}a^2$ and $\frac{d^2v}{da^2} = -\frac{3}{2}a$

Now, $\frac{dv}{da} = 0$ gives, $12 - \frac{3}{4}a^2 = 0$ or, $a^2 = 16$ or, $a = 4$

For, $a = 4$, $\frac{d^2v}{da^2} = -\frac{3}{2} \times 4 = -6 < 0$

Therefore, volume v is maximum when a = 4 and then, $h = \frac{48-a^2}{4a} = \frac{48-16}{16} = \frac{32}{16} = 2$

So each side of the square tank is 4 mt. and the height is 2 mt. **(Ans)**

(b) If the area of a circle increases at a uniform rate then show that the rate of increase of its perimeter varies inversely as the radius.

**Solution :** Let r be the radius of the circle.

Then by the problem, $\frac{d}{dt}\left(\pi r^2\right) = k$, constant

or, $2\pi r \frac{dr}{dt} = k$ or, $\frac{dr}{dt} = \frac{k}{2\pi r}$ .................... (1)

Now perimeter of the circle, $C = 2\pi r$

$\therefore \frac{dC}{dt} = 2\pi \frac{dr}{dt} = 2\pi\left(\frac{k}{2\pi r}\right) = \frac{k}{r}$ $\therefore \frac{dC}{dt} \propto \frac{1}{r}$, which shows that the rate of increase of its perimeter varies inversely as the radius. **(Proved)**

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104(N)

JUNE – 2021

**MATHEMATICS – I**

Time Allowed: 3 Hours  Full Marks : 60

Answer to Question No.1 is compulsory and Answer any four Questions from Group – A, B & C, taking at least one from each group.

1. Choose the correct answer from the given alternatives (any twenty) : $20 \times 1$

(i) The value of $\log_3 63 - \log_{27} 343$ is (a) 0 (b) 1 (c) 2 (d) None of these.

**Solution:** Given, $\log_3 63 - \log_{27} 343 = \frac{\log 63}{\log 3} - \frac{\log 343}{\log 27}$

$$= \frac{\log 63}{\log 3} - \frac{\log 7^3}{\log 3^3} = \frac{\log 63}{\log 3} - \frac{3\log 7}{3\log 3} = \frac{\log 63}{\log 3} - \frac{\log 7}{\log 3}$$

$$= \frac{\log 63 - \log 7}{\log 3} = \frac{\log \frac{63}{7}}{\log 3} = \frac{\log 9}{\log 3} = \frac{\log 3^2}{\log 3} = \frac{2\log 3}{\log 3} = 2$$

**Ans.** (c)

(ii) If $\frac{1}{\log_x 4} + 1 = \frac{2}{\log_2 4}$ then x = (a) –1 (b) 0 (c) 1 (d) None of these.

**Solution:** Given, $\frac{1}{\log_x 4} + 1 = \frac{2}{\log_2 4}$ or, $\log_4 x + \log_4 4 = 2\log_4 2$

or, $\log_4(4x) = \log_4 2^2$ or, $\log_4(4x) = \log_4 4$

or, $4x = 4$ or, $x = 1$

**Ans.** (c)

(iii) The value of $\frac{i^{2021} + i^{2020}}{i^{2021} - i^{2020}}$ is – (a) –1 (b) 0 (c) 1 (d) None of these.

**Solution:** Given, $\frac{i^{2021} + i^{2020}}{i^{2021} - i^{2020}} = \frac{i^{2020}(i+1)}{i^{2020}(i-1)} = \frac{i+1}{i-1}$

$$= \frac{(i+1)^2}{(i-1)(i+1)} = \frac{i^2 + 2i + 1}{i^2 - 1} = \frac{-1 + 2i + 1}{-1 - 1} = \frac{2i}{-2} = -i$$

**Ans.** (d)

(iv) The value of $(\omega^{2021} + \omega^{2020} - 1)(\omega^{2021} + \omega^{2020} + 1)$ is – (a) –1 (b) 0 (c) 1 (d) None of these.

**Solution:** Given, $(\omega^{2021} + \omega^{2020} - 1)(\omega^{2021} + \omega^{2020} + 1)$

We know, $\omega^{2021} = (\omega^3)^{673}\omega^2 = \omega^2$ [since, $\omega^3 = 1$] Similarly, $\omega^{2020} = (\omega^3)^{673}\omega = \omega$

Therefore, $(\omega^2 + \omega - 1)(\omega^2 + \omega + 1) = 0$ [since, $\omega^2 + \omega + 1 = 0$]

**Ans.** (b)

(v) If the roots of the equation $ax^2 + bx + c = 0$ are reciprocal to each other then –

(a) a = b (b) b = c (c) c = a (d) None of these.

**Solution:** Since, the roots of the equation $ax^2 + bx + c = 0$ are reciprocal to each other then, product of the roots = 1 or, $\frac{c}{a} = 1$ or, c = a

**Ans.** (c)

(vi) If $2 - 3i$ is a root of the equation $x^2 - px + q = 0$ then the value of p is –

(a) 4 (b) 3 (c) 2 (d) None of these.

**Solution:** By the problem, $2 - 3i$ and $2 + 3i$ are the two roots of the equation $x^2 - px + q = 0$.

Therefore, $2 - 3i + 2 + 3i = p$ or, $p = 4$

**Ans.** (a)

(vii) The fifth term of the expansion of $(1 + x)^6$ is – (a) $5x^4$ (b) $6x^4$ (c) $7x^4$ (d) None of these.

**Solution:** The fifth term of the expansion of $(1 + x)^6$ is

$$t_5 = t_{4+1} = {}^6C_4\, x^4 = \frac{6!}{4! \times 2!}x^4 = \frac{6 \times 5 \times 4!}{4! \times 2}x^4 = 15x^4$$

**Ans.** (d)

(viii) The middle term in the expansion of $\left(x^2 + \frac{1}{x^2}\right)^8$ is – (a) 70 (b) 80 (c) 90 (d) None of these.

**Solution :** Number of terms in the expansion of $\left(x^2 + \frac{1}{x^2}\right)^8$ is $(8 + 1) = 9$, which is an odd number.

$\therefore$ only middle term is $\left(\frac{8}{2} + 1\right) = 5$th term.

$$\text{Now, } t_5 = t_{4+1} = {}^8C_4\left(x^2\right)^{8-4} \cdot \left(\frac{1}{x^2}\right)^4 = \frac{8!}{4! \times 4!} \times x^8 \times \frac{1}{x^8}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4!}{4 \times 3 \times 2 \times 1 \times 4!} = 70$$

**Ans.** (a)

(ix) If the vectors $2\vec{i}-\vec{j}+5\vec{k}$ and $a\vec{i}-2\vec{k}$ are perpendicular then a =

(a) 2 (b) –5 (c) 5 (d) None of these.

**Solution :** By the problem, $2\vec{i}-\vec{j}+5\vec{k}$ and $a\vec{i}-2\vec{k}$ are perpendicular.

Therefore, 2a – 10 = 0 or, 2a = 10 or, a = 5

**Ans.** (c)

(x) If $\vec{a}=\vec{i}+\vec{j}+\vec{k}$ and $\vec{b}=\vec{i}-\vec{j}+\vec{k}$ then $\vec{a}\cdot\vec{b}=$(a) 1 (b) 0 (c) –1 (d) None of these.

**Solution :** Given, $\vec{a}=\vec{i}+\vec{j}+\vec{k}$ and $\vec{b}=\vec{i}-\vec{j}+\vec{k}$

Therefore, $\vec{a}\cdot\vec{b}=1-1+1=1$

**Ans.** (a)

(xi) The value of $\cos1^0 \cos2^0 \cos3^0$ ..................$\cos100^0$ is – (a) 1 (b) 0 (c) –1 (d) None of these.

**Solution :** Given, $\cos1^0 \cos2^0 \cos3^0$.......$\cos100^0$ = $\cos1^0 \cos2^0 \cos3^0$.......$\cos90^0$......$\cos100^0$ =0

**Ans.** (b)

(xii) If $(\vec{a}\times\vec{b})\cdot\vec{c}=0$ for the vector $\vec{a}=\alpha\vec{i}+\vec{k}$, $\vec{b}=\vec{i}-5\vec{j}+\vec{k}$ and $\vec{c}=-2\vec{j}$ then α =

(a) 1 (b) 0 (c) –1 (d) None of these.

**Solution :** Given, $\vec{a}=\alpha\vec{i}+\vec{k}$, $\vec{b}=\vec{i}-5\vec{j}+\vec{k}$ and $\vec{c}=-2\vec{j}$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \alpha & 0 & 1\\ 1 & -5 & 1\end{vmatrix}=5\vec{i}-(\alpha-1)\vec{j}-5\alpha\vec{k}$$

$$\therefore (\vec{a}\times\vec{b})\cdot\vec{c}=-\{5\vec{i}-(\alpha-1)\vec{j}-5\alpha\vec{k}\}\cdot 2\vec{j}=2(\alpha-1)$$

By the problem, 2(α – 1) = 0 or, α = 1

**Ans.** (a)

(xiii) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$ = (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) None of these.

**Solution :** Given, $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)=\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

**Ans.** (c)

(xiv) In triangle ABC, if a = 1, b = 2 and A = $30^0$, then B =

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) None of these.

**Solution :** We know, $\frac{a}{\sin A}=\frac{b}{\sin B}$ or, $\frac{1}{\sin 30^0}=\frac{2}{\sin B}$

Therefore, $\sin B = 2\sin 30^0 = 2\times\frac{1}{2} = 1$ or, $B = \frac{\pi}{2}$

**Ans.** (a)

(xv) The function $f(x) = e^{\cos(3x^2+2)}$ is – (a) even function (b) odd function (c) periodic function (d) None is true.

**Solution :** Given, $f(x) = e^{\cos(3x^2+2)}$

$\therefore f(-x) = e^{\cos\{3(-x)^2+2\}} = e^{\cos\{3x^2+2\}} = f(x)$

Therefore, f(x) is an even function.

**Ans.** (a)

(xvi) If the vectors $\vec{i}-\vec{j}+\vec{k}$ and $\vec{i}+a\vec{j}+\vec{k}$ are collinear then a =

(a) 1 (b) –1 (c) 0 (d) None of these.

**Solution :** Since, $\vec{i}-\vec{j}+\vec{k}$ and $\vec{i}+a\vec{j}+\vec{k}$ are collinear

Therefore, $\vec{i}+a\vec{j}+\vec{k} = x(\vec{i}-\vec{j}+\vec{k})$, which gives,

1 = x, a = –x, 1 = x i. e., a = –1

**Ans.** (b)

(xvii) The dot product of the vectors $\vec{i}+\vec{k}$ and $\vec{j}$ is – (a) 0 (b) 1 (c) –1 (d) None of these.

**Solution :** The dot product of the vectors $\vec{i}+\vec{k}$ and $\vec{j} = (\vec{i}+\vec{k})\cdot\vec{j} = 0$

**Ans.** (a)

(xviii) The period of the function $f(x) = \cos\frac{x}{3}$ is – (a) $2\pi$ (b) $4\pi$ (c) $6\pi$ (d) None of these.

**Solution :** Given, $f(x) = \cos\frac{x}{3}$

Since, $f(6\pi+x) = \cos\frac{6\pi+x}{3} = \cos\left(2\pi+\frac{x}{3}\right) = \cos\frac{x}{3}$ so, period of the given function is $6\pi$.

**Ans.** (c)

(xix) If $f(x-1) = x^2 - x + 1$ then f(x) =

(a) $x^2 + x + 1$ (b) $x^2 - x + 1$ (c) $x^2 + x - 1$ (d) None of these.

**Solution :** Given, $f(x-1) = x^2 - x + 1$

Replacing x by x + 1 we get,

$f(x+1-1) = (x+1)^2 - (x+1) + 1$

or, $f(x) = x^2 + 2x + 1 - x - 1 + 1 = x^2 + x + 1$

**Ans.** (a)

(xx) $\underset{x \to 0}{\text{Lt}} \dfrac{x}{\sin^{-1} x}$ is – (a) –1 (b) 0 (c) 1 (d) None of these.

**Solution :** Let, $\sin^{-1} x = y \therefore x = \sin y$ and $\because x \to 0 \therefore y \to 0$

$$\therefore \underset{x \to 0}{\text{Lt}} \frac{x}{\sin^{-1} x} = \underset{y \to 0}{\text{Lt}} \frac{\sin y}{y} = 1$$

**Ans.** (c)

(xxi) The function $f(x) = \dfrac{1}{2021}$ is –

(a) Continuous (b) Discontinuous (c) Both (a) and (b) (d) None of these.

**Solution :** Given, $f(x) = \dfrac{1}{2021}$ is a constant function and hence continuous.

**Ans.** (a)

(xxii) The derivative of sinx with respect to cosx is – (a) tanx (b) cotx (c) –cotx (d) None of these.

**Solution :** Let u = sinx and v = cosx

Differentiating both sides with respect to x we get,

$\dfrac{du}{dx} = \cos x$ and $\dfrac{dv}{dx} = -\sin x$ Therefore, $\dfrac{du}{dv} = \dfrac{du}{dx} \Big/ \dfrac{dv}{dx} = \dfrac{\cos x}{-\sin x} = -\cot x$

**Ans.** (c)

(xxiii) The maximum value of sinx cosx is – (a) 0 (b) 1 (c) $\frac{1}{2}$ (d) None of these.

**Solution :** We know, $\sin x \cos x = \frac{1}{2}(2 \sin x \cos x) = \frac{1}{2} \sin 2x$

Since, maximum value of sin2x = 1, therefore maximum value of sinx.cosx = $\frac{1}{2}$

**Ans.** (c)

(xxiv) The value of $(\sin 30^0 + \cos 30^0) - (\sin 60^0 + \cos 60^0)$ is – (a) 0 (b) 1 (c) –1 (d) None of these.

**Solution :** Given, $(\sin 30^0 + \cos 30^0) - (\sin 60^0 + \cos 60^0) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = 0$

**Ans.** (a)

(xxv) The value of $\tan 1^0 \tan 2^0 \tan 3^0 \ldots\ldots\ldots \tan 89^0$ is – (a) 0 (b) 1 (c) –1 (d) None of these.

**Solution :** Given expression, $\tan 1^0 \tan 2^0 \tan 3^0 \ldots\ldots\ldots\ldots\ldots \tan 89^0$

$= (\tan 1^0 \tan 89^0)(\tan 2^0 \tan 88^0)(\tan 3^0 \tan 87^0) \ldots\ldots\ldots\ldots (\tan 44^0 \tan 46^0) \tan 45^0$

$= (\tan 1^0 \cot 1^0)(\tan 2^0 \cot 2^0)(\tan 3^0 \cot 3^0) \ldots\ldots\ldots\ldots (\tan 44^0 \cot 44^0) \tan 45^0$

$= 1$ [since $\tan 89^0 = \tan(90^0 - 1^0) = \cot 1^0$ etc.]

**Ans.**(b)

## Group – A

2.(a) If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$ then show that $x^x y^y z^z = 1$

**Solution:** Given, $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y} = k$ (say)

$\therefore \log x = k(y-z), \log y = k(z-x), \log z = k(x-y)$

$\therefore x \log x + y\log y + z\log z = xk(y-z) + yk(z-x) + zk(x-y)$

or, $\log x^x + \log y^y + \log z^z = k(xy - zx + yz - xy + zx - yz)$

or, $\log(x^x y^y z^z) = k \times 0 = 0$ or, $x^x y^y z^z = e^0 = 1$ **(Proved)**

(b) Find the term independent of x in the expansion of $\left(9x^2 - \frac{1}{3x}\right)^{12}$

**Solution :** Let (r + 1)th term is independent of x in the expansion of $\left(9x^2 - \frac{1}{3x}\right)^{12}$

Here, (r + 1)th term, $t_{r+1} = {}^{12}C_r \cdot \left(9x^2\right)^{12-r} \cdot \left(-\frac{1}{3x}\right)^r = {}^{12}C_r \cdot 9^{12-r} \cdot \left(-\frac{1}{3}\right)^r \cdot x^{24-3r}$

$\therefore 24 - 3r = 0$ or, $3r = 24$, or, $r = 8$

$\therefore$ (r + 1)th term = 9th term is independent of x.

and the required term is

$= {}^{12}C_8 \cdot 9^{12-8} \cdot \left(-\frac{1}{3}\right)^8 = \frac{12!}{8! \cdot 4!} \cdot 9^4 \cdot \frac{1}{3^8} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{8! \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot 3^8 \cdot \frac{1}{3^8} = 11 \cdot 5 \cdot 9 = 495$ **(Ans)**

3.(a) If $\alpha$ and $\beta$ are the roots of the quadratic equation $2x^2 + x - 1 = 0$, find the equation whose roots are $2\alpha + 1$ and $2\beta + 1$.

**Solution :** $\because$ $\alpha, \beta$ are the roots of $2x^2 + x - 1 = 0$

$\therefore \alpha + \beta = -\frac{1}{2}, \alpha\beta = -\frac{1}{2}$

$\therefore (2\alpha + 1) + (2\beta + 1) = 2(\alpha + \beta) + 2 = 2\left(-\frac{1}{2}\right) + 2 = -1 + 2 = 1$

and $(2\alpha + 1)(2\beta + 1) = 4\alpha\beta + 2(\alpha + \beta) + 1$

$= 4\left(-\frac{1}{2}\right) + 2\left(-\frac{1}{2}\right) + 1 = -2 - 1 + 1 = -2$

$\therefore$ the required equation is

$x^2$ – (sum of the roots)x + product of the roots = 0

or, $x^2 - x - 2 = 0$ **(Ans)**

(b) If $x+\frac{1}{x}=2\cos\theta$, $y+\frac{1}{y}=2\cos\phi$, using De Moivre's theorem, show that

$$x^m y^n+\frac{1}{x^m y^n}=2\cos(m\theta+n\phi).$$

**Solution :** Given $2\cos\theta = x+\frac{1}{x}$, or, $2x\cos\theta = x^2+1$

or, $x^2 - 2x\cos\theta + 1 = 0$

$\therefore x=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4\times1\times1}}{2\times1}$ or, $x=\frac{2\cos\theta\pm2\sqrt{-\left(1-\cos^2\theta\right)}}{2}=\cos\theta\pm i\sin\theta$

Taking +ve sign only in place of $\pm$, we get $x = \cos\theta + i\sin\theta$,

Similarly, $y = \cos\phi + i\sin\phi$

Now $x^m\cdot y^n = (\cos\theta + i\sin\theta)^m\cdot(\cos\phi + i\sin\phi)^n$

$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$ [By De Moivre's Theorem]

$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$

$\therefore x^{-m}\cdot y^{-n} = [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$

$= \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$ [By De Moivre's Theorem]

$\therefore x^m y^n + x^{-m}y^{-n} = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$

or, $x^m y^n+\frac{1}{x^m y^n}=2\cos(m\theta+n\phi)$ **(Proved)**

4.(a) Show that the points whose position vectors are $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ form a right angle triangle.

**Solution:** $\overrightarrow{OA} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$, where O is origin.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) - (4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

And $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) - (3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$|\overrightarrow{AB}| = \sqrt{2^2+1^2+2^2} = 3$, $|\overrightarrow{BC}| = \sqrt{1^2+1^2+4^2} = 3\sqrt{2}$ and $|\overrightarrow{CA}| = \sqrt{1^2+2^2+2^2} = 3$

Therefore, $|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = 3^2 + 3^2 = 18 = \left(3\sqrt{2}\right)^2 = |\overrightarrow{BC}|^2$

or $|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{BC}|^2 \Rightarrow$ ABC is a right–angled triangle, whose $\angle A = 90^0$ **(Proved)**

(b) If two vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$.

**Solution:** Given, $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ or, $|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

or, $|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

or, $2\vec{a}\cdot\vec{b} = -2\vec{a}\cdot\vec{b}$ or, $4\vec{a}\cdot\vec{b} = 0$ or, $\vec{a}\cdot\vec{b} = 0$ whish shows that the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, i. e., the angle between $\vec{a}$ and $\vec{b}$ is $90^0$. **(Ans)**

5.(a) A particle is acted on by a force 4i + j – 3k and 3i + j – k is displaced from the point A(1, 2, 3) to thr point B(5, 4, 1). Find the work done by the forces on the particle.

**Solution :** Given forces are 4i + j – 3k and 3i + j – k.

Therefore, the resultant force = (4i + j – 3k) + (3i + j – k) = 7i + 2j – 4k.

Since, the particle is displaced from the point A(1, 2, 3) to the point B(5, 4, 1), the displacement vector = (5 – 1)i + (4 – 2)j + ( 1 – 3)k = 4i + 2j – 2k

Hence the required work done

= (7i + 2j – 4k) . (4i + 2j – 2k) = 28 + 4 + 8 = 40 unit of work. **(Ans)**

(b) If $2\vec{i}+3\vec{j}+4\vec{k}$ and $3\vec{i}+4\vec{j}+7\vec{k}$ are the diagonals of a parallelogram then find the area of the parallelogram.

**Solution :** Given diagonals of the parallelogram are $2\vec{i}+3\vec{j}+4\vec{k}$ and $3\vec{i}+4\vec{j}+7\vec{k}$

$\therefore$ the required vector area of the parallelogram

$$= \frac{1}{2}(2\vec{i}+3\vec{j}+4\vec{k})\times(3\vec{i}+4\vec{j}+7\vec{k}) = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{vmatrix}$$

$$= \frac{1}{2}\{(21-16)\vec{i} - (14-12)\vec{j} + (8-9)\vec{k}\} = \frac{1}{2}(5\vec{i} - 2\vec{j} - \vec{k})$$

$\therefore$ the required scalar area $= \frac{1}{2}|5\vec{i} - 2\vec{j} - \vec{k}| = \frac{1}{2}\sqrt{5^2 + (2)^2 + (-1)^2} = \frac{1}{2}\sqrt{30}$ sq. unit. **(Ans)**

## <u>Group – B</u>

6.(a) If 2tanα = 3tanβ prove that $\tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}$

**Solution :** 2 tanα = 3 tanβ or, $\tan\alpha = \frac{3}{2}\tan\beta$

$$\therefore \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}=\frac{\frac{3}{2}\tan\beta-\tan\beta}{1+\frac{3}{2}\tan\beta\cdot\tan\beta}=\frac{3\tan\beta-2\tan\beta}{2+3\tan^2\beta}=\frac{\tan\beta}{2+3\tan^2\beta}$$

$$=\frac{\frac{\sin\beta}{\cos\beta}}{2+\frac{3\sin^2\beta}{\cos^2\beta}}=\frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta}=\frac{2\sin\beta\cos\beta}{2\cdot\left(2\cos^2\beta\right)+3\cdot\left(2\sin^2\beta\right)}$$

$$=\frac{\sin 2\beta}{2\cdot(1+\cos 2\beta)+3\cdot(1-\cos 2\beta)}=\frac{\sin 2\beta}{2+2\cos 2\beta+3-3\cos 2\beta}=\frac{\sin 2\beta}{5-\cos 2\beta}$$ **(Proved)**

(b) If cosecA + secA = cosecB + secB, prove that $\tan\frac{A+B}{2}=\cot A\cot B$.

**Solution :** Given, cosecA + secA = cosecB + secB

or, secA – secB = cosecB – cosecA

or, $\frac{1}{\cos A}-\frac{1}{\cos B}=\frac{1}{\sin B}-\frac{1}{\sin A}$ or, $\frac{\cos B-\cos A}{\cos A\cos B}=\frac{\sin A-\sin B}{\sin A\sin B}$

or, $\frac{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}{\cos A\cos B}=\frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\sin A\sin B}$

or, $\frac{\sin\frac{A+B}{2}}{\cos A\cos B}=\frac{\cos\frac{A+B}{2}}{\sin A\sin B}$ or, $\frac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}}=\frac{\cos A\cos B}{\sin A\sin B}$

or, $\tan\frac{A+B}{2}=\cot A\cot B$ **(Proved)**

7.(a) If $\cos^2\theta-\sin^2\theta=\tan^2\alpha$, show that $\cos^2\alpha-\sin^2\alpha=\tan^2\theta$.

**Solution :** Given, $\cos^2\theta-\sin^2\theta=\tan^2\alpha$ or, $\frac{\cos^2\theta-\sin^2\theta}{1}=\frac{\sin^2\alpha}{\cos^2\alpha}$

or, $\frac{1-\cos^2\theta+\sin^2\theta}{1+\cos^2\theta-\sin^2\theta}=\frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha+\sin^2\alpha}$ [by componendo and dividendo]

or, $\frac{\sin^2\theta+\sin^2\theta}{\cos^2\theta+\left(1-\sin^2\theta\right)}=\frac{\cos^2\alpha-\sin^2\alpha}{1}$ [$\because \sin^2x+\cos^2x=1$]

or, $\frac{2\sin^2\theta}{2\cos^2\theta}=\cos^2\alpha-\sin^2\alpha$ or, $\cos^2\alpha-\sin^2\alpha=\tan^2\theta$ **(Proved)**

(b) If $\tan\theta = \sec 2\alpha$ then prove that, $\sin 2\theta = \dfrac{1-\tan^4\alpha}{1+\tan^4\alpha}$

**Solution :** $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{2\,.\sec 2\alpha}{1+\sec^2 2\alpha} = \dfrac{2\cos 2\alpha}{\cos^2 2\alpha + 1} = \dfrac{2\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}}{\dfrac{(1-\tan^2\alpha)^2}{(1+\tan^2\alpha)^2}+1}$

$= \dfrac{2(1+\tan^2\alpha)(1-\tan^2\alpha)}{(1+\tan^2\alpha)^2+(1-\tan^2\alpha)^2} = \dfrac{2(1-\tan^4\alpha)}{2(1+\tan^4\alpha)}$

$\therefore \sin 2\theta = \dfrac{1-\tan^4\alpha}{1+\tan^4\alpha}$ **(Proved)**

8.(a) Solve : $7\cos^2\theta + 3\sin^2\theta = 4$ for $0 \le \theta \le 2\pi$

**Solution :** $7\cos^2\theta + 3\sin^2\theta = 4$ or, $7\cos^2\theta + 3\sin^2\theta = 4(\sin^2\theta + \cos^2\theta)$

or, $7\cos^2\theta + 3\sin^2\theta = 4\sin^2\theta + 4\cos^2\theta$ or, $3\cos^2\theta = \sin^2\theta$

or, $\tan^2\theta = 3$ or, $\tan\theta = \pm\sqrt{3}$ or, $\tan\theta = \tan(\pm 60^o)$

$\therefore \theta = n\pi \pm 60^o$, where $n = 0, \pm1, \ldots\ldots\ldots\ldots\ldots\ldots$

For $n = 0$, $\theta = \pm 60^o$

For $n = 1$, $\theta = \pi \pm 60^o = 180^o + 60^o, 180^o - 60^o = 240^o, 120^o$

For $n = 2$, $\theta = 2\pi \pm 60^o = 2\pi + 60^o, 2\pi - 60^o = 420^o, 300^o$

$\therefore$ the required solutions are $\theta = 60^o, 240^o, 120^o, 300^o$ **(Ans)**

(b) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then show that $x + y + z = xyz$.

**Solution :** Given, $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

or, $\tan^{-1}\left(\dfrac{x+y+z-xyz}{1-yz-zx-xy}\right) = \pi$ or, $\dfrac{x+y+z-xyz}{1-yz-zx-xy} = \tan\pi = 0$

or, $x + y + z - xyz = 0$ or, $x + y + z = xyz$ **(Proved)**

9.(a) Evaluate : $\underset{x\to 0}{\text{Lt}} \dfrac{2^{2x}-3^{3x}}{x}$

**Solution :** $\underset{x\to 0}{\text{Lt}} \dfrac{2^{2x}-3^{3x}}{x} = \underset{x\to 0}{\text{Lt}} \dfrac{4^x-27^x}{x} = \underset{x\to 0}{\text{Lt}} \dfrac{(4^x-1)-(27^x-1)}{x}$

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$$= \underset{x\to 0}{Lt}\frac{4^x-1}{x} - \underset{x\to 0}{Lt}\frac{27^x-1}{x} = \log_e 4 - \log_e 27 \left[\because \underset{x\to 0}{Lt}\frac{a^x-1}{x} = \log_e a\right]$$

$$= \log_e\left(\frac{4}{27}\right) \textbf{ (Ans)}$$

(b) If $f(x) = |x|$, prove that $f(x)$ is continuous at $x = 0$.

**Solution :** We know,

$$f(x) = |x| = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$$

Now,

$$\underset{x\to 0-}{Lt} f(x) = \underset{x\to 0-}{Lt}(-x) = 0 \qquad [\because f(x) = -x, \text{ when } x < 0]$$

$$\underset{x\to 0+}{Lt} f(x) = \underset{x\to 0+}{Lt} x = 0 \qquad [\because f(x) = x, \text{ when } x > 0]$$

and $f(0) = 0$

$$\therefore \underset{x\to 0-}{Lt} f(x) = \underset{x\to 0+}{Lt} f(x) = f(0) \quad \text{i.e., } \underset{x\to 0}{Lt} f(x) = f(0)$$

Hence, from definition of continuity $f(x)$ is continuous at $x = 0$. **(Proved)**

10.(a) If $x^y = e^{x-y}$, prove that, $\frac{dy}{dx} = \frac{\log x}{(\log ex)^2}$

**Solution :** $x^y = e^{x-y}$

Taking logarithm of both sides we get,

$y\log x = (x - y)\log e = x - y$ $[\because \log e = 1]$ or, $x = y + y\log x = y(1 + \log x)$

$$\text{or, } y = \frac{x}{1+\log x} \quad \text{or, } \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{1+\log x}\right) = \frac{(1+\log x)\times\frac{d}{dx}(x) - x\times\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

[ differentiating both sides with respect to x ]

$$\text{or, } \frac{dy}{dx} = \frac{1+\log x - x\times\frac{1}{x}}{(1+\log x)^2} = \frac{1+\log x - 1}{(\log e + \log x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\log x}{\{\log(ex)\}^2} \quad \textbf{(Proved)}$$

(b) Differentiate $\sec^{-1}\frac{1}{2x^2-1}$ with respect to $\sqrt{1-x^2}$

**Solution :**

Let, $u=\sec^{-1}\frac{1}{2x^2-1}=\sec^{-1}\frac{1}{2\cos^2\theta-1}=\sec^{-1}\frac{1}{\cos 2\theta}=\sec^{-1}\sec 2\theta=2\theta=2\cos^{-1}x$

[ let, $x = \cos\theta$ ]

$\therefore \frac{du}{dx}=2\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{2}{\sqrt{1-x^2}}$ [differentiating both sides with respect to x]

Again let, $v=\sqrt{1-x^2}$

$\therefore \frac{dv}{dx}=\frac{d}{dx}\left(\sqrt{1-x^2}\right)=\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}\left(1-x^2\right)=\frac{-2x}{2\sqrt{1-x^2}}=-\frac{x}{\sqrt{1-x^2}}$

[differentiating both sides with respect to x]

$$\therefore \frac{du}{dv}=\frac{du}{dx}\bigg/\frac{dv}{dx}=\frac{-\frac{2}{\sqrt{1-x^2}}}{-\frac{x}{\sqrt{1-x^2}}}=\frac{2}{x}\ \textbf{(Ans)}$$

11.(a) If $y=\left(\sin^{-1}x\right)^2$ prove that, $(1-x^2)y_2-xy_1-2=0$

**Solution :** Given, $y=(\sin^{-1}x)^2$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx}=2\left(\sin^{-1}x\right)\times\frac{d}{dx}\left(\sin^{-1}x\right)=2\sin^{-1}x\times\frac{1}{\sqrt{1-x^2}}$$

or, $\sqrt{1-x^2}\frac{dy}{dx}=2\sin^{-1}x$ or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=4\left(\sin^{-1}x\right)^2$ [squaring both sides]

or, $\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=4y$ or, $\frac{d}{dx}\left\{\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2\right\}=4\frac{dy}{dx}$

[differentiating both sides with respect to x ]

or, $\left(1-x^2\right)\times 2\frac{dy}{dx}\times\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\times(-2x)=4\frac{dy}{dx}$

or, $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$

or, $(1-x^2)y_2-xy_1-2=0$ **(Proved)**

11.(b) Find the value of x for which the function $2x^3 - 21x^2 + 36x - 20$ has maximum and minimum values and also find the maximum and minimum values of the function.

**Solution :** Let, $f(x) = 2x^3 - 21x^2 + 36x - 20$

$\therefore$ $f'(x) = 6x^2 - 42x + 36$ and $f''(x) = 12x - 42$

Now, $f'(x) = 0$ gives, $6x^2 - 42x + 36 = 0$ or, $x^2 - 7x + 6 = 0$

or, $(x - 6)(x - 1) = 0$ or, $x = 1, 6$

Now, at $x = 1$, $f''(x) = 12 - 42 = -30 < 0$ and

at $x = 6$, $f''(x) = 72 - 42 = 30 > 0$ which shows that the given function is maximum and minimum at $x = 1$ and $x = 6$ respectively.

Maximum value, $f(1) = 2 - 21 + 36 - 20 = 38 - 41 = -3$ and

Minimum value, $f(6) = 2 \cdot 6^3 - 21 \cdot 6^2 + 36 \cdot 6 - 20 = 432 - 756 + 216 - 20 = -128$ **(Ans)**

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APRIL – 2022

## MATHEMATICS – I

Time Allowed: 3 Hours                                                    Full Marks : 60

Answer the following questions from Group – A, B & C as directed.

### Group – A

1. Choose the correct alternatives (any ten) : $10 \times 1$

(i) The value of $\log_b a \times \log_c b \times \log_d c$ is – (a) $\log_c a$ (b) $\log_d c$ (c) $\log_b a$ (d) $\log_d a$

**Solution:** Given, $\log_b a \times \log_c b \times \log_d c = \frac{\log a}{\log b} \times \frac{\log b}{\log c} \times \frac{\log c}{\log d} = \frac{\log a}{\log d} = \log_d a$

**Ans.** (d)

(ii) The value of $z^{3\log_z a}$ is equal to – (a) $a^2$ (b) $a^3$ (c) a (d) $z^a$

**Solution:** Given, $z^{3\log_z a} = z^{\log_z a^3} = a^3$

**Ans.** (b)

(iii) The equation $\log_a x + \log_a(1 + x) = 0$ may be written as –

(a) $x^2 + x + 1 = 0$ (b) $x^2 - x + 1 = 0$ (c) $x^2 + x - 1 = 0$, (d) None of these.

**Solution:** Given, $\log_a x + \log_a(1 + x) = 0$ or, $\log_a x(1 + x) = 0$ or, $x(1 + x) = a^0$

or, $x(1 + x) = 1$ or, $x^2 + x - 1 = 0$

**Ans.** (c)

**(iv)** If one root of the equation $x^2 + 6x + m = 0$ is 1, then the value of m is –

(a) 7 (b) –7 (c) 6 (d) None of these.

**Solution:** Since, one root of the equation $x^2 + 6x + m = 0$ is 1,

then $1^2 + 6 + m = 0$ or, $m = -7$

**Ans.** (b)

(v) In the expansion of $(1 + px)^{10}$ the coefficient of $x^2$ is –

(a) ${}^{10}C_2$ (b) ${}^{10}C_2\, p$ (c) ${}^{10}C_2\, p^2$ (d) None of these.

**Solution:** Given, $(1 + px)^{10} = 1 + {}^{10}C_1\, px + {}^{10}C_2\, p^2x^2 + \ldots\ldots\ldots\ldots + p^{10}x^{10}$

Therefore, the coefficient of $x^2$ is ${}^{10}C_2\, p^2$

**Ans.** (c)

(vi) Square root of –7 is (a) 49 (b) 7i (c) $\sqrt{7}\,i$ (d) None of these.

**Solution:** Since, $\left(\sqrt{7}\,i\right)^2 = 7i^2 = -7$ [since $i^2 = -1$], therefore square root of –7 is $\sqrt{7}\,i$

**Ans.** (c)

(vii) Which one of the following is impossible ?

(a) $0 < \cos\theta < 1$ (b) $\operatorname{cosec}^2\theta < 1$ (c) $-1 \le \sin\theta \le 1$ (d) $\sin\theta = 0$.

**Solution:** We know, $\operatorname{cosec}\theta \ge 1$ and $\operatorname{cosec}\theta \le -1$ therefore, $\operatorname{cosec}^2\theta < 1$ is impossible.

**Ans.** (b)

(viii) If $\tan A = \frac{7}{5}$ and $\tan B = \frac{5}{7}$ then the value of $\cot(A - B)$ is –

(a) 0 (b) $\frac{35}{12}$ (c) $\frac{5}{12}$ (d) None of these.

**Solution:** Given, $\tan A = \frac{7}{5}$ and $\tan B = \frac{5}{7}$

Therefore, $\cot(A - B) = \dfrac{1}{\tan(A-B)} = \dfrac{1 + \tan A \tan B}{\tan A - \tan B} = \dfrac{1 + \frac{7}{5} \times \frac{5}{7}}{\frac{7}{5} - \frac{5}{7}} = \dfrac{1+1}{\frac{49-25}{35}} = \dfrac{70}{24} = \dfrac{35}{12}$

**Ans.** (b)

(ix) If $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$ and $\tan 2\theta = \frac{3}{4}$ then the value of $\tan\theta$ is –

(a) 3 (b) –3 (c) 2 (d) None of these.

**Solution:** Given, $\tan 2\theta = \frac{3}{4}$ or, $\dfrac{2\tan\theta}{1 - \tan^2\theta} = \dfrac{3}{4}$

or, $8\tan\theta = 3 - 3\tan^2\theta$ or, $3\tan^2\theta + 8\tan\theta - 3 = 0$ or, $3\tan^2\theta + 9\tan\theta - \tan\theta - 3 = 0$

or, $3\tan\theta(\tan\theta + 3) - 1(\tan\theta + 3) = 0$ or, $(\tan\theta + 3)(3\tan\theta - 1) = 0$

or, $\tan\theta = -3, \frac{1}{3}$

Since, $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$, therefore, $\tan\theta = -3$

**Ans.** (b)

(x) Which of the following does not exist ?

(a) $\sin^{-1}(0.6)$ (b) $\operatorname{cosec}^{-1}(6.6)$ (c) $\cot^{-1}(10)$ (d) $\sec^{-1}(0.3)$.

**Solution:** We know, $\sec^{-1}x$ is defined when $x \ge 1$ and $x \le -1$

Therefore, $\sec^{-1}(0.3)$ does not exist.

**Ans.** (d)

(xi) Which of the following is true ?

(a) sinx is a periodic function of period $\pi$ (b) cosx is an odd function.

(c) $x^3$ is an even function. (d) tanx is a periodic function of period $\pi$.

**Solution:** Since, $\tan(\pi + x) = \tan x$, therefore, tanx is a periodic function of period $\pi$ is true.

**Ans.** (d)

(xii) $x \to 2$ does imply – (a) $x < 2$ (b) $x > 2$ (c) $x = 2$ (d) x is nearer to 2.

**Solution:** $x \to 2$ does imply, x is nearer to 2.

**Ans.** (d)

(xiii) $\frac{d}{dx}(\log_{10} \sin x)$ is equal to – (a) cotx (b) $\frac{\cot x}{\log_{10} e}$ (c) $\frac{\cot x}{\log_e 10}$ (d) None of these.

**Solution:** Given, $\frac{d}{dx}(\log_{10} \sin x) = \frac{d}{dx}\left(\frac{\log_e \sin x}{\log_e 10}\right) = \frac{1}{\log_e 10} \cdot \frac{d}{dx}(\log_e \sin x)$

$= \frac{1}{\log_e 10} \cdot \frac{1}{\sin x} \frac{d}{dx}(\sin x) = \frac{1}{\log_e 10} \cdot \frac{1}{\sin x} \cdot \cos x = \frac{\cot x}{\log_e 10}$

**Ans.** (c)

(xiv) Which of the following is not a vector ?

(a) Velocity (b) Displacement (c) Mass (d) None of these.

**Solution:** Mass is not a vector.

**Ans.** (c)

(xv) $\vec{a} \times \vec{b}$ is a vector –

(a) Parallel to $\vec{a}$ (b) Perpendicular to both $\vec{a}$ and $\vec{b}$.

(c) Parallel to $\vec{b}$ (d) Perpendicular to $\vec{a}$

**Solution:** $\vec{a} \times \vec{b}$ is equal to a vector, perpendicular to both $\vec{a}$ and $\vec{b}$

**Ans.** (b)

2. Fill in the blanks (any ten) : $10 \times 1$

(i) If $\log_{10}a + \log_{10}b = \log_{10}(a + b)$, then b is equal to ________

**Solution :** Given, $\log_{10}a + \log_{10}b = \log_{10}(a + b)$

or, $\log_{10}(ab) = \log_{10}(a + b)$ or, $ab = a + b$ or, $b(a - 1) = a$ or, $b = \frac{a}{a-1}$ **(Ans)**

**(ii)** If one root of the equation $x^2 - 8x + m = 0$ is 2, then the other root is ________

**Solution :** By the problem $x = 2$ must satisfy the equation $x^2 - 8x + m = 0$.

Therefore, $4 - 16 + m = 0$ or, $m = 12$

So the given equation becomes,

$x^2 - 8x + 12 = 0$ or, $(x - 2)(x - 6) = 0$ or, $x = 2, 6$

Hence, the other root is 6 **(Ans)**

(iii) The number of terms in the expansion of $(a + b)^n$, n is a positive integer, is ________

**Solution :** Since, the power is n, therefore the number of terms in the expansion of $(a + b)^n$ is $(n + 1)$ **(Ans)**

(iv) Conjugate of the complex number 5 is ________

**Solution :** Given, complex number is $5 = 5 + i\,0$.

Therefore, the required conjugate complex number is $5 - i\,0 = 5$ **(Ans)**

(v) If z is purely imaginary, then $z+\bar{z}$ is ________

**Solution :** Since, z is purely imaginary, let $z = i\,x$, x is real.

Therefore, $z+\bar{z} = i\,x - i\,x = 0$ **(Ans)**

(vi) The value of $\cos1^0\cos2^0\cos3^0........................\cos123^0\cos124^0\cos125^0$ is ________

**Solution :** Given, $\cos1^0\cos2^0\cos3^0........................\cos123^0\cos124^0\cos125^0$

$= \cos1^0\cos2^0\cos3^0............\cos90^0............\cos123^0\cos124^0\cos125^0 = 0$ [since, $\cos90^0 = 0$ ]

(vii) Minimum value of $(\sin\theta + \cos\theta)$ is ________

**Solution :** Given, $\sin\theta+\cos\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right)$

$$= \sqrt{2}\left(\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right)= \sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$

We know, $-1 \le \sin\left(\theta+\frac{\pi}{4}\right) \le 1$ or, $-\sqrt{2} \le \sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right) \le \sqrt{2}$

or, $-\sqrt{2} \le \sin\theta+\cos\theta \le \sqrt{2}$

Which shows that, minimum value of $\sin\theta + \cos\theta$ is $-\sqrt{2}$ **(Ans)**

(viii) If $\tan\theta = 2$, then the value of $\cos 2\theta$ is ________

**Solution :** Given, $\tan\theta = 2$

Therefore, $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-2^2}{1+2^2} = -\frac{3}{5}$ **(Ans)**

(ix) $\cos\left(\sin^{-1}\frac{1}{2}+\sec^{-1}2\right)$ is ________

**Solution :** Given, $\cos\left(\sin^{-1}\frac{1}{2}+\sec^{-1}2\right) = \cos\left(\operatorname{cosec}^{-1}2+\sec^{-1}2\right) = \cos\frac{\pi}{2} = 0$ **(Ans)**

(x) Value of $\underset{x\to 0}{\text{Lt}}\ \frac{x}{3x^2-2x}$ is ________

**Solution :** Given, $\underset{x\to 0}{\text{Lt}}\ \frac{x}{3x^2-2x} = \underset{x\to 0}{\text{Lt}}\ \frac{x}{x(3x-2)} = \underset{x\to 0}{\text{Lt}}\ \frac{1}{3x-2}\ [x \neq 0 \because x \to 0] = -\frac{1}{2}$ **(Ans)**

(xi) If $y = \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$ then $\frac{dy}{dx}$ is ________

**Solution :** Given, $y = \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \tan^{-1}\sqrt{\frac{2\sin^2 x}{2\cos^2 x}} = \tan^{-1}\sqrt{\tan^2 x} = \tan^{-1}(\tan x) = x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x) = 1$ **(Ans)**

(xii) If $y = \left(\tan^{-1}x\right)^2$ and $\frac{dy}{dx} = 2\tan^{-1}x\cdot f(x)$ then $f(x)$ is ________

**Solution :** Given, $y = \left(\tan^{-1}x\right)^2$

Differentiating both sides with respect to x we get,

or, $\frac{dy}{dx} = 2\tan^{-1}x\cdot\frac{1}{1+x^2}$

Again, given $\frac{dy}{dx} = 2\tan^{-1}x\cdot f(x)$

Therefore, $2\tan^{-1}x\cdot f(x) = 2\tan^{-1}x\cdot\frac{1}{1+x^2}$

or, $f(x) = \frac{1}{1+x^2}$ **(Ans)**

(xiii) If $\vec{a} = -\vec{i} + 2\vec{j}$, $\vec{b} = 3\vec{i} - 2\vec{j}$ and $\vec{c} = 5\vec{j}$ then $|\vec{a} + \vec{b} - 2\vec{c}|$ is ____________

**Solution :** Given, $\vec{a} = -\vec{i} + 2\vec{j}$, $\vec{b} = 3\vec{i} - 2\vec{j}$ and $\vec{c} = 5\vec{j}$

$$\therefore |\vec{a} + \vec{b} - 2\vec{c}| = |(-\vec{i} + 2\vec{j}) + (3\vec{i} - 2\vec{j}) - 10\vec{j}| = |2\vec{i} - 10\vec{j}| = \sqrt{2^2 + (-10)^2} = \sqrt{104} \text{ (Ans)}$$

(xiv) If the position vectors of two points A and B are respectively $\vec{i} + 2\vec{j} - 3\vec{k}$ and $4\vec{i} - 5\vec{j} + 6\vec{k}$ then the unit vector in the direction of $\overrightarrow{AB}$ is ____________

**Solution :** Let, $\overrightarrow{OA} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\overrightarrow{OB} = 4\vec{i} - 5\vec{j} + 6\vec{k}$ where O is origin.

Therefore, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\vec{i} - 5\vec{j} + 6\vec{k}) - (\vec{i} + 2\vec{j} - 3\vec{k}) = 3\vec{i} - 7\vec{j} + 9\vec{k}$

Therefore, the required unit vector is

$$\pm \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \pm \frac{3\vec{i} - 7\vec{j} + 9\vec{k}}{|3\vec{i} - 7\vec{j} + 9\vec{k}|} = \pm \frac{3\vec{i} - 7\vec{j} + 9\vec{k}}{\sqrt{3^2 + (-7)^2 + 9^2}} = \pm \frac{3\vec{i} - 7\vec{j} + 9\vec{k}}{\sqrt{9 + 49 + 81}} = \pm \frac{3\vec{i} - 7\vec{j} + 9\vec{k}}{\sqrt{139}} \text{ (Ans)}$$

(xv) The angle between the vectors $2\vec{i} + 3\vec{j} + \vec{k}$ and $2\vec{i} - \vec{j} - \vec{k}$ is ____________

**Solution :** Let, $\theta$ be the angle between the vectors $2\vec{i} + 3\vec{j} + \vec{k}$ and $2\vec{i} - \vec{j} - \vec{k}$.

$$\therefore \cos\theta = \frac{(2\vec{i} + 3\vec{j} + \vec{k}) \cdot (2\vec{i} - \vec{j} - \vec{k})}{|2\vec{i} + 3\vec{j} + \vec{k}| \cdot |2\vec{i} - \vec{j} - \vec{k}|} = \frac{2 \times 2 + 3 \times (-1) + 1 \times (-1)}{\sqrt{2^2 + 3^2 + 1^2} \cdot \sqrt{2^2 + (-1)^2 + (-1)^2}} = \frac{4 - 3 - 1}{\sqrt{14} \cdot \sqrt{6}} = \frac{4 - 4}{\sqrt{14} \cdot \sqrt{6}} = 0$$

$$\therefore \theta = \frac{\pi}{2} = 90^0 \text{ (Ans)}$$

3. Answer the following questions (any ten) : 10 × 1

(i) If $\log_{10} a = r$ then find the value of $a^{\frac{3}{r}}$

**Solution :** Given, $\log_{10} a = r$ or, $a = 10^r$

Therefore, $a^{\frac{3}{r}} = (10^r)^{\frac{3}{r}} = 10^3 = 1000$ **(Ans)**

(ii) If $4 - 3i$ be a root of the equation $px^2 + qx + 1 = 0$, where p, q are real, find the value of p and q.

**Solution :** Since $4 - 3i$ is a root of the equation $px^2 + qx + 1 = 0$, so $4 + 3i$ is the other root.

Therefore, $4 - 3i + 4 + 3i = -\frac{q}{p}$ or, $8 = -\frac{q}{p}$ or, $q = -8p$ .................................. (1)

and $(4 - 3i)(4 + 3i) = \frac{1}{p}$ or, $16 + 9 = \frac{1}{p}$ or, $p = \frac{1}{25}$ and $q = -\frac{8}{25}$ [ by (1) ] **(Ans)**

(iii) If the roots of the equation $x^2 - px + q = 0$ be in the ratio 1 : 2 then find the relation between p and q.

**Solution :** Since the ratio of the roots is 1 : 2, let a, 2a be the roots.

Therefore, $a + 2a = p$ or, $3a = p$ or, $a = \frac{p}{3}$ ........................................ (1)

and $a \cdot 2a = q$ or, $2a^2 = q$ or, $2\left(\frac{p}{3}\right)^2 = q$ [ by (1) ] or, $2p^2 = 9q$ **(Ans)**

(iv) Find the middle term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^8$.

**Solution :** Middle term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^8$ is $\left(\frac{8}{2}+1\right)$th = 5th term.

$$\text{Now } t_5 = t_{4+1} = {}^8C_4\left(2x^2\right)^{8-4}\left(-\frac{1}{x}\right)^4 = \frac{8!}{4!\cdot 4!}\cdot 2^4 x^8 \cdot \frac{1}{x^4}$$

$$= \frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!\cdot 4\cdot 3\cdot 2\cdot 1}\cdot 2^4 x^4 = 1120x^4 \quad \textbf{(Ans)}$$

(v) What in the amplitude of the complex number –10.

**Solution :** We know, $-10 = -10 + 0\,i$

Therefore, amplitude $= \tan^{-1}\left(\frac{0}{-10}\right) = \tan^{-1} 0 = 0^0$ **(Ans)**

(vi) If cot3x cot5x = 1, then find the value of cot4x.

**Solution :** Given, cot3x cot5x = 1 or, cot3x = tan5x or, cot3x = $\cot(90^0 - 5x)$

or, $3x = 90^0 - 5x$ or, $8x = 90^0$ or, $4x = 45^0$

Therefore, $\cot 4x = \cot 45^0 = 1$ **(Ans)**

(vii) Find the value of $\tan 27^0 + \tan 18^0 + \tan 27^0 \tan 18^0$

**Solution :** We know, $\tan 45^0 = 1$ or, $\tan(27^0 + 18^0) = 1$

or, $\frac{\tan 27^0 + \tan 18^0}{1 - \tan 27^0 \tan 18^0} = 1$ or, $\tan 27^0 + \tan 18^0 = 1 - \tan 27^0 \tan 18^0$

or, $\tan 27^0 + \tan 18^0 + \tan 27^0 \tan 18^0 = 1$ **(Ans)**

(viii) If $\cos(x - y) + 1 = 0$, then find the value of cosx + cosy.

**Solution :** Given, $\cos(x - y) + 1 = 0$ or, $\cos(x - y) = -1 = \cos\pi$

Therefore, $x - y = \pi$ or, $x = \pi + y$

Now, $\cos x + \cos y = \cos(\pi + y) + \cos y = -\cos y + \cos y = 0$ **(Ans)**

(ix) What is the value of $\sec^{-1}(-\sqrt{2})$

**Solution :** Given, $\sec^{-1}(-\sqrt{2}) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(-\cos\frac{\pi}{4}\right)$

$= \cos^{-1}\left\{\cos\left(\pi - \frac{\pi}{4}\right)\right\} = \cos^{-1}\left(\cos\frac{3\pi}{4}\right) = \frac{3\pi}{4}$ **(Ans)**

(x) Evaluate : $\underset{x \to 1}{\text{Lt}} \frac{\sqrt{x}-1}{x-1}$

**Solution :** Given, $\underset{x \to 1}{\text{Lt}} \frac{\sqrt{x}-1}{x-1} = \underset{x \to 1}{\text{Lt}} \frac{\sqrt{x}-1}{(\sqrt{x}+1)(\sqrt{x}-1)} = \underset{x \to 1}{\text{Lt}} \frac{1}{\sqrt{x}+1} [\because x \to \therefore x \neq 1]$

$= \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}$ **(Ans)**

(xi) $f(x) = \frac{2x^2-8}{x-2}$ is undefined at x = 2, then find the value of f(2).

**Solution :** Given, $f(x) = \frac{2x^2-8}{x-2} = \frac{2(x^2-4)}{x-2} = \frac{2(x+2)(x-2)}{x-2} = 2(x+2);\ x \neq 2$

Therefore, $f(2) = 2(2 + 2) = 8$ **(Ans)**

(xii) If $y = \cot^{-1}x + \cot^{-1}\frac{1}{x}$, find the value of $\frac{dy}{dx}$.

**Solution :** Given, $y = \cot^{-1}x + \cot^{-1}\frac{1}{x} = \cot^{-1}x + \tan^{-1}x = \frac{\pi}{2}$

Differentiating both sides with respect to x we get,

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$ **(Ans)**

(xiii) Force $\vec{F} = 2\vec{i} - \vec{j} - \vec{k}$ is applied to an object making a displacement $\vec{d} = 3\vec{i} + 2\vec{j} - 5\vec{k}$. Find the work done.

**Solution :** Given force $\vec{F} = 2\vec{i} - \vec{j} - \vec{k}$ and displacement $\vec{d} = 3\vec{i} + 2\vec{j} - 5\vec{k}$

Therefore, the required work down,

$\vec{F} \cdot \vec{d} = (2\vec{i} - \vec{j} - \vec{k}) \cdot (3\vec{i} + 2\vec{j} - 5\vec{k})$ = (6 – 2 + 5) unit = 9 unit. **(Ans)**

(xiv) A force $\vec{F} = 2\vec{i} + \vec{j} - \vec{k}$ acts at a point whose position vector is $2\vec{i} - \vec{j}$, find the moment of $\vec{F}$ about the origin.

**Solution :** Here, $\vec{r} = \overrightarrow{OP} = 2\vec{i} - \vec{j}$ where O is origin, P is a point whose position vector is $2\vec{i} - \vec{j}$.

Given force, $\vec{F} = 2\vec{i} + \vec{j} - \vec{k}$

Therefore, the required moment,

$$\vec{M} = \vec{r} \times \vec{F} = (2\vec{i} - \vec{j}) \times (2\vec{i} + \vec{j} - \vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix}$$

$= (1-0)\vec{i} - (-2-0)\vec{j} + (2+2)\vec{k} = \vec{i} + 2\vec{j} + 4\vec{k}$ **(Ans)**

(xv) If $\vec{a} = 2\vec{i} - \vec{j} + \lambda\vec{k}$ and $\vec{b} = 4\vec{i} - 2\vec{j} + 3\vec{k}$ are collinear, then find the value of $\lambda$.

**Solution :** Given, $\vec{a} = 2\vec{i} - \vec{j} + \lambda\vec{k}$ and $\vec{b} = 4\vec{i} - 2\vec{j} + 3\vec{k}$ are collinear.

Therefore, $2\vec{i} - \vec{j} + \lambda\vec{k} = x(4\vec{i} - 2\vec{j} + 3\vec{k})$, x is a scalar.

Which gives, $2 = 4x$, $-1 = -2x$, $\lambda = 3x$

or, $x = \frac{1}{2}$ and $\lambda = \frac{3}{2}$ **(Ans)**

## Group – B

4. Answer the following questions (any six) : 6 × 2

(i) Evaluate : $\log_{y^3} x \times \log_{z^3} y \times \log_{x^3} z$

**Solution :** Given, $\log_{y^3} x \times \log_{z^3} y \times \log_{x^3} z$

$$= \frac{\log x}{\log y^3} \times \frac{\log y}{\log z^3} \times \frac{\log z}{\log x^3} = \frac{\log x}{3\log y} \times \frac{\log y}{3\log z} \times \frac{\log z}{3\log x} = \frac{1}{27} \text{ (Ans)}$$

(ii) The roots $\alpha$ and $\beta$ of the equation $x^2 - px + q = 0$ are such that $2\beta + \alpha = 0$. Then find the relation between p and q.

**Solution :** Since, $\alpha$ and $\beta$ are the roots of the equation $x^2 - px + q = 0$

Therefore, $\alpha + \beta = p$ ......... (1) and $\alpha\beta = q$ .......... (2)

Again, given $2\beta + \alpha = 0$ or, $\beta + \beta + \alpha = 0$ or, $\beta + p = 0$ [by (1)[ or, $\beta = -p$

From (1), $\alpha - p = p$ or, $\alpha = 2p$

From (2), $2p(-p) = q$ or, $2p^2 + q = 0$, is the required relation between p and q. **(Ans)**

(iii) If x, y are real and $x + 3i$ and $-2 + iy$ are conjugate to each other, find the value of x and y.

**Solution :** $\because$ $x + 3i$ and $-2 + iy$ are conjugate to each other,

$\therefore$ $x + 3i = -2 - iy \Rightarrow x = -2, y = -3$ **(Ans)**

(iv) If $|z - 5| = 5$ where $z = x + 3i$, x and y being real, then find the locus of the point (x, y).

**Solution :** Given, $|z - 5| = 5$ where $z = x + 3i$

or, $|x + 3i - 5| = 5$ or, $|x - 5 + 3i| = 5$

or, $\sqrt{(x-5)^2+3^2} = 5$ or, $(x - 5)^2 + 3^2 = 5^2$ or, $x^2 - 10x + 25 + 9 = 25$

or, $x^2 - 10x + 9 = 0$ is the required locus of the point (x, y). **(Ans)**

(v) Find the value of $\tan 1^0 \tan 2^0 \tan 3^0$ ....................... $\tan 89^0$

**Solution :** Given expression, $\tan 1^0 \tan 2^0 \tan 3^0$ ................. $\tan 89^0$

$= (\tan 1^0 \tan 89^0)(\tan 2^0 \tan 88^0)(\tan 3^0 \tan 87^0)$ ............. $(\tan 44^0 \tan 46^0) \tan 45^0$

$= (\tan 1^0 \cot 1^0)(\tan 2^0 \cot 2^0)(\tan 3^0 \cot 3^0)$ ............. $(\tan 44^0 \cot 44^0) \tan 45^0$

$= 1$ **(Ans)** [since $\tan 89^0 = \tan(90^0 - 1^0) = \cot 1^0$ etc.]

(vi) Prove that $\tan A + \frac{1}{\tan A} = 2\operatorname{cosec} 2A$

**Solution :** L. H. S $= \tan A + \frac{1}{\tan A} = \frac{\tan^2 A + 1}{\tan A} = \frac{\sec^2 A}{\tan A} = \frac{\cos A}{\cos^2 A \sin A}$

$= \frac{2}{2\sin A \cos A} = \frac{2}{\sin 2A} = 2\operatorname{cosec} 2A$ **(Proved)**

(vii) If $180^0 < \theta < 270^0$ and $\sin\theta = \frac{3}{5}$ find the value of $\cos\theta$.

**Solution :** Given, $\sin\theta = \frac{3}{5}$

$\therefore \cos\theta = \pm\sqrt{1-\left(\frac{3}{5}\right)^2} = \pm\sqrt{1-\frac{9}{25}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

Since, $180^0 < \theta < 270^0$ i. e., $\theta$ lies in the third quadrant,

$\therefore \cos\theta = -\frac{4}{5}$ **(Ans)**

(viii) Evaluate : $\underset{x \to 0}{\mathrm{Lt}} \frac{\sin^{-1} x}{x}$

**Solution :** Given, $\underset{x \to 0}{\mathrm{Lt}} \frac{\sin^{-1} x}{x}$

[Let, $\sin^{-1}x = \theta$, then $x = \sin\theta$; When $x \to 0$, $\theta \to 0$]

Therefore, $\underset{x \to 0}{\mathrm{Lt}} \frac{\sin^{-1} x}{x} = \underset{\theta \to 0}{\mathrm{Lt}} \frac{\theta}{\sin\theta} = \frac{1}{\underset{\theta \to 0}{\mathrm{Lt}} \frac{\sin\theta}{\theta}} = \frac{1}{1} = 1$ **(Ans)**

(ix) If $f(x)=\dfrac{1}{1+x^{b-c}+x^{a-c}}+\dfrac{1}{1+x^{c-b}+x^{a-b}}+\dfrac{1}{1+x^{c-a}+x^{b-a}}$ find $f'(0)$.

**Solution :** Given, $f(x)=\dfrac{1}{1+x^{b-c}+x^{a-c}}+\dfrac{1}{1+x^{c-b}+x^{a-b}}+\dfrac{1}{1+x^{c-a}+x^{b-a}}$

$$=\frac{x^c}{x^c+x^b+x^a}+\frac{x^b}{x^b+x^c+x^a}+\frac{x^a}{x^a+x^c+x^b}=\frac{x^c+x^b+x^a}{x^c+x^b+x^a}=1$$

Therefore, $f'(x)=0$ and hence $f'(0)=0$ **(Ans)**

(x) If the diagonals of a parallelogram are represented by the vectors 3i + j – 2k and i + 3j – 4k, then find its area.

**Solution :** Since, the diagonals of a parallelogram are represented by the vectors 3i + j – 2k and i + 3j – 4k, therefore the required vector area of the parallelogram

$$=\frac{1}{2}(3i+j-2k)\times(1+3j-4k)=\frac{1}{2}\begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & 3 & -4 \end{vmatrix}$$

$$=\frac{1}{2}\{(-4+6)i-(-12+2)j+(9-1)k\} = =\frac{1}{2}(2i+10j+8k)=i+5j+4k$$

Therefore, the required scalar area $=|i+5j+4k|=\sqrt{1^2+5^2+4^2}=\sqrt{42}$ sq. unit. **(Ans)**

## Group – C

**5.** Answer the following question (any one) : 1 × 6

**(i)** If one root of the equation $x^2 + rx - s = 0$ is square of other, prove that, $r^3 + s^2 + 3sr - s = 0$.

**Solution :** Let $\alpha$, $\alpha^2$ be the roots of the equation $x^2 + rx - s = 0$

$\therefore \alpha+\alpha^2=-\frac{r}{1}=-r$ -------- (1), and $\alpha\cdot\alpha^2=-\frac{s}{1}=-s$ or, $\alpha^3=-s$ -------- (2)

Cubing both sides of (1) we get,

$(\alpha+\alpha^2)^3=(-r)^3$ or, $\alpha^3+(\alpha^2)^3+3\alpha\cdot\alpha^2(\alpha+\alpha^2)=-r^3$

or, $\alpha^3+(\alpha^3)^2+3\alpha^3(\alpha+\alpha^2)=-r^3$ or, $-s+(-s)^2+3(-s)(-r)=-r^3$

or, $-s+s^2+3rs=-r^3$ or, $r^3+s^2+3sr-s=0$. **(Proved)**

**(ii)** Form a quadratic equation with real coefficients, whose one root is (2 – 3i).

**Solution :** Given, one root is 2 – 3i. Therefore other root is 2 + 3i.

Now, sum of the roots = (2 + 3i) + (2 – 3i) = 4

product of the roots = (2 + 3i)(2 – 3i) = 4 + 9 [ since, $i^2 = -1$ ] = 13

Therefore, the required quadratic equation is,

$x^2$ – (sum of the roots) x + p-roduct of the roots = 0

or, $x^2 - 4x + 13 = 0$ **(Ans)**

6. Answer the following question (any one) : 1 × 6

(i) If $x+iy=\sqrt{\frac{a+ib}{c+id}}$, then prove that $\left(x^2+y^2\right)^2=\frac{a^2+b^2}{c^2+d^2}$ where x, y, a, b, c, d all are real.

**Solution :** $x+iy=\sqrt{\frac{a+ib}{c+id}}$

or, $x^2-y^2+2ixy=\frac{ac+ibc-iad-i^2bd}{c^2-i^2d^2}$

or, $x^2-y^2+2ixy=\frac{ac+ibc-iad-i^2bd}{c^2-i^2d^2}$

or, $\left(x^2-y^2\right)+2ixy=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$, $\left[\because i^2=-1\right]$

or, $\left(x^2-y^2\right)+2ixy=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}$

Equating real and imaginary parts from both sides we get,

$\therefore x^2-y^2=\frac{ac+bd}{c^2+d^2}$, $2xy=\frac{bc-ad}{c^2+d^2}$

$\therefore \left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+4x^2y^2$

$=\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{bc-ad}{c^2+d^2}\right)^2$

$=\frac{(ac+bd)^2+(bc-ad)^2}{\left(c^2+d^2\right)^2}=\frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{\left(c^2+d^2\right)^2}$

$$=\frac{c^2(a^2+b^2)+d^2(a^2+b^2)}{(c^2+d^2)^2}=\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}$$

$$\therefore (x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$$ **(Proved)**

(ii) If $z_1 = 3 + 5i$, $z_2 = 1 - i$, verify the relation $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$

**Solution:** Given, $z_1 = 3 + 5i$, $z_2 = 1 - i$

Therefore, $\frac{z_1}{z_2}=\frac{3+5i}{1-i}=\frac{(3+5i)(1+i)}{(1+i)(1-i)}=\frac{3+5i+3i-5}{1+1}, \left[\because i^2=1\right]$

or, $\frac{z_1}{z_2}=\frac{-2+8i}{2}=-1+4i$

$\therefore \left|\frac{z_1}{z_2}\right|=|-1+4i|=\sqrt{(-1)^2+4^2}=\sqrt{17}$

Again, $\frac{|z_1|}{|z_2|}=\frac{|3+5i|}{|1-i|}=\frac{\sqrt{3^2+5^2}}{\sqrt{1^2+(-1)^2}}=\sqrt{\frac{34}{2}}=\sqrt{17}$

$\therefore \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$ **(Proved)**

7. Answer the following question (any one) : 1 × 6

(i) If the 4th term in the expansion of $\left(px+\frac{1}{x}\right)^n$ is independent of x, find the value of n.

Also calculate p if the 4th term be $\frac{5}{2}$.

**Solution:** Here, 4th term in the expansion of $\left(px+\frac{1}{x}\right)^n$ is

$t_4=t_{3+1}={}^nC_3\cdot(px)^{n-3}\cdot\left(\frac{1}{x}\right)^3={}^nC_3\cdot p^{n-3}\cdot x^{n-6}$

$\therefore n-6=0$ or, $n=6$ **(Ans)**

**2nd part :** By the problem, ${}^nC_3\cdot p^{n-3}=\frac{5}{2}$; n = 6 here.

or, ${}^6C_3\cdot p^{6-3}=\frac{5}{2}$ or, $\frac{6!}{3!\cdot 3!}\cdot p^3=\frac{5}{2}$ or, $\frac{6\cdot 5\cdot 4\cdot 3!}{3\cdot 2\cdot 3!}\cdot p^3=\frac{5}{2}$

or, $20p^3=\frac{5}{2}$ or, $p^3=\frac{1}{8}$ or, $p=\frac{1}{2}$ **(Ans)**

(ii) Prove that $\frac{1}{\log_a(bc)+1}+\frac{1}{\log_b(ca)+1}+\frac{1}{\log_c(ab)+1}=1$

Solution : L. H. S = $\frac{1}{\log_a(bc)+1}+\frac{1}{\log_b(ca)+1}+\frac{1}{\log_c(ab)+1}$

$=\frac{1}{\log_a(bc)+\log_a a}+\frac{1}{\log_b(ca)+\log_b b}+\frac{1}{\log_c(ab)+\log_c c}$ [$\log_a a = 1$ e. t. c.]

$=\frac{1}{\log_a(abc)}+\frac{1}{\log_b(abc)}+\frac{1}{\log_c(abc)}$

$=\log_{abc} a+\log_{abc} b+\log_{abc} c = \log_{abc}(abc)=1=$ R. H. S. **(Proved)**

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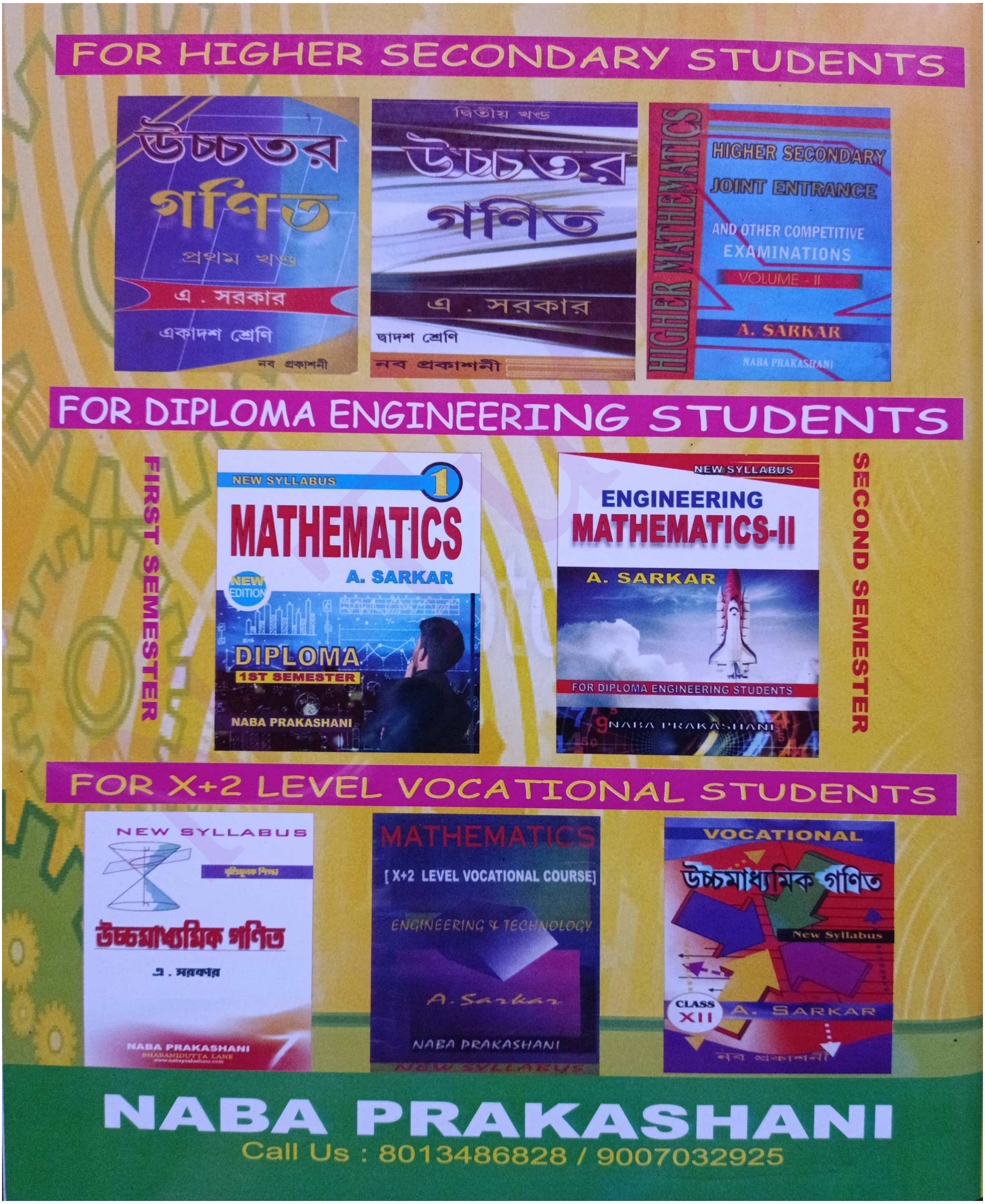
N. Tute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